Harmonic analysis on polytopes and cones

São Paulo school of advanced science on algorithms, combinatorics, and optimization

Thursday, July 21, 2016

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Lecture 3.

Lattice point enumeration in polytopes: Discrete Volumes

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Lattice point enumeration in polytopes:
Discrete Volumes

One definition of a discrete volume of a polytope P is given by

$$L_P := \#\{\mathbb{Z}^d \cap P\}.$$

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Lattice point enumeration in polytopes: Discrete Volumes

One definition of a discrete volume of a polytope P is given by

$$L_P := \#\{\mathbb{Z}^d \cap P\}.$$

If we dilate P by a real number t first, and then count, we get:

$$L_P(t) := \#\{\mathbb{Z}^d \cap tP\}.$$

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Theorem (Ehrhart, 1950's)

Suppose P is an integer polytope.

Then

$$L_P(t) = \mathrm{vol}(P)t^d + c_{d-1}t^{d-1} + \cdots + c_1 t + 1,$$

a polynomial in the discrete positive integer variable t.

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a polynomial in the discrete positive integer variable t.

Ehrhart also showed that $c_{d-1} = \frac{1}{2}\text{vol}(\partial P)$, once we normalize the boundary of P with respect to the appropriate integer sublattice passing through the affine span of each facet of P.

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Theorem (Ehrhart, 1950's)

Suppose P is a rational polytope. Then

$$L_P(t) = \mathrm{vol}(P)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_1(t)t + c_0(t),$$

where each coefficient $c_k(t)$ is a periodic function of $t \in \mathbb{Z}_{>0}$.

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Theorem (Ehrhart, 1950's)

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$$L_P(t) = \mathrm{vol}(P)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_1(t)t + c_0(t),$$

where each coefficient $c_k(t)$ is a periodic function of $t \in \mathbb{Z}_{>0}$.

A lot of current research is focused on these coefficients. Intuitively, each coefficient $c_k(t)$ gives us information about the k-skeleton of P.

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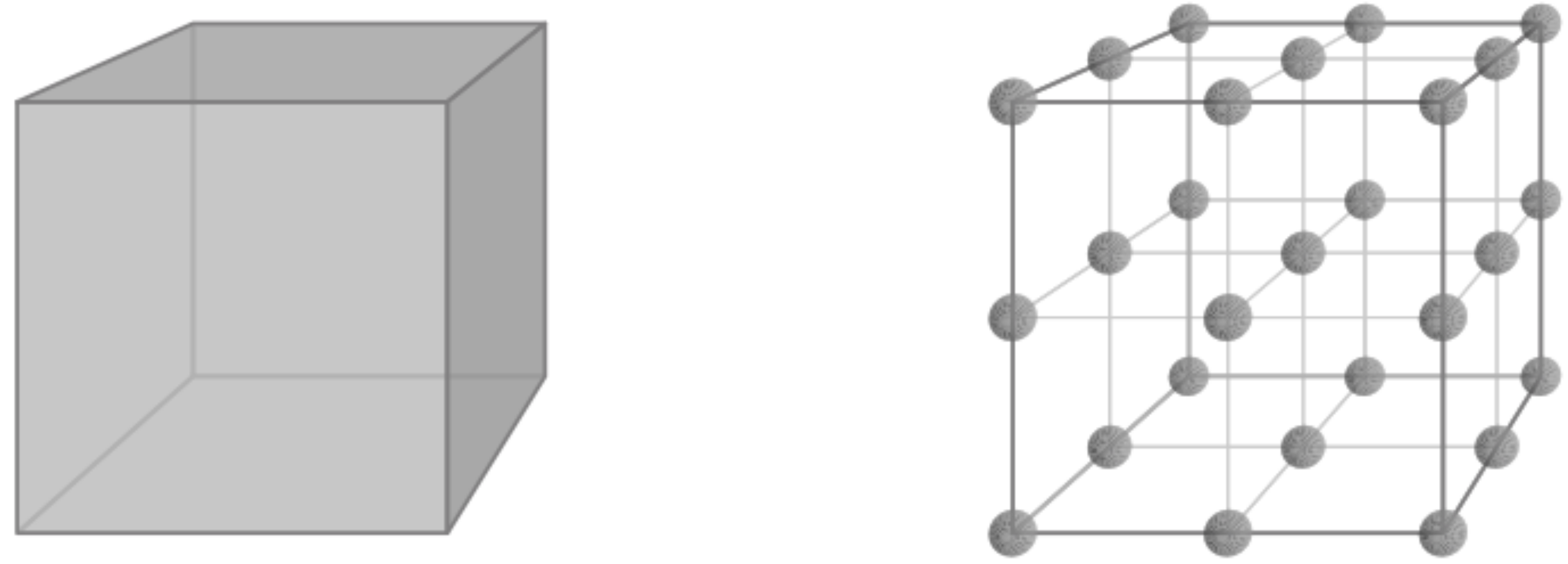
shameless plug:



Continuous and discrete volume.

M. Beck and S. Robins, Computing the continuous discretely: integer point enumeration in polytopes, Springer UTM series, 2'nd edition, 2015.

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We recall again that the **solid angle** at any point $x \in \mathbb{R}^d$, of a d-dim'l polytope $P \subset \mathbb{R}^d$, is defined by:

$$\omega_P(x) := \frac{\mathrm{vol}(S_\epsilon(x) \cap P)}{\mathrm{vol}(S_\epsilon(x))},$$

where $S_\epsilon(x)$ is a small sphere of radius ϵ, centered at x.

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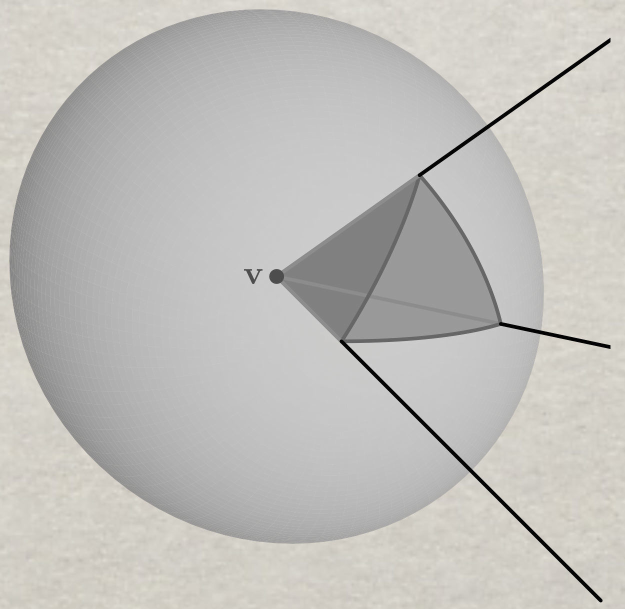

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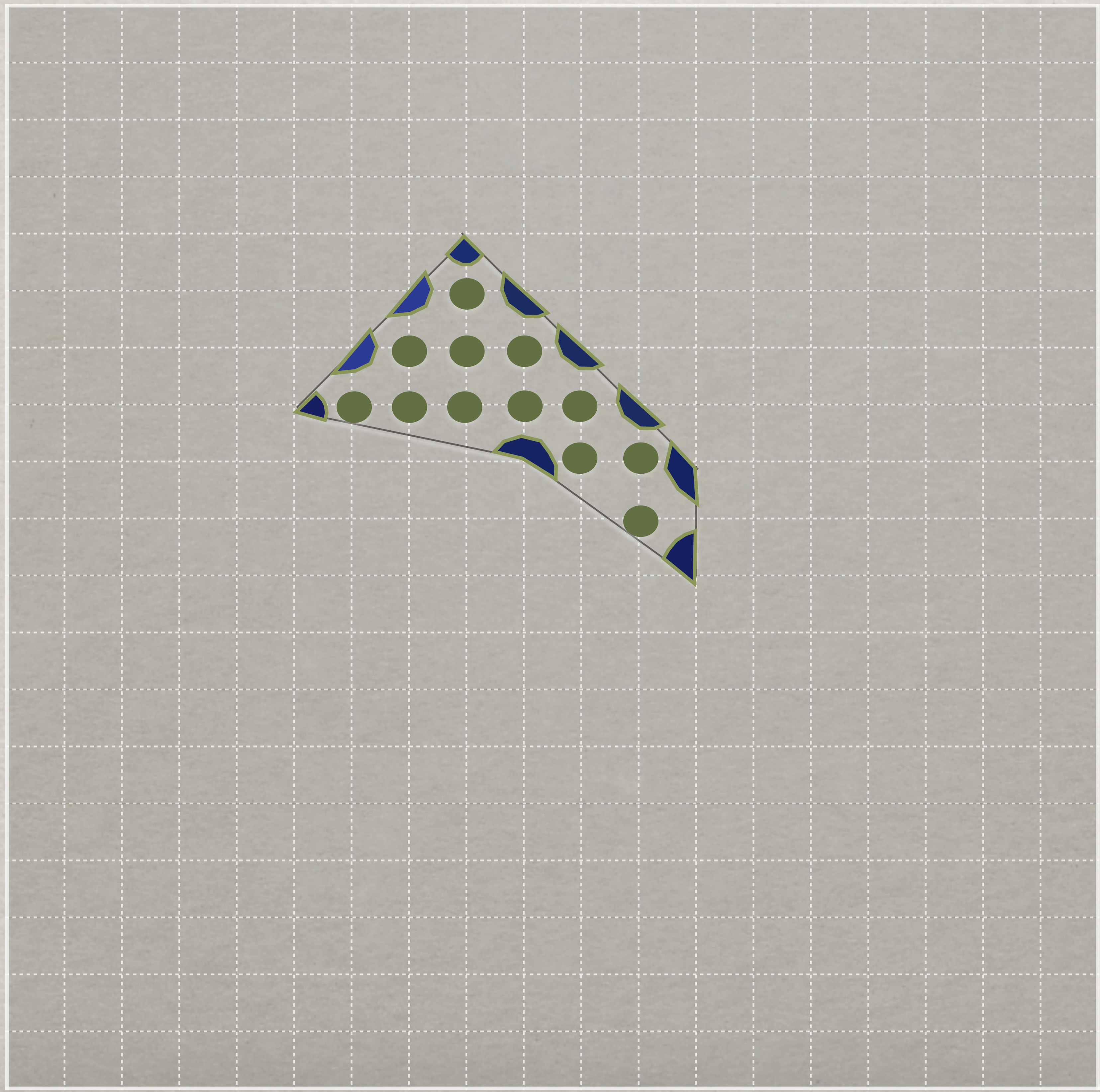
We'd like to formalize and develop these pictures:



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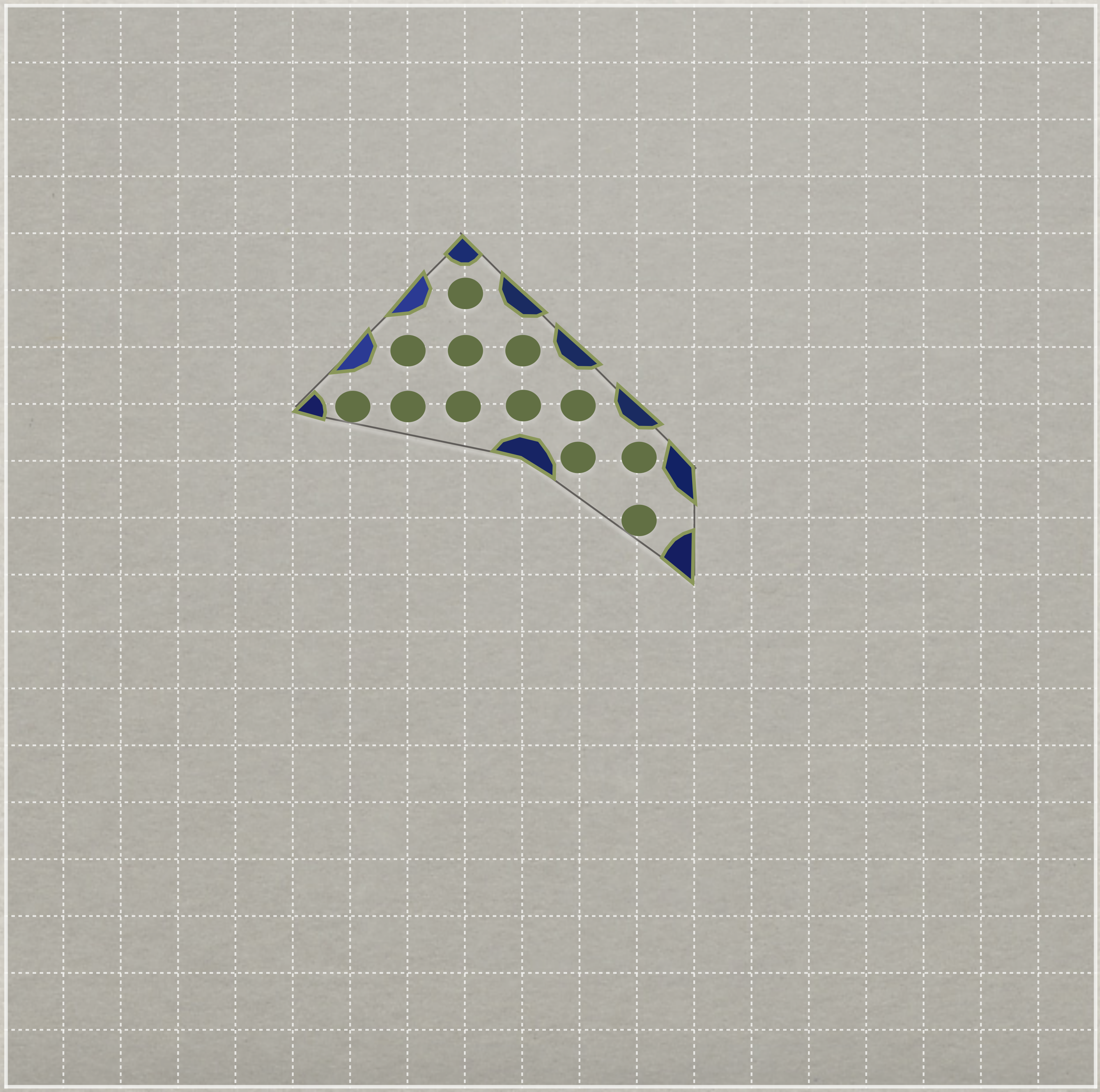
We'd like to formalize and develop these pictures:

So today we lift the carpet.....and see what is underneath the carpet.



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As before, we form the solid angle sum at all integer points:

$$A_P(t) := \sum_{n \in \mathbb{Z}^d} \omega_{tP}(n),$$

initially defined for all positive integer values of t.

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Theorem (Ehrhart, Macdonald)

Suppose $P \subset \mathbb{R}^d$ is a d-dimensional integer polytope.

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Theorem (Ehrhart, Macdonald)

Suppose $P \subset \mathbb{R}^d$ is a d-dimensional integer polytope.
Then the solid angle sum $A_P(t)$ is a polynomial for positive integral values of t:

$$A_P(t) = \mathrm{vol}(P)t^d + a_{d-2}t^{d-2} + a_{d-4}t^{d-4} + \cdots + a_1 t,$$

when d is odd, and

$$A_P(t) = \mathrm{vol}(P)t^d + a_{d-2}t^{d-2} + a_{d-4}t^{d-4} + \cdots + a_2 t^2,$$

when d is even.

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Next, we link solid angles to Harmonic analysis.

Consider the classical Gaussian:

$$G_\epsilon(x) := \frac{1}{\epsilon^{d/2}} e^{\frac{1}{\epsilon}||x||^2},$$

defined for all $x \in \mathbb{R}^d$ and for any fixed $\epsilon > 0$.

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We use this sequence of functions to "smooth" the indicator function of P. The idea is to use the convolution of functions, defined by:

$$(f * g)(x) := \int_{\mathbb{R}^d} f(u)g(x-u)du.$$

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Lemma. (folklore) For each $n \in \mathbb{R}^d$, we have

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$$\sum_{n \in \mathbb{Z}^d} (1_P * G_\epsilon)(n) = \sum_{m \in \mathbb{Z}^d} (\widehat{1_P * G_\epsilon})(m)$$

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using the standard fact that the Fourier transform of a Gaussian is another Gaussian (Exercise 1).

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$$\lim_{\epsilon \to 0} \sum_{n \in \mathbb{Z}^d} (1_P * G_\epsilon)(n) = \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{1}_P(m) e^{-\epsilon ||m||^2}$$

Using the Lemma, we now have:

$$\sum_{n \in \mathbb{Z}^d} \omega_P(n) = \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{1}_P(m) e^{-\epsilon ||m||^2}$$

combinatorial geometry

computing Fourier transforms of polytopes

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Dilating P by any real number $t > 0$, we arrive at:

$$\sum_{n \in \mathbb{Z}^d} \omega_{tP}(n) = \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d} \hat{1}_{tP}(m) e^{-\epsilon ||m||^2}$$

$$= \lim_{\epsilon \to 0} t^d \sum_{m \in \mathbb{Z}^d} \hat{1}_P(tm) e^{-\epsilon ||m||^2}$$

Exercise 3. Show that $\hat{1}_{tP}(m) = t^d \hat{1}_P(tm)$.

We notice that the $m = 0$ contribution from the RHS gives us $\hat{1}_P(0) = \mathrm{vol}(P)$, so that we have:

$$A_P(t) = t^d \mathrm{vol}(P) + \lim_{\epsilon \to 0} \sum_{m \in \mathbb{Z}^d - \{0\}} \hat{1}_P(tm) e^{-\epsilon ||m||^2}$$

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Note that we have now defined this solid angle sum for all real, positive t.

So we see how the fun begins! The next step is to sift out the "non-generic" frequencies m that are orthogonal to the facets of P, and it turns out that these m's will give the next "error" term.

And so on....

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Brion's Theorem (the continuous version)

We recall again the Brianchon-Gram relations for any polytope:

Theorem. (Brianchon-Gram)

$$1_P = \sum_{F \subset P} (-1)^{\dim F} 1_{K_F}$$

where 1_{K_F} is the indicator function of the tangent cone to F.

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Theorem (Brion, 1988)

Given a convex, simple d-dim'l polytope P, with vertex set V, and known local tangent cone data at each vertex $v_j \in V$, we have:

$$\int_P e^{2\pi i \langle z, x\rangle} dx = \sum_{v \in V} \int_{K_v} e^{2\pi i \langle z, x\rangle} dx,$$

where K_v is the vertex tangent cone at the vertex $v \in P$.

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These integrals in Brion's identity are to be thought of as the meromorphic continuation of themeselves, which are "rational-exponential" functions given below.

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The Goal

Our goal: to find a "global" proof for Brion's identity, which means a global approach to the Fourier transform of a polytope.

We use Gaussians and elementary Fourier methods.

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Proof. (SR)

Step 1. Fix $z \in \mathbb{C}^d$, multiply both sides of the Brianchon-Gram identity by the function

$$e^{2\pi i \langle z, x \rangle - \epsilon ||x||^2},$$

and integrate in x, over $\mathbb{R}^d$.

Proof.

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and integrate in x, over $\mathbb{R}^d$.

The presence of a fixed $\epsilon > 0$ now allows us to proceed with a global analysis of all cones simultaneously, avoiding the usual problems of the 'magic' of disjoint domains of convergence.

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Proof.

$$1_P(x) \;=\; \sum_{F \subset P} (-1)^{\dim F} \; 1_{K_F}(x)$$

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$$\int_{\mathbb{R}^d} 1_P(x)\, e^{2\pi i\langle x,z\rangle - \epsilon||x||^2} dx = \int_{\mathbb{R}^d} \sum_{F\subset P} (-1)^{\dim F}\, 1_{K_F}(x)\, e^{2\pi i\langle x,z\rangle - \epsilon||x||^2} dx$$

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$$\int_P e^{2\pi i \langle x, z\rangle - \epsilon ||x||^2}\, dx \;=\; \sum_{F \subset P} (-1)^{\dim F} \int_{K_F} e^{2\pi i \langle x, z\rangle - \epsilon ||x||^2} dx$$

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Proof. Step 2.

We need to compute the limit as $\epsilon \to 0$:

The left-hand side is easy, and we still have to contend with the Right-hand side:

$$\int_P e^{2\pi i\langle x,z\rangle - \epsilon||x||^2} dx = \sum_{F \subset P} (-1)^{\dim F} \int_{K_F} e^{2\pi i\langle x,z\rangle - \epsilon||x||^2} dx$$

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Proof. Step 3.

We now have to compute the limit for each tangent cone:

$$\lim_{\epsilon \to 0} \int_{K_F} e^{2\pi i \langle x, z\rangle - \epsilon ||x||^2} dx$$

We analyze each tangent cone integral based on the following distinction among tangent cones:

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We analyze each tangent cone integral based on the following distinction among tangent cones:

If there exists a vector $v \in \mathbb{R}^d$ such that

$$K_F + v = K_F,$$

then we call K_F a **type I tangent cone**.
Otherwise, we call K_F a **type II tangent cone**.

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Note: A type I tangent cone is also called a "translation-invariant" cone.

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Proof. Step 3.

From step 2, we must make a local computation for each tangent cone K_F, according to:

$$\int_P e^{2\pi i\langle x,z\rangle} dx = \lim_{\epsilon\to 0} \sum_{F\subset P} (-1)^{dim F} \int_{K_F} e^{2\pi i\langle x,z\rangle - \epsilon||x||^2}$$

For the invariant tangent cones (type I), it is easy to show that their Fourier-Laplace transform vanishes.
For the type II tangent cones, it is less straightforward to compute the nontrivial limit, but not that bad (we'll use iterated integration by parts).

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Proof. Step 4.

Lemma.

(a) Let K_v be a d-dim'l simplicial pointed cone, (type II) with vertex v and with edge vectors $w_1, \ldots, w_d \in \mathbb{R}^d$.

Then

$$\lim_{\epsilon \to 0} \int_{K_v} e^{2\pi i \langle x, z\rangle - \epsilon ||x||^2} dx = \left(\frac{-1}{2\pi i}\right)^d \frac{e^{2\pi i \langle v, z\rangle} |\det K_v|}{\prod_{k=1}^d \langle w_k(v), z\rangle}.$$

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(b) For any type I cone K_F, we have

$$\lim_{\epsilon \to 0} \int_{K_F} e^{2\pi i \langle x, z\rangle - \epsilon ||x||^2} dx = 0.$$

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Proof. Step 4.

Here we use integration by parts, (iterated d times).

To see why part (a) of the Lemma is true, consider the 1-dimensional case:

$$\int_0^\infty e^{2\pi i x z - \epsilon x^2} dx =$$

Proof. Step 4.

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To see why part (a) of the Lemma is true, consider the 1-dimensional case:

$$\int_0^\infty e^{2\pi i x z - \epsilon x^2} dx = e^{-\epsilon x^2} \frac{e^{2\pi i x z}}{2\pi i z}\Big|_0^\infty - \int_0^\infty \frac{e^{2\pi i x z}}{2\pi i z}(-2\epsilon x) e^{-\epsilon x^2} dx$$

$$= \frac{-1}{2\pi i z} + \frac{\epsilon}{\pi i z}\int_0^\infty x e^{2\pi i x z - \epsilon x^2} dx$$

$$= \frac{-1}{2\pi i z} + \frac{\sqrt{\epsilon}}{\pi i z}\int_0^\infty u e^{2\pi i \frac{u}{\sqrt{\epsilon}} z - u^2} du,$$

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$$= \frac{-1}{2\pi i z} + \frac{\epsilon}{\pi i z} \int_0^\infty x e^{2\pi i x z - \epsilon x^2} dx$$

$$= \frac{-1}{2\pi i z} + \frac{\sqrt{\epsilon}}{\pi i z} \int_0^\infty u e^{2\pi i \frac{u}{\sqrt{\epsilon}} z - u^2} du,$$

where we have used the substitution $u := \sqrt{\epsilon}\, x$ in the last step. Therefore,

$$\lim_{\epsilon \to 0} \int_0^\infty e^{2\pi i x z - \epsilon x^2} dx = \frac{-1}{2\pi i z}.$$

$\square$

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Summary

1. This proof of the (continuous) Brion identity uses a global analytic approach, as opposed to all the previous local approaches taken before.

2. Only one fixed value of $z \in \mathbb{C}^d$ is needed to allow all integrals to simultaneously converge, namely any value of z for which $\langle w, \Re(z)\rangle \neq 0$, or for which $\langle w, \Im(z)\rangle \neq 0$, for all edge vectors w of P.

In other words, if either the real or imaginary part of z lies outside this finite union of hyperplanes, it suffices. Hence, no meromorphic continuation of any transforms is required.

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Exercises.

Exercise 1.

Consider the classical Gaussian:

$$G_\epsilon(x) := \frac{1}{\epsilon^{d/2}} e^{\frac{1}{\epsilon}||x||^2},$$

defined for all $x \in \mathbb{R}^d$ and for any fixed $\epsilon > 0$.
Prove that its Fourier transform equals

$$\hat{G}_\epsilon(n) = e^{\epsilon||x||^2}.$$

Exercise 2.

Prove part (b) of the Lemma, namely that if we have a tangent cone K_F which is translation invariant by some vector v, then

$$\lim_{\epsilon \to 0} \int_{K_F} e^{2\pi i \langle x, z\rangle - \epsilon||x||^2} dx = 0.$$

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Exercises.

Exercise 3.

Suppose that P is a polytope, and t a positive real number.

$$\text{Show that } \hat{1}_{tP}(m) = t^d \hat{1}_P(tm).$$

Exercise 4.

More generally, suppose that $M \in GL_d(\mathbb{R})$, and P any polytope.

$$\text{Show that } \hat{1}_{M(P)}(m) = |\det(M)|\, \hat{1}_P(M^t m),$$

where $M(P)$ is the image of P under the linear transformation M, and where M^t is the transpose of the matrix M.

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Exercises.

Exercise 5.

Show that if c is a constant, then $\hat{f}(x+c) = \hat{f}(x)e^{2\pi i x c}$.

Exercise 6.

Show that $\widehat{(f * g)}(x) = \hat{f}(x)\hat{g}(x)$.

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Thank You

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