Harmonic analysis on polytopes and cones

São Paulo school of advanced science on algorithms, combinatorics, and optimization

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Lecture 4.

The discrete volume we would like to develop is the solid angle sum

$$A_P(t) := \sum_{n \in \mathbb{Z}^d} \omega_{tP}(n),$$

where t is now extended to all real positive numbers.

It was known from the 1960's that we have a structure theorem for such combinatorial-geometric polynomials.

Theorem (Ehrhart, Macdonald)

Suppose $P \subset \mathbb{R}^d$ is a d-dimensional integer polytope.

Then the solid angle sum $A_P(t)$ is a polynomial for positive integral values of t:

$$A_P(t) = \mathrm{vol}(P)t^d + a_{d-2}t^{d-2} + a_{d-4}t^{d-4} + \cdots + a_1 t,$$

when d is odd, and

$$A_P(t) = \mathrm{vol}(P)t^d + a_{d-2}t^{d-2} + a_{d-4}t^{d-4} + \cdots + a_2 t^2,$$

when d is even.

In lecture 3, we had the following:

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$$A_P(t) = t^d \mathrm{vol}(P) + t^d \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_P(t\xi) e^{-\epsilon ||\xi||^2}.$$

This representation is valid for any positive real $t > 0$.
Our job now is to continue to develop the theory of $\hat{1}_P(t\xi)$.

Next, we'll use Stoke's formula to do so.

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More details of the analysis

Applying Stokes' formula to $\hat{1}_P$, we get:

$$A_P(t) = t^d \text{vol}(P) + t^d \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d - \{0\}} \hat{1}_P(t\xi) e^{-\epsilon ||\xi||^2}$$

$$= t^d \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d} \left(\text{vol}(P) e^{i\Phi_\xi} \; 1_{\{0\}}(\xi) + \; \frac{-1}{2\pi i} \sum_{G \in \partial P} \frac{\langle t\xi, N_P(G) \rangle}{\|t\xi\|^2} \hat{G}(t\xi) \; 1_{\mathbb{R}^d - \{0\}}(\xi) \right)$$

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$$= t^d \mathrm{vol}(P) + \left(\frac{-1}{2\pi i} \right) t^{d-1} \sum_{G \in \partial P} \lim_{\epsilon \to 0} \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \frac{\langle \xi, N_P(G) \rangle}{\|\xi\|^2} \hat{G}(t\xi).$$

Ok, so where do these steps come from?

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Theorem. Combinatorial Stoke's Formula

(1) If $\mathrm{proj}_F(\xi) = 0$, then

$$\hat{1}_F(\xi) = \mathrm{vol}(F) e^{i\Phi_\xi},$$

and $\phi_\xi(x) := -2\pi\langle \xi, x\rangle := \Phi_\xi$ is constant on F.

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(2) If $\mathrm{proj}_F(\xi) \neq 0$, then

$$\hat{1}_F(\xi) = \frac{-1}{2\pi i} \sum_{G \in \partial F} \frac{\langle \mathrm{proj}_F(\xi), N_F(G)\rangle}{\|\mathrm{proj}_F(\xi)\|^2} \hat{1}_G(\xi).$$

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Let F be a polytope in $\mathbb{R}^d$ whose dimension satisfies $1 \le \dim(F) \le n$.

For each (codimension-one) facet $G \subset F$ let $N_F(G)$ be the unit normal vector to G that points out of F.

Let $\phi_\xi(x) := -2\pi\langle \xi, x\rangle$ denote the real linear phase function in the integral formula of the definition of $\hat{1}_F$.

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Historical notes

This combinatorial version of Stokes' formula was used by Burton Randol in the mid 1960's, in the context of (asymptotic) lattice point enumeration for irrational polytopes.

This same formula was used by Alexander Barvinok, in the 1990's (and again in 2006), for lattice point enumeration, in particular to obtain complexity results.

The same formula was also used by Skriganov in the late 1990's to obtain lattice point enumeration results in polytopes, together with ergodic techniques, for the purposes of average-case discrepancy results over polytopes.

Skriganov's analysis involves ergodicity over the space of all d-dim'l lattices:

$$SL_d(\mathbb{R})/SL_d(\mathbb{Z}).$$

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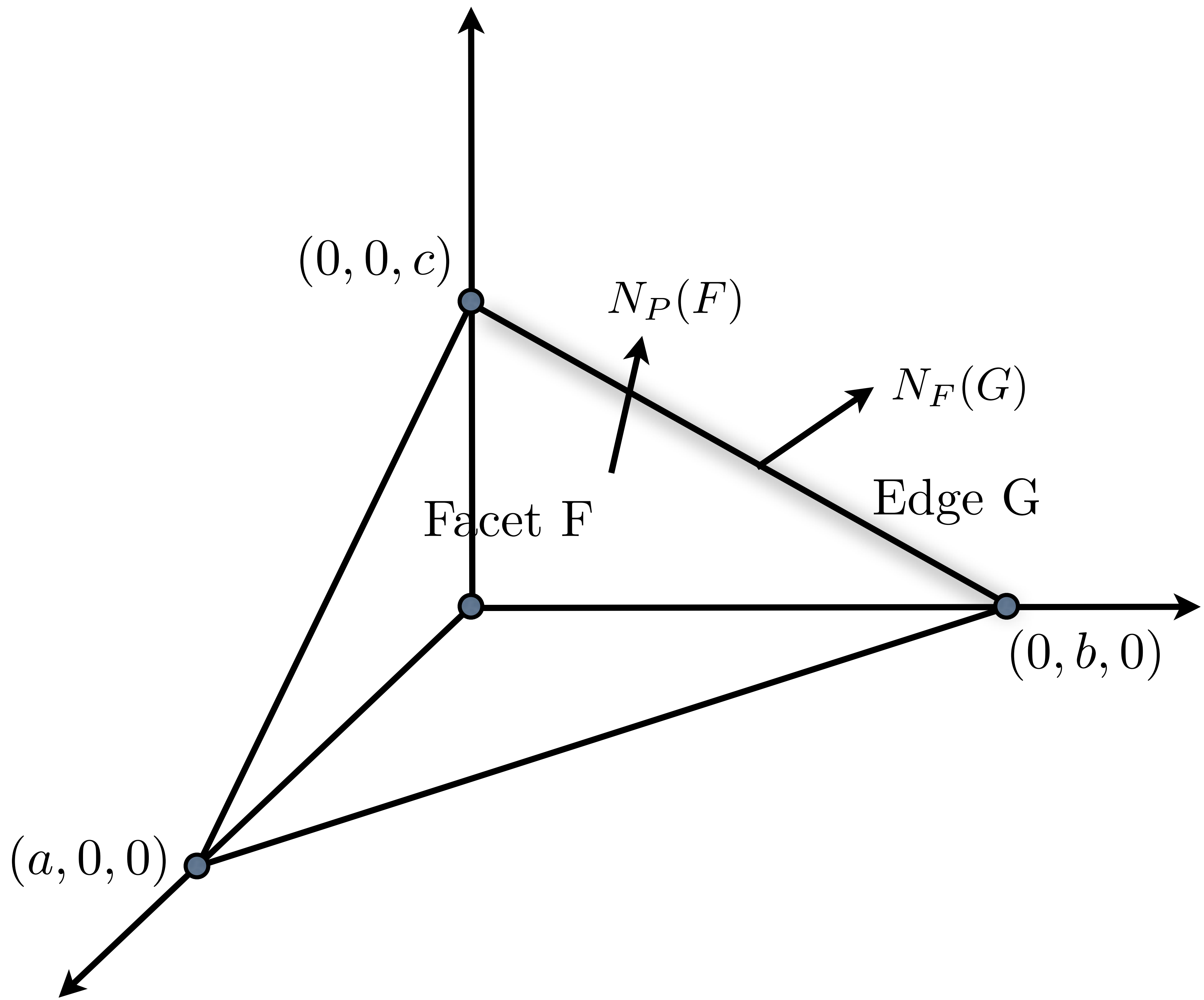

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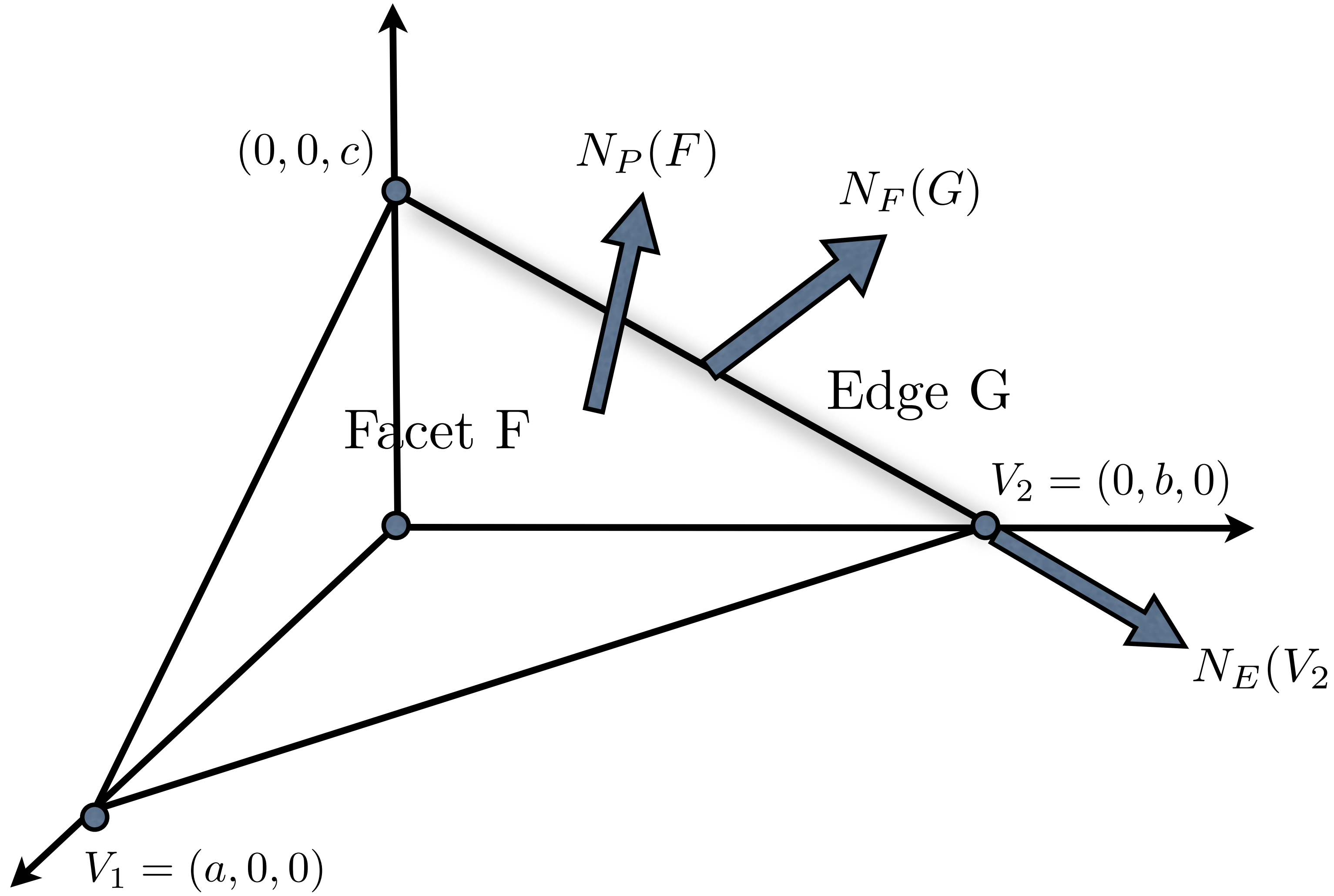
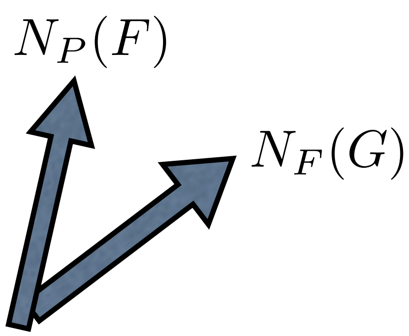
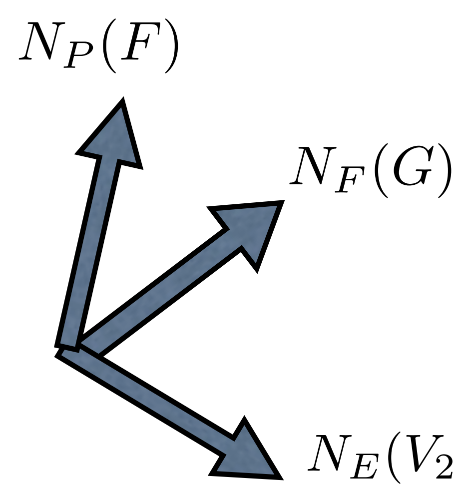
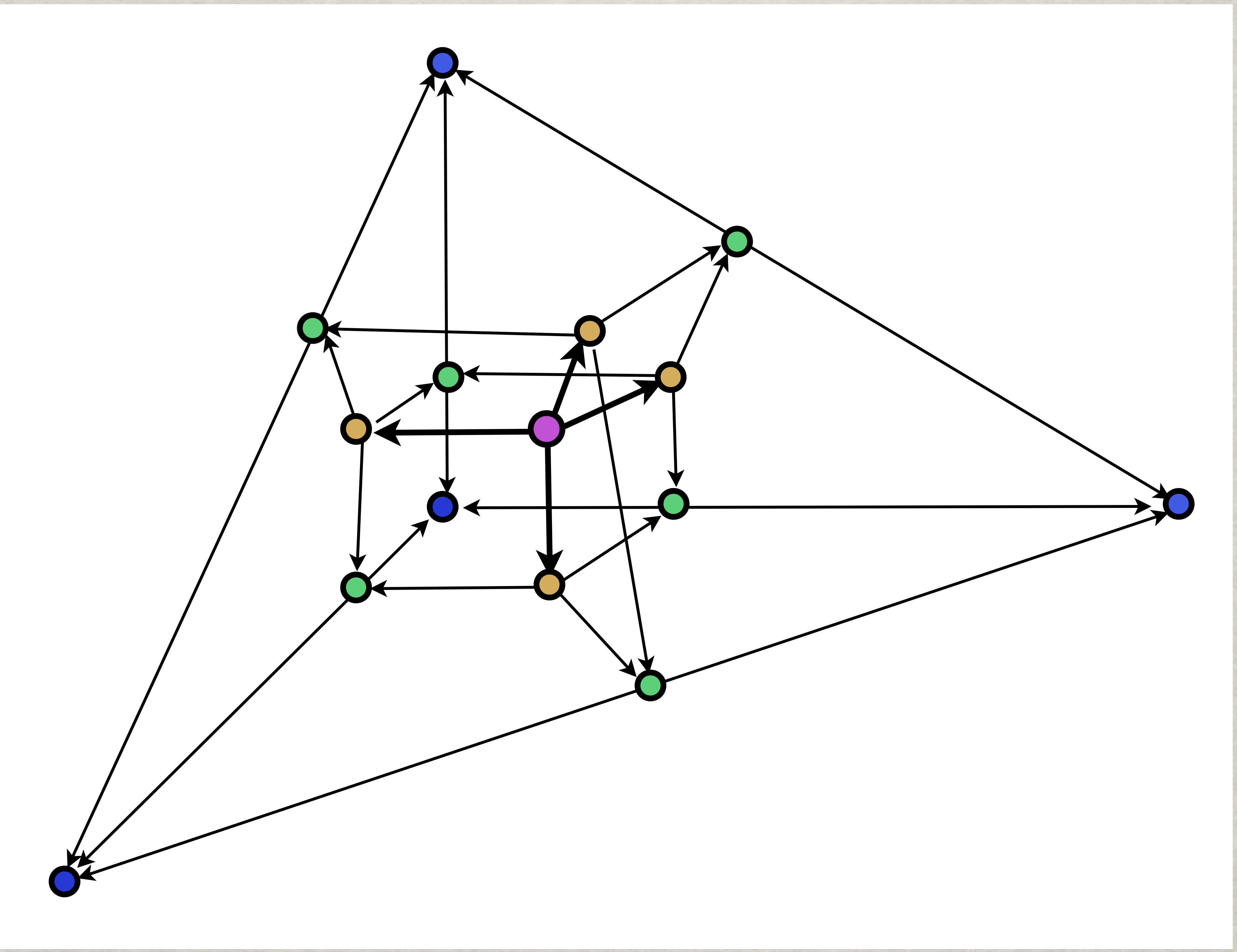




An orthogonal frame formed by normals to faces of P after 2 iterations of the discrete Stoke formula



An orthogonal frame formed by normals to faces of P after 3 iterations of the discrete Stoke formula



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This visual gives a symbolic depiction of the face poset of P, here drawn as a suggestive directed graph. We can see all the (rooted) flags, beginning from a symbolic vertex in the center, marked with the color purple.

Here the flags that terminate with the yellow vertices have length 1, those that terminate with the green vertices have length 2, and those that terminate with the blue vertices have length 3.

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We recall our goal here: To find explicit formulations for all of the coefficients of the solid angle polynomial (in t):

$$A_P(t) := \sum_{n \in \mathbb{Z}^d} \omega_{tP}(n),$$

in terms of the geometry of P.

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in terms of the geometry of P.

Theorem. Let P be a rational polytope.

Then for all positive real numbers t,

$$A_P(t) = \mathrm{vol}(P)t^d + a_{d-1}(t)t^d + \cdots + a_0(t),$$

where each coefficient is a periodic function of t, admitting a period of 1.

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Theorem (2016, Diaz, Le Quang, R)

Let P be any real polytope. Then the codimension-1 quasi-coefficient of the solid angle sum $A_P(t)$ has the following closed form:

$$a_{d-1}(t) = - \sum_{\substack{F \text{ a facet of } P \\ with\ v_F \neq 0}} \frac{\mathrm{vol}(F)}{\|v_F\|} \bar{B}_1(\langle v_F, x_F \rangle t),$$

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where v_F is the primitive integer vector which is an outward-pointing normal vector to F, x_F is any point lying in the affine span of F, and t is any positive real number.

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where v_F is the primitive integer vector which is an outward-pointing normal vector to F, x_F is any point lying in the affine span of F, and t is any positive real number.

The first periodic Bernoulli polynomial is defined by:

$$\bar{B}_1(x) := \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

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(If there does not exist a primitive integer vector that is normal to F, then by convention F contributes 0 to this sum)

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Incidentally, the Fourier series for the first periodic Bernoulli polynomial $\bar{B}_1(x)$ is quite useful:

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Incidentally, the Fourier series for the first periodic Bernoulli polynomial $\bar{B}_1(x)$ is quite useful:

(Exercise 1) Show that $\bar{B}(\langle v_F, x_F \rangle t)$ is independent of the choice of x_F lying in the affine span of F.

(Exercise 2) $$\bar{B}_1(x) = \frac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i n}}{n}.$$

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Theorem (2016, Diaz, Le Quang, R)

Let P be a d-dimensional real polytope in $\mathbb{R}^d$, and let t be a positive real number. Then we have

$$A_P(t) = \sum_{i=0}^{d} a_i(t) t^i,$$

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where

$$a_i(t) := \lim_{\epsilon \to 0} \sum_{l(T)=d-i} \sum_{\xi \in \mathbb{Z}^d \cap S(T)} R_T(\xi) E_T(t\xi)\, e^{-\pi \epsilon \|\xi\|^2}.$$

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Let's describe the notation carefully.

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The notation

What is the admissible set $S(\mathbf{T})$?

Let $\mathbf{T}$ be any flag in the face poset of P, given by

$$T := (P \to F_1 \to F_2, \ldots, \to F_{k-1} \to F_k),$$

so that by definition $\dim(F_j) = d - j$.

We define the **admissible set** $S(T)$ of the flag $\mathbf{T}$ to be the set of all vectors $\xi \in \mathbb{R}^d$ that are orthogonal to the tangent space of F_k but not orthogonal to the tangent space of F_{k-1}.

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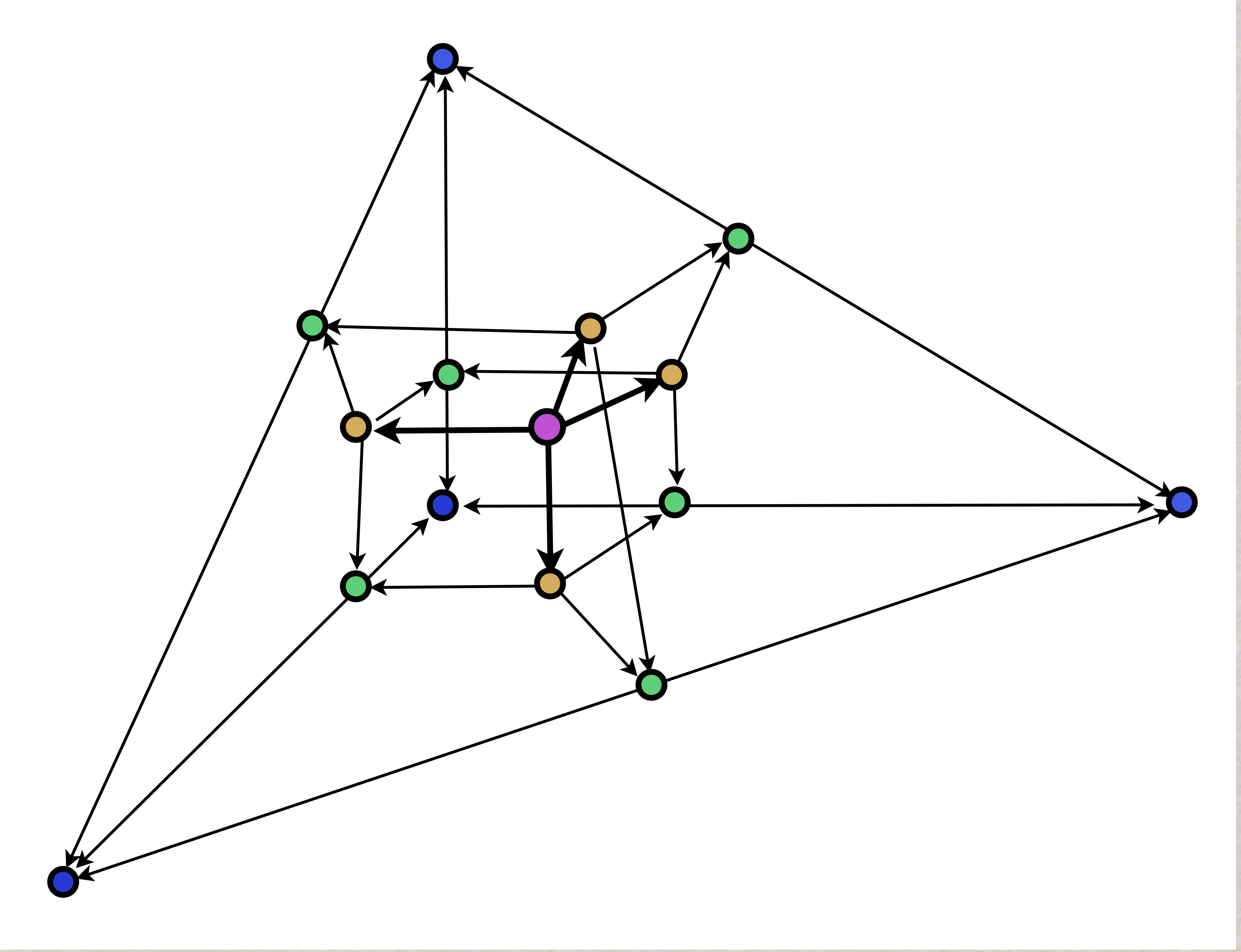

The face poset of P, where P is a tetrahedron, showing our flags T as directed paths.

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What is the rational function weight $R(\mathbf{T})$?

We attach a weight to each edge of the face poset of P:

$$W_{(F,G)}(\xi) := \frac{-1}{2\pi i} \frac{\langle \mathrm{proj}_F(\xi), N_F(G) \rangle}{\|\mathrm{proj}_F(\xi)\|^2}.$$

We notice that these weights are rational functions of ξ, and homogeneous of degree -1.

We define the rational weight $R_{\mathbf{T}}(\xi) := R_{(P \to \ldots \to F_{k-1} \to F_k)}(\xi)$ to be the product of weights associated to all the flags $\mathbf{T}$ of length one, times the Hausdorff volume of F_k (the last node of the flag $\mathbf{T}$).

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What is the exponential function $E(\mathbf{T})$?

The exponential weight $E_{\mathbf{T}}(\xi) = E_{(P \to \ldots \to F_{k-1} \to F_k)}(\xi)$ is defined to be the evaluation of $e^{-2\pi i \langle \xi, x \rangle}$ at any point x on the face F_k:

$$E_{\mathbf{T}}(\xi) := e^{-2\pi i \langle \xi, x_0 \rangle},$$

for any $x_0 \in F_k$. We note that the inner product $\langle \xi, x_0 \rangle$ does not depend on the position of $x_0 \in F_k$.

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Exercises.

Exercise 1. Show that $\bar{B}(\langle v_F, x_F \rangle t)$ is independent of the choice of x_F lying in the affine span of F.

Exercise 2. Show that $\bar{B}_1(x) = \dfrac{1}{2\pi i} \sum_{n \in \mathbb{Z}} \dfrac{e^{2\pi i n}}{n}.$

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Exercises.

Exercise 3.

Let P be a full-dimensional real polytope in $\mathbb{R}^d$.
Then the solid angle sum $A_P(t)$ satisfies the functional identity:

$$A_P(-t) = (-1)^{\dim(P)} A_P(t),$$

valid for all nonzero real numbers t.

(so in particular we can extend the solid angle polynomial to all real numbers t)

Exercises.

Exercise 4.

Work out the full solid angle sum (which is a quasi-polynomial of the real parameter t) for the triangle whose vertices are $(0,0)$, $(1,0)$, and $(0,1)$.

Exercise 5.

Work out the full solid angle sum for the Octahedron whose vertices are $(1,0,0)$, $(-1,0,0)$, $(0,1,0)$, $(0,-1,0)$,$(0,0,1)$, $(0,0,-1)$.

Exercise 6.

Compute the solid angle sum, valid for all real t, for the cross polytope

$$\diamond := \{x \in \mathbb{R}^d \mid |x_1| + \cdots + |x_d| \leq 1\}.$$

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Thank You

Reference: www.mathematicaguidebooks.org/soccer/