Applied Hilbert's Nullstellensatz for Combinatorial Optimization

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Abstract

Various problems in Combinatorial Optimization can be modeled using *Systems of Polynomial Equations*. For instance, it is possible to model problems such as finding the *chromatic number* or the *stability number* of graphs using these systems. A well known result of Hilbert (1893) characterize the infeasibility of systems of polynomial equations by the feasibility of a *(large)* system of *linear equations*. We study the properties of the solutions of these later systems and give a concrete study for the case of determining the existence of k*-colorings* of graphs.

Colorings

Let $G = (V, E)$ be a graph and let $k \geq 2$ be an integer. We say that G is k*-colorable* if it is possible to label its vertices using k colors in such a way that no pair of adjacent vertices are labeled with the same color.

Bayer (1982) showed that it is possible to determine the existence of a k-colorings using systems of polynomial equations.

Theorem [Bayer]. *$G = (V, E)$ is k-colorable if and only if the system of polynomial equations*

$$\begin{aligned} p_u(x) &:= x_u^k - 1 = 0 & u \in V, \\ q_{uv}(x) &:= \sum_{r=0}^{k-1} x_u^{k-r} x_v^r = 0 & \{u, v\} \in E, \end{aligned} \tag{1}$$

has a solution over the complex numbers.

Hilbert's Nullstellensatz

Let $p_1, p_2, \ldots, p_m \in \mathbb{C}[x_1, \ldots, x_n]$ be polynomials on n variables over the complex numbers. Hilbert (1893) proved that the system

$$p_1(x) = p_2(x) = \cdots = p_m(x) = 0 \tag{2}$$

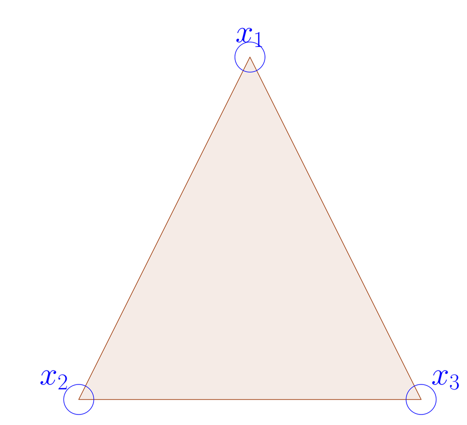
has no complex solutions if and only if there exist polynomials $r_1, \ldots, r_m \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$r_1(x)p_1(x) + \cdots + r_m(x)p_m(x) = 1. \tag{3}$$

The polynomials $r_1, r_2, \ldots, r_m$ are called **Nullstellensatz Certificates** and it is possible to find them using systems of *linear* equations.

In particular, one can use Hilbert's Nullstellensatz to determine the non-k-colorability of graphs using Bayer's Formulation.

Example: K_3 is not 2-colorable



$$\begin{aligned} p_u(x) &:= x_u^2 - 1 = 0 & u \in \{1, 2, 3\}, \\ q_{uv}(x) &:= 1 + x_u x_v = 0 & \{u, v\} \in E, \end{aligned} \tag{4}$$

Then,

$$-\frac{1}{2}x_2x_3(x_1^2 - 1) + \frac{1}{2}(1 + x_1x_2) + \frac{1}{2}(1 + x_2x_3) - \ldots - \frac{1}{2}x_1x_2(1 + x_1x_3) = 1.$$

And K_3 has a Nullstellensatz certificate *of degree* two for its non-2-colorability.

Open Problem

Theorem [De Loera et al. (2008)] *For $k \geq 3$. If $P \neq NP$, then for every d there exists a non-k-colorable graph with minimal Nullstellensatz Certificate of degree $\geq d$.*

Find a non-3-colorable graph with *minimal* Nullstellensatz Certificate of degree $d > 6$.

Duality

One idea to attack the above problem is to ask instead: *when does not exist a Nullstellensatz Certificate of degree d?*. As the existence of a Nullstellensatz Certificates is determined by systems of linear equations, we can use Freadholm's Theorem of the Alternative to show the non-existence of certificates of certain degree.

Theorem [R., Tunçel]. *Let $G = (V, E)$ be a non-3-colorable graph. G has a Nullstellensatz Certificate of degree d if and only if every solution to the system*

$$\lambda_\alpha + \lambda_{\alpha+e_u-e_v} + \lambda_{\alpha-e_u+e_v} = 0, \quad \forall \{u, v\} \in E \tag{D}$$

with $\alpha \in \mathbb{Z}_3^V$ and $|\alpha| \leq d$, satisfies $\lambda_0 = 0$.

Lower Bounds

It is possible to use the dual system to derive obtain lower bounds on the Nullstellensatz Certificates. In particular, in the cases when the graph is sparse enough we obtain the following result.

Theorem [R., Tunçel]. *Suppose that G is a non-3-colorable graph with minimal Nullstellensatz Certificate of degree d^*. If G has no cycles of length less than or equal to 6, then*

$$d^* \geq 6.$$

We believe that a graph with *girth* at least eight will be enough to force $d^* > 6$ in the above theorem.

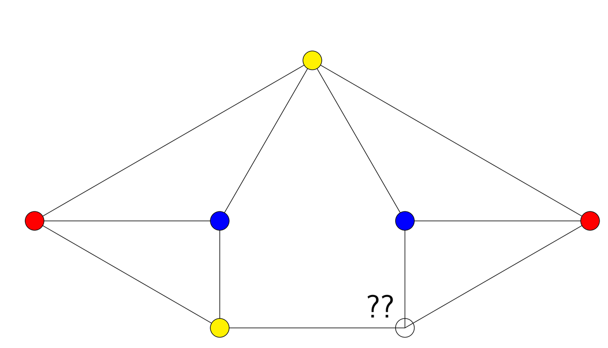


Figure 1: A non-3-colorable graph.

Upper Bounds

It is possible to find upper bounds for the Nullstellensatz Certificates of graphs having a rich 3-cycle structure.

Lemma [R., Tunçel]. *Suppose that λ is a solution to* (D). *If u, v, w are the vertices of a 3-cycle, then*

$$\lambda_\alpha = \lambda_{\alpha+e_u+e_v+e_w}$$

for all α such that $|\alpha|, |\alpha + e_u + e_v + e_w| \leq d + k$.

For instance, it is possible to use the above results to prove that the graph of Figure 1 has a Nullstellensatz Certificate of degree 3 (Ask me how!).