Lecture 2.

Part I. History of tilings (1-tilings)

Part II. Multi-tilings (k-tilings), recent results

Part III. Harmonic analysis approaches/ideas

Part I. History of tilings (1-tilings)

What kind of tilings?

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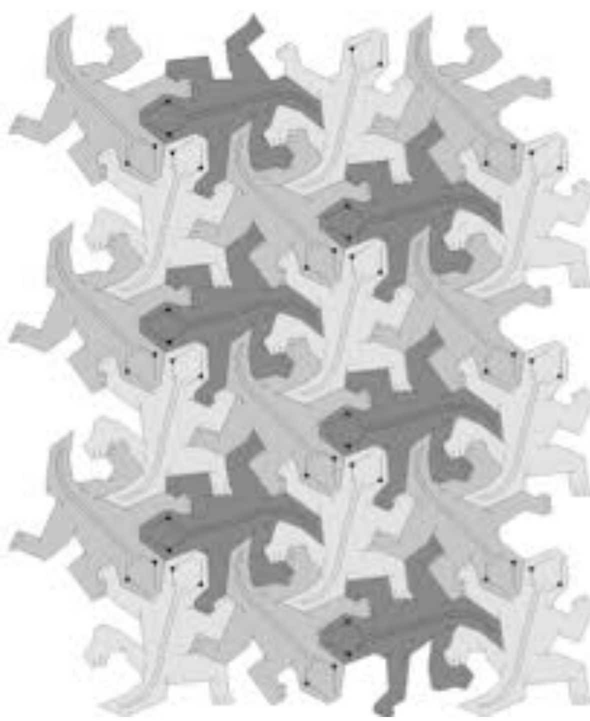
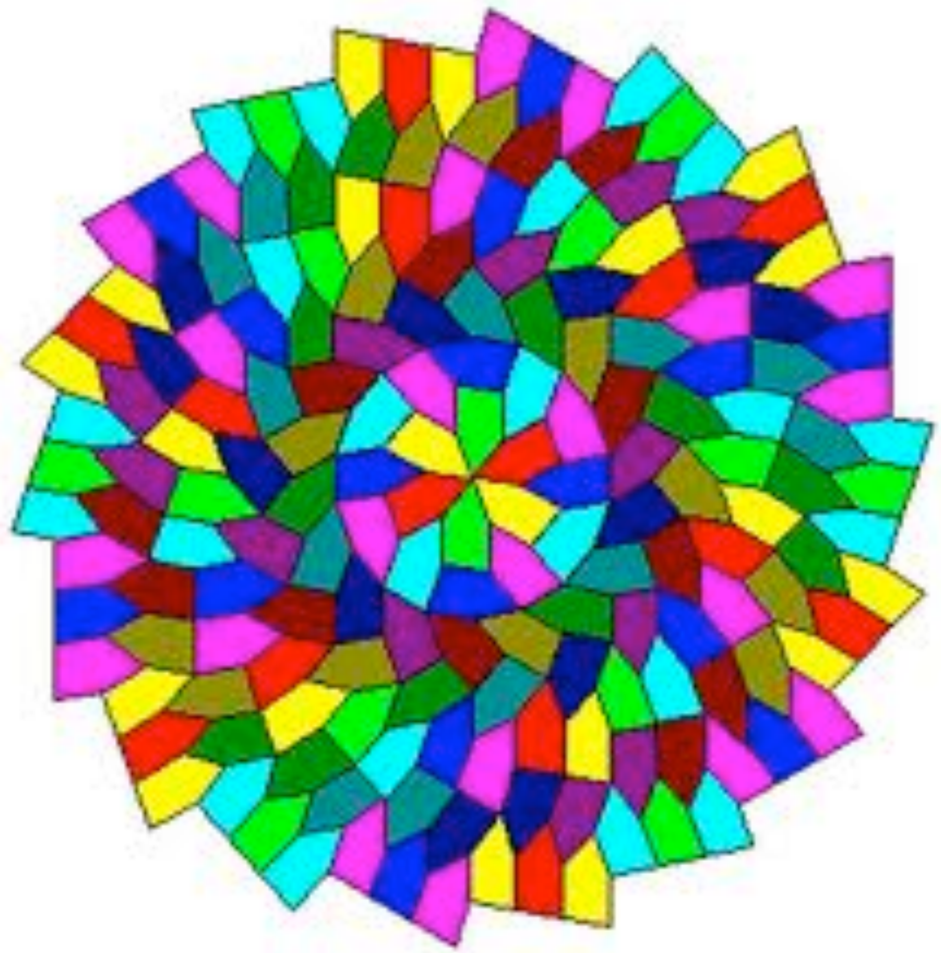


? Not for this theory

Part I. History of tilings (1-tilings)

Aperiodic tiling with one object: The Hirschhorn tiling

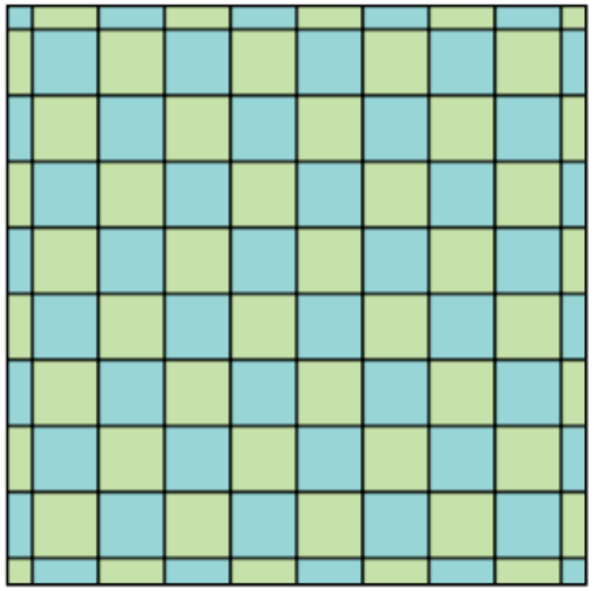
(Michael Hirschhorn, 1976, UNSW)

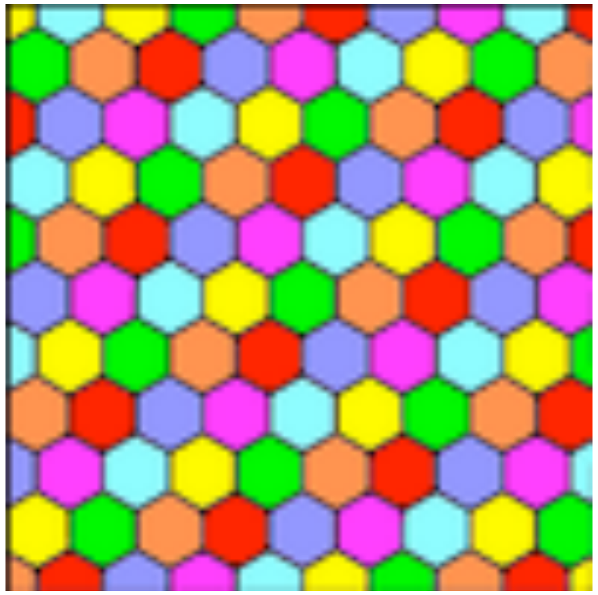


Part I. History of tilings (1-tilings)

What kind of tilings do we study here?

1. We fix one object
2. Even more, we focus on translational tilings
3. Finally, we invoke the assumption that our object is convex.





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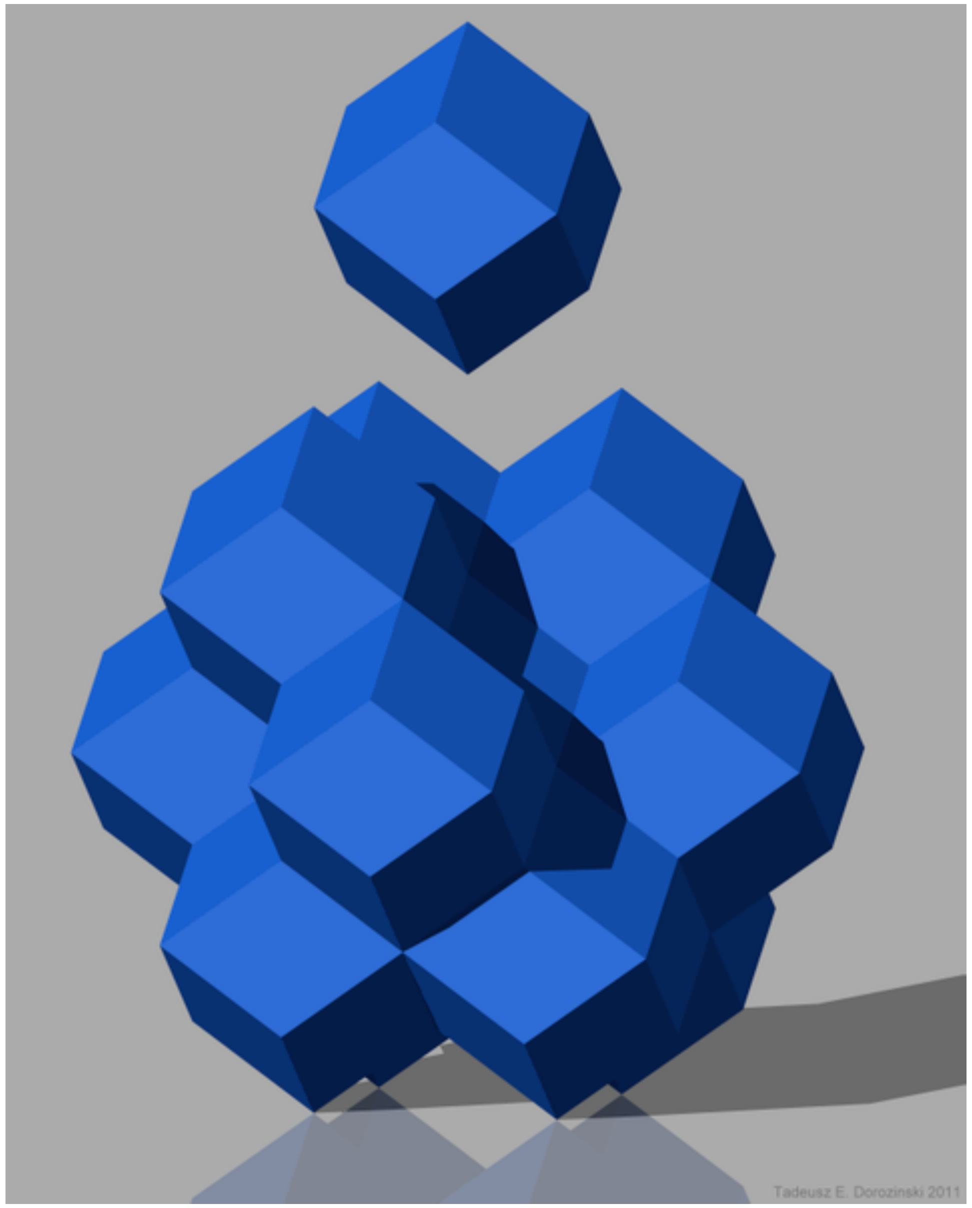
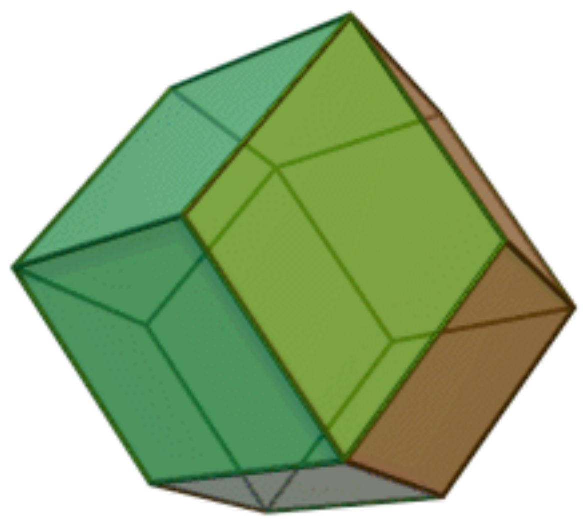


So we consider translations by one convex object P (necessarily a polytope), and we tile Euclidean space by a set of **discrete** translation vectors Λ, so that (almost) every point gets covered exactly once.

Example.



This Fedorov solid (also known as a Rhombic Dodecahedron) tiles $\mathbb{R}^3$

Indicator functions

Definition.

Given any set $P \subset \mathbb{R}^d$, we define

$$1_P(x) := \begin{cases} 1 & \text{if } x \in P \\ 0 & \text{if } x \notin P. \end{cases}$$

Definition.

We say that a polytope P is symmetric (about the origin) if for any $x \in P$, we also have $-x \in P$.

So to be a bit more Bourbaki about it, we may write:

Definition.

We say that P tiles $\mathbb{R}^d$ with the discrete multi set of vectors Λ if

$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = 1,$$

for all $v \notin \partial P + \Lambda$.

Question 1. What is the structure of a polytope P that tiles all of Euclidean space by translations, with some discrete set of vectors Λ?

For example, when is it a zonotope? What do its facets look like?

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Question 2. What is the structure of the discrete set of vectors Λ?

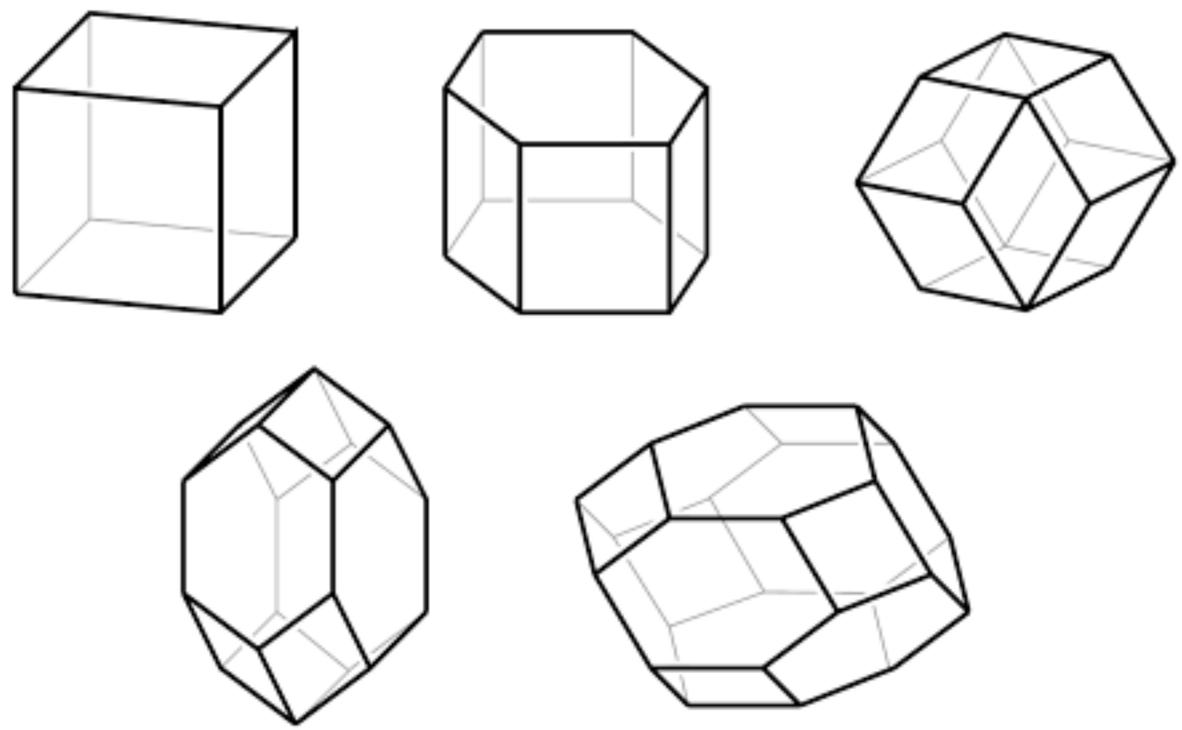
For example, does Λ have to be a lattice? When? Why?

Can Λ be a finite union of lattices?

1-tilings in $\mathbb{R}^3$

Theorem. (Fedorov, 1885) There are 5 different combinatorial types of convex bodies that tile $\mathbb{R}^3$.

Nikolai Fyodorovich Fedorov



1-tilings in $\mathbb{R}^d$

What about higher dimensions? Can we "classify" all polytopes that tile $\mathbb{R}^d$ by translations?

Minkowski gives a partial answer

The first results for tiling Euclidean space in general dimension were given by Hermann Minkowski.

Minkowski gave necessary conditions for a polytope to tile $\mathbb{R}^d$.

Minkowski's result

Theorem. (Minkowski, 1897)
If a convex polytope P tiles $\mathbb{R}^d$ by translations, then:

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What's that?

Zonotopes

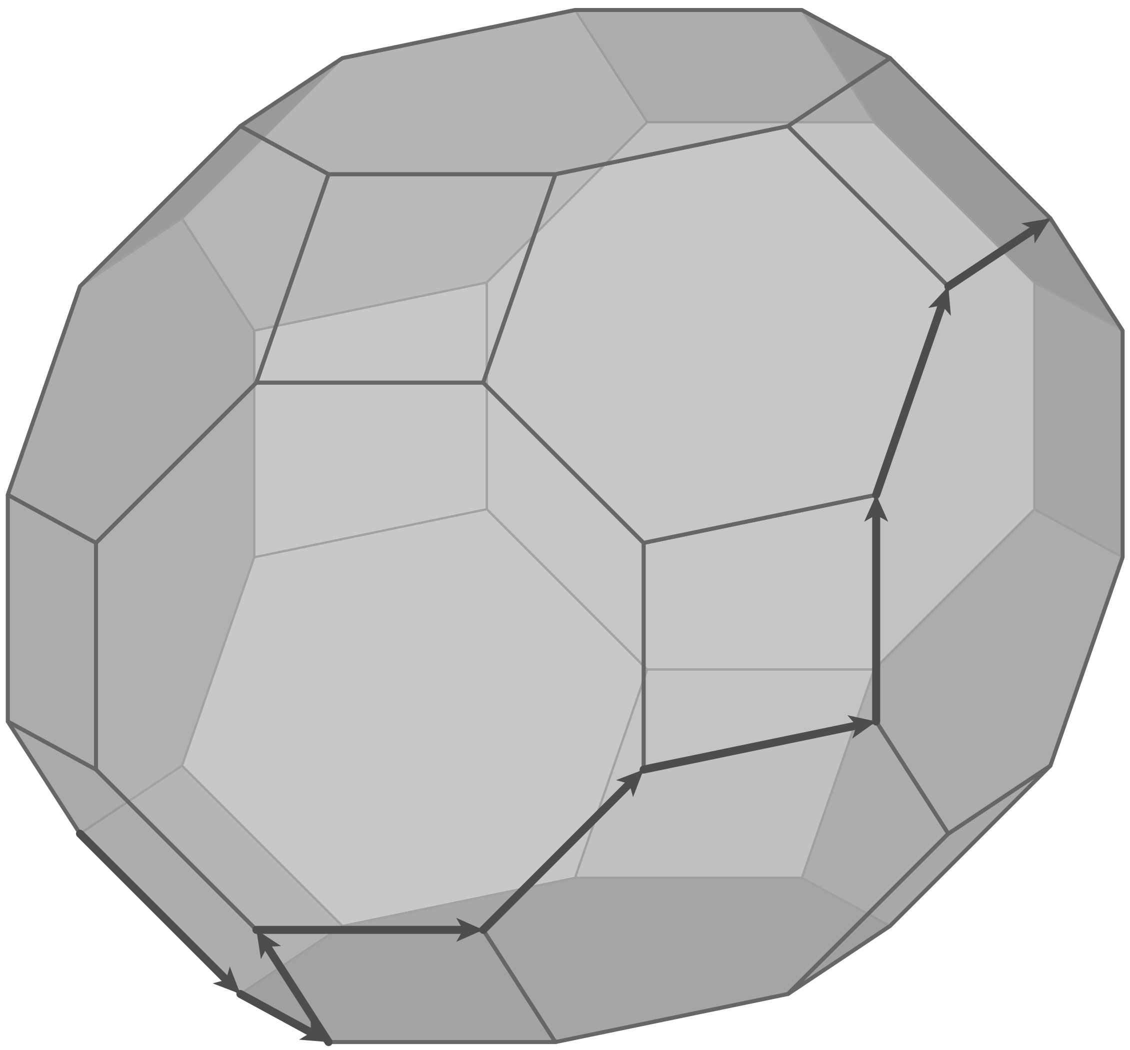
Definition.

A Zonotope is a polytope P with the following equivalent properties:
1. All of the faces of P are centrally symmetric

2. P is the Minkowski sum of a finite number of line-segments

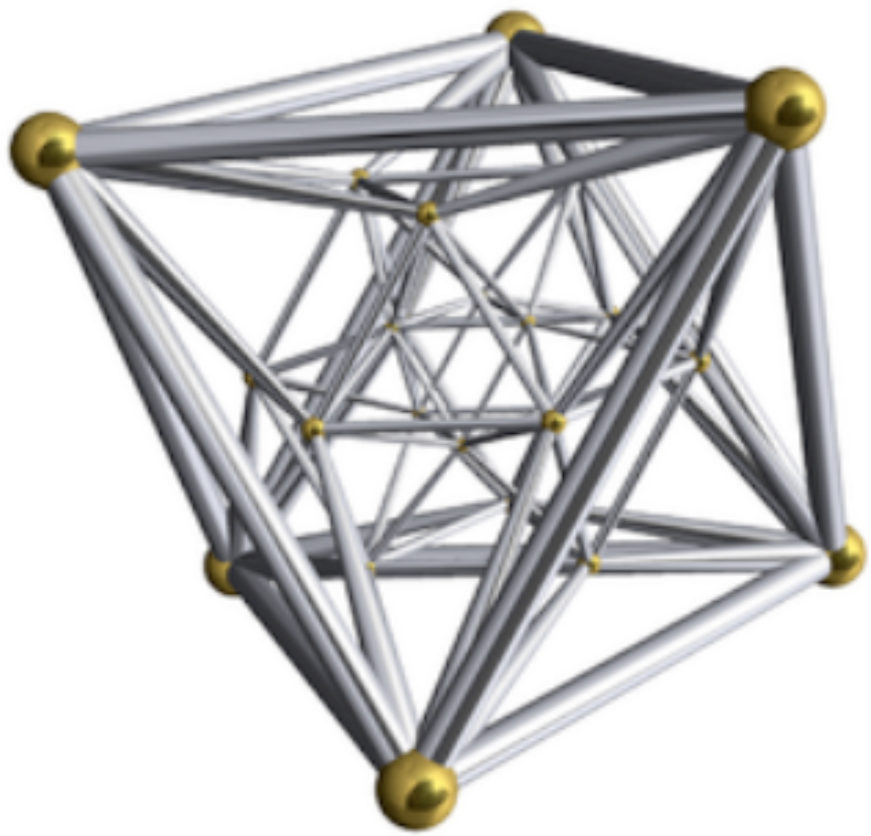
3. P is the affine image of some n-dimensional cube $[0,1]^n$.

Example. A zonotope with 9 generators



This is the projection of a 9-dimensional cube into $\mathbb{R}^3$

The 24-cell, a source of counterexamples



The 24-cell is a 4-dimensional polytope, arising as the Voronoi cell of the lattice $D_4 \subset \mathbb{R}^4$.

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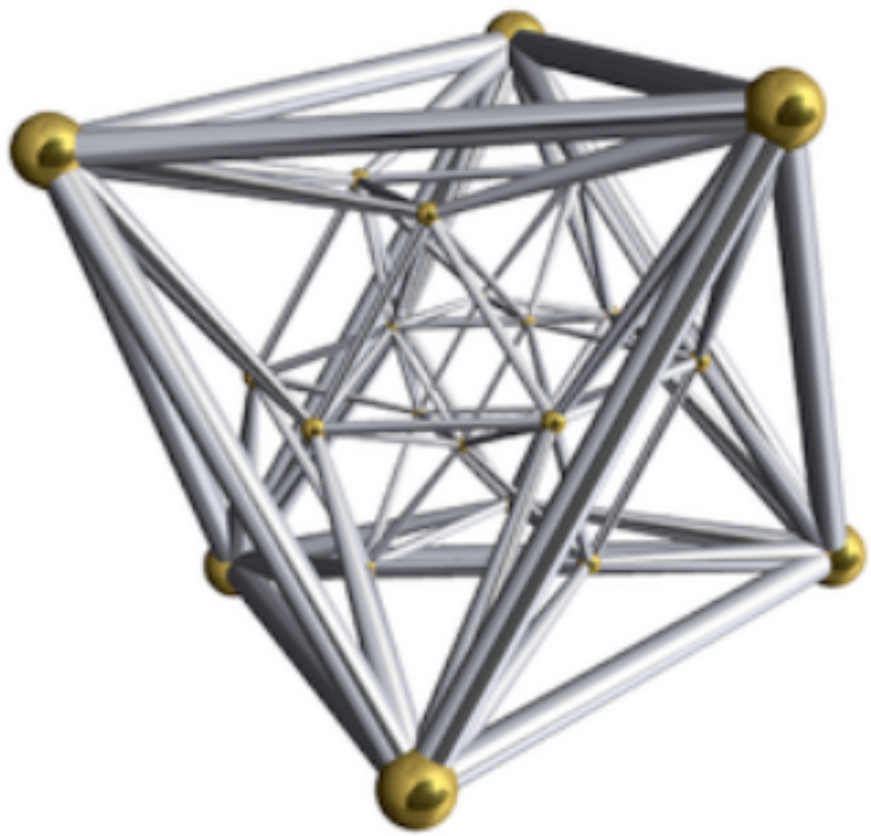

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The lattice D_4 is defined by:

$$D_4 := \{\mathbf{x} \in \mathbb{Z}^{\mathbf{d}} \mid \sum_{\mathbf{k=1}}^{\mathbf{d}} \mathbf{x_k} \equiv \mathbf{0} \bmod \mathbf{2}\}$$

It tiles $\mathbb{R}^4$ but it is **not** a zonotope.

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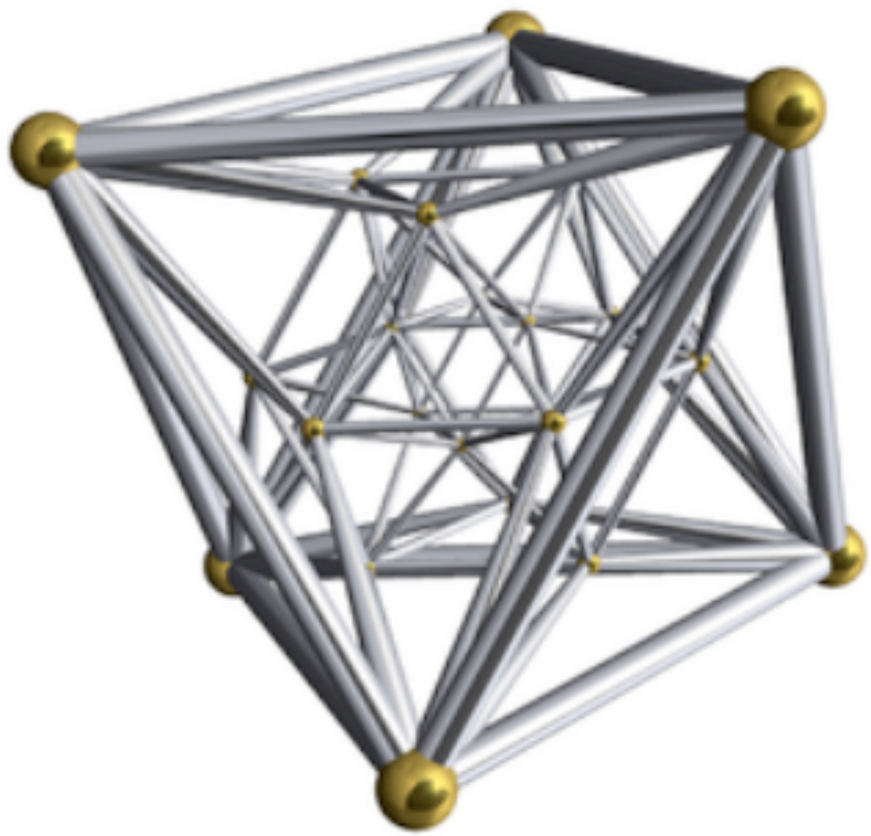

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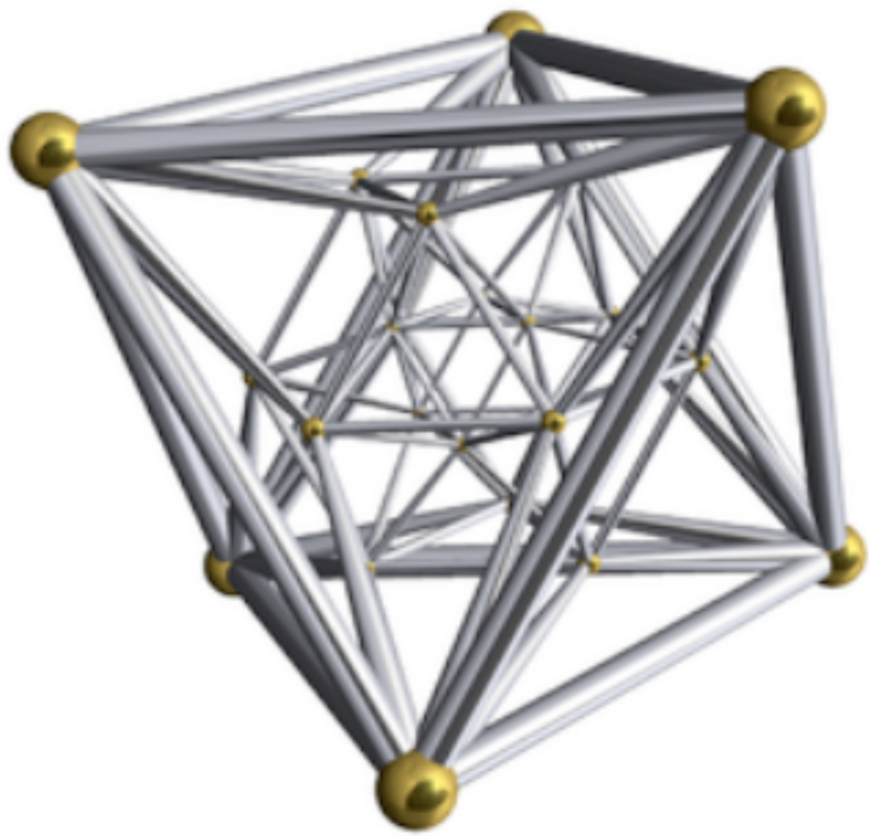

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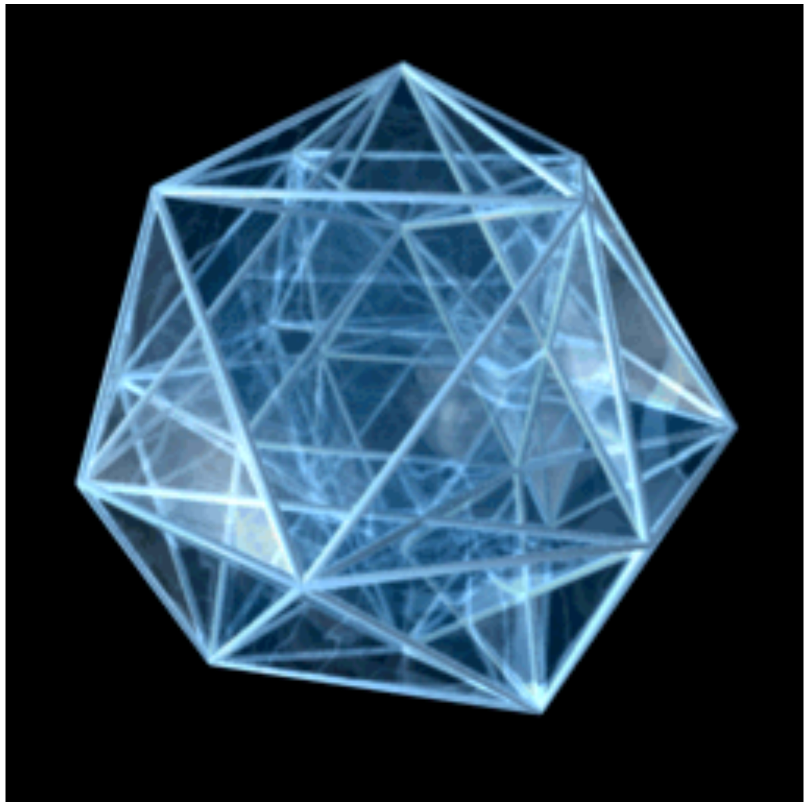
Quiz. why not?

Answer. It has a face which is not centrally symmetric.

Def. A Voronoi cell (at the origin) of any lattice $\mathcal{L}$ is defined to be

$$\{\mathbf{x} \in \mathbb{R}^{\mathbf{d}} \mid \mathbf{d}(\mathbf{x}, \mathbf{0}) \leq \mathbf{d}(\mathbf{x}, \mathbf{l}), \text{ for all } \mathbf{l} \in \mathcal{L}\}$$

This region is, almost by definition, a polytope (why?).



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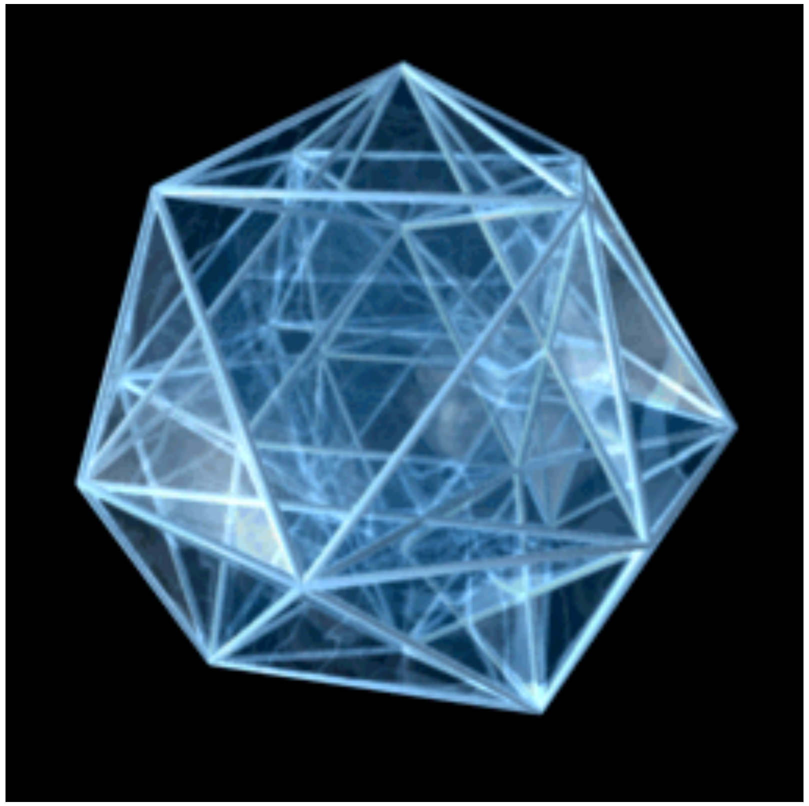

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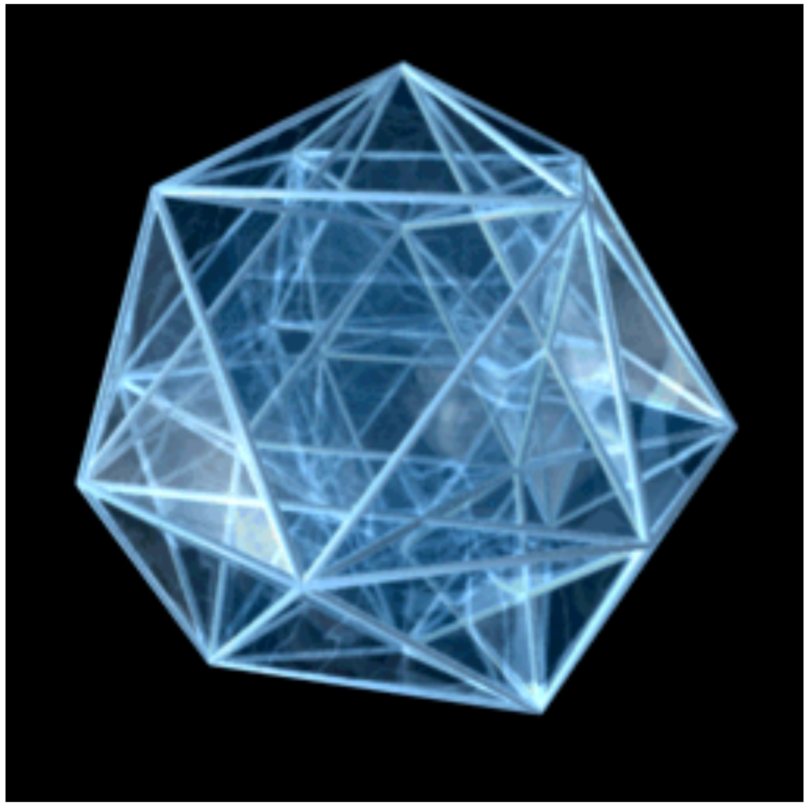

The Venkov-McMullen result, a converse to Minkowski

After 50 years passed, a converse to Minkowski's Theorem was found.

Theorem. (Minkowski, 1897; Venkov, 1954; McMullen, 1980)

A convex polytope P tiles $\mathbb{R}^d$ by translations if and only if:

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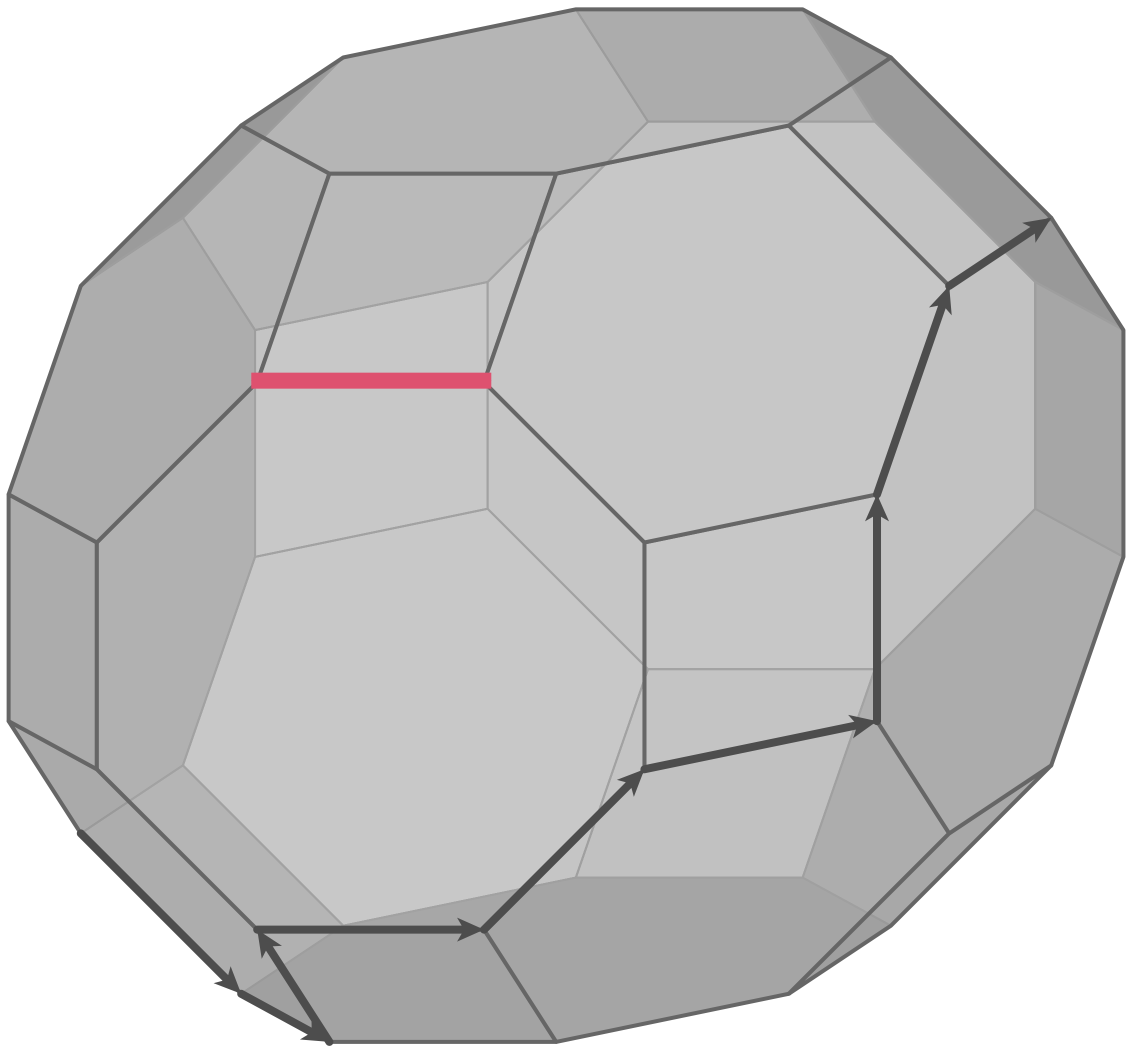
Example. The red belt for this zonotope consists of 8 faces (1-dimensional faces).



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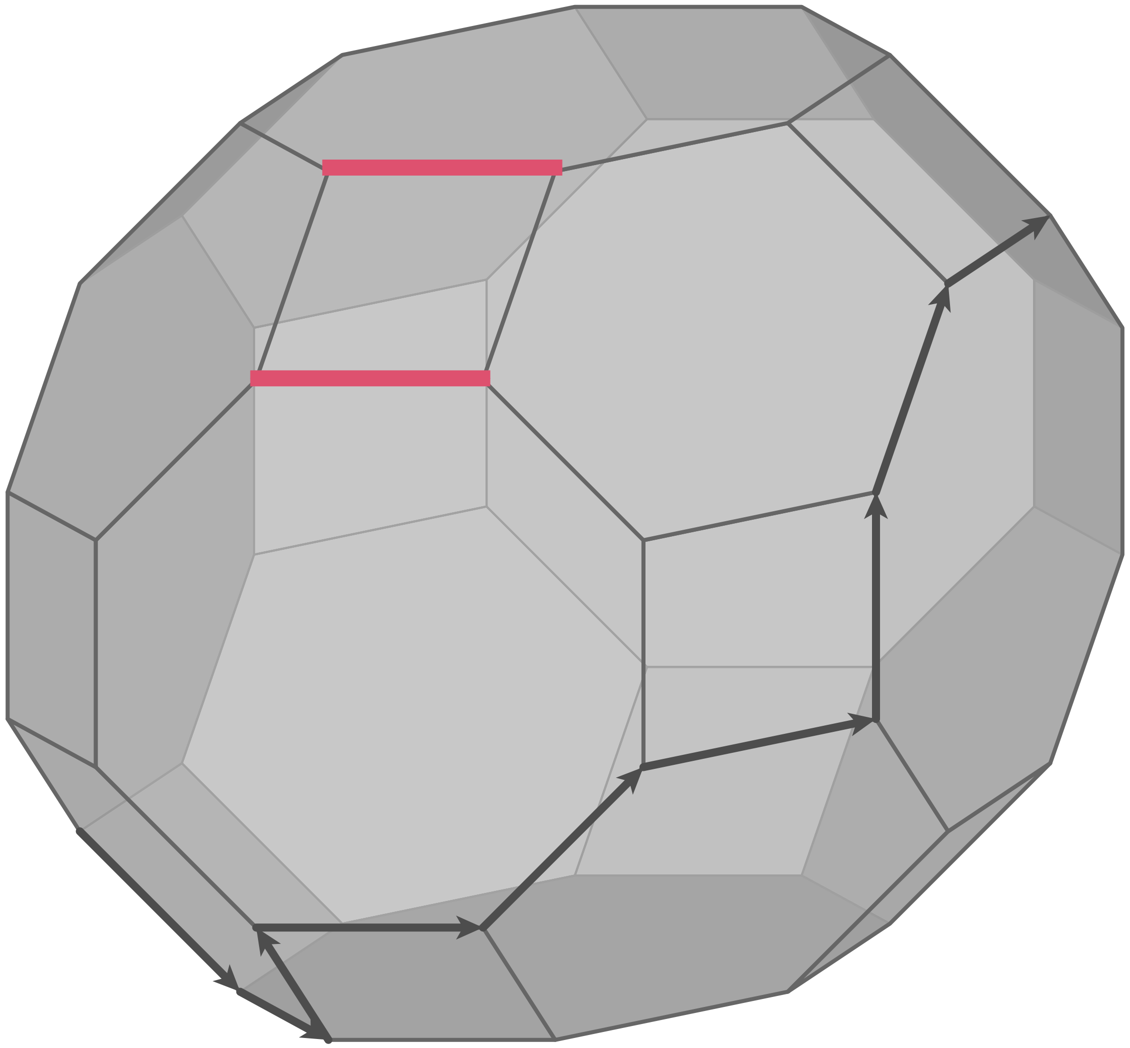

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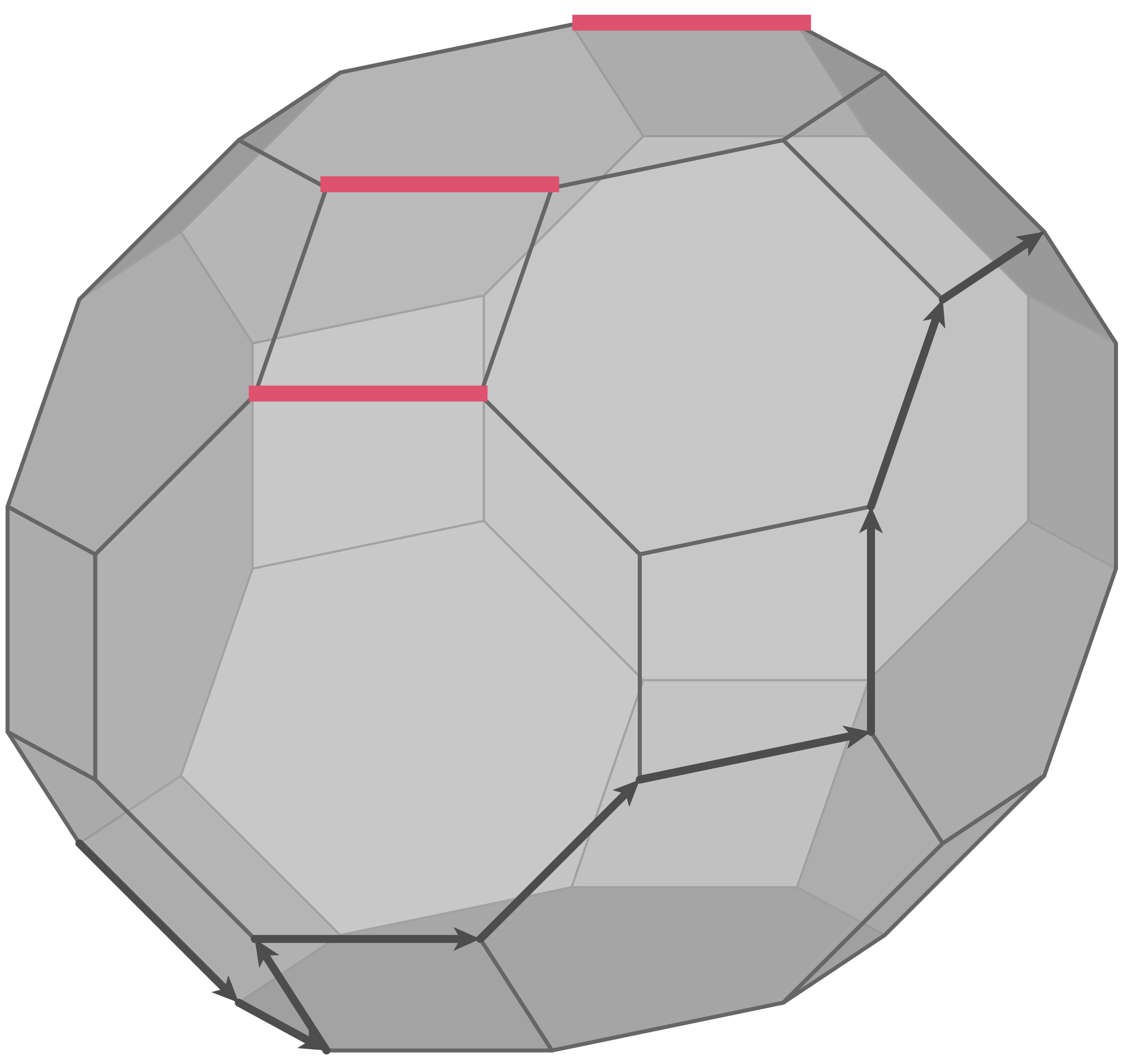

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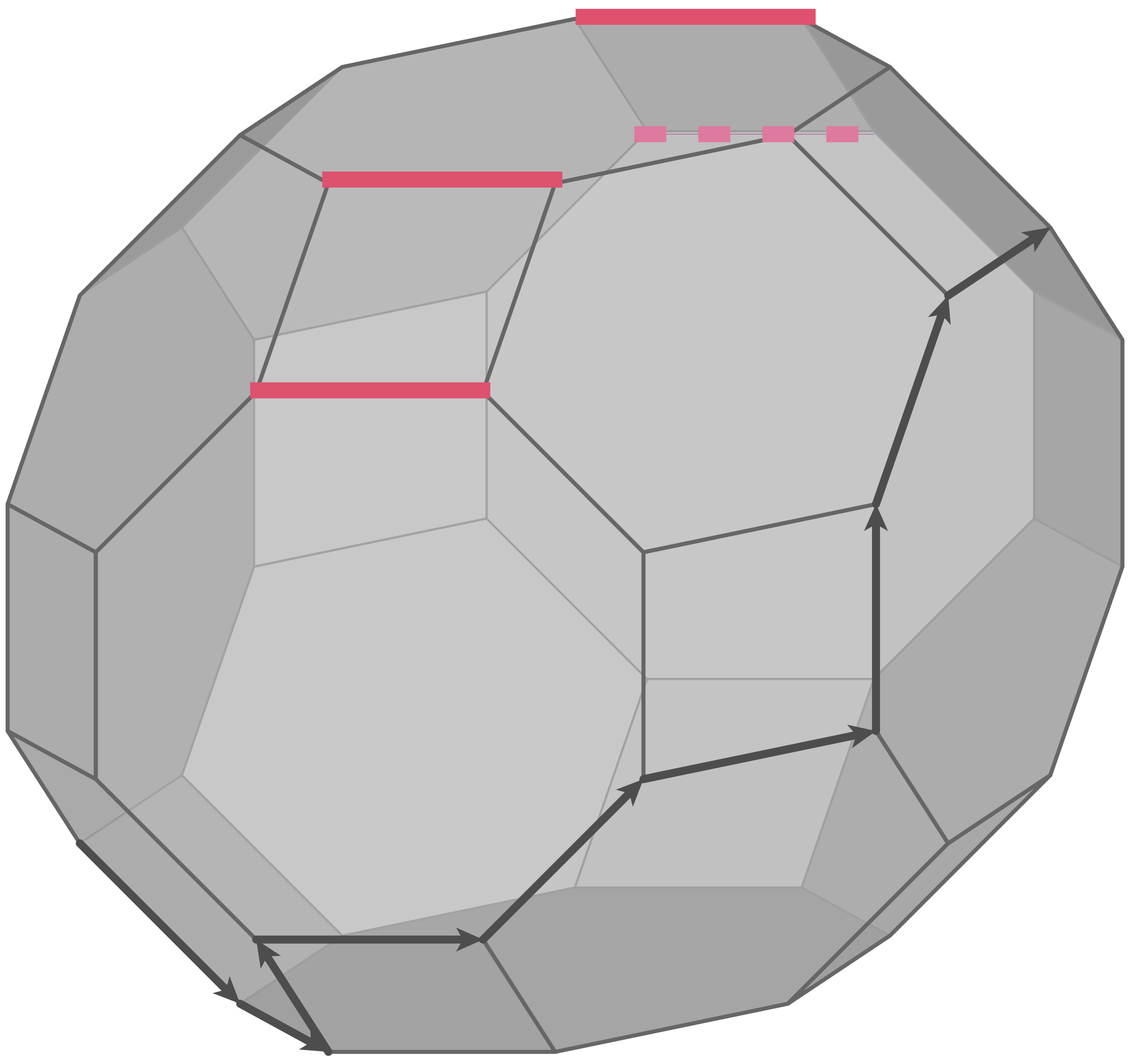

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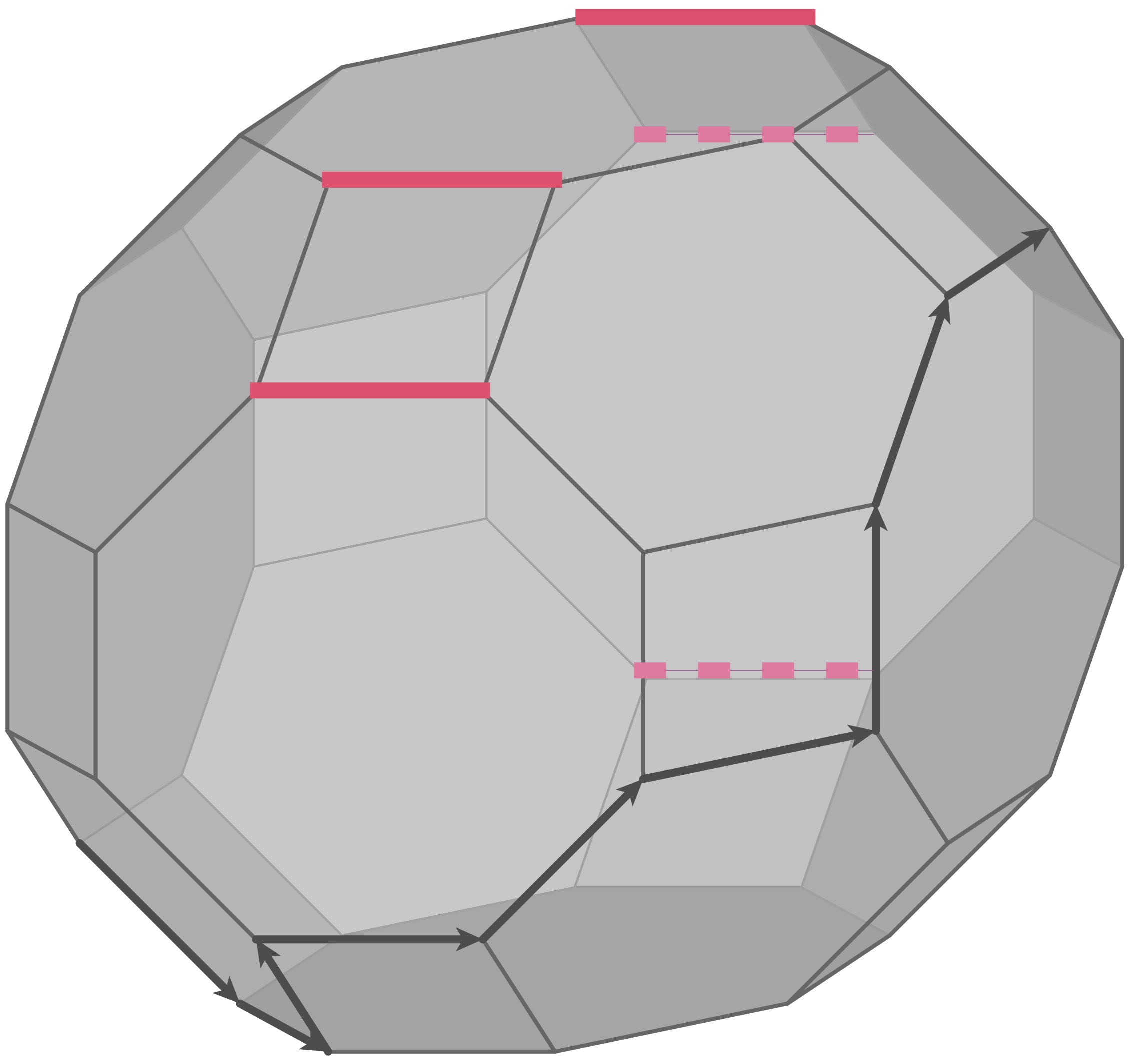

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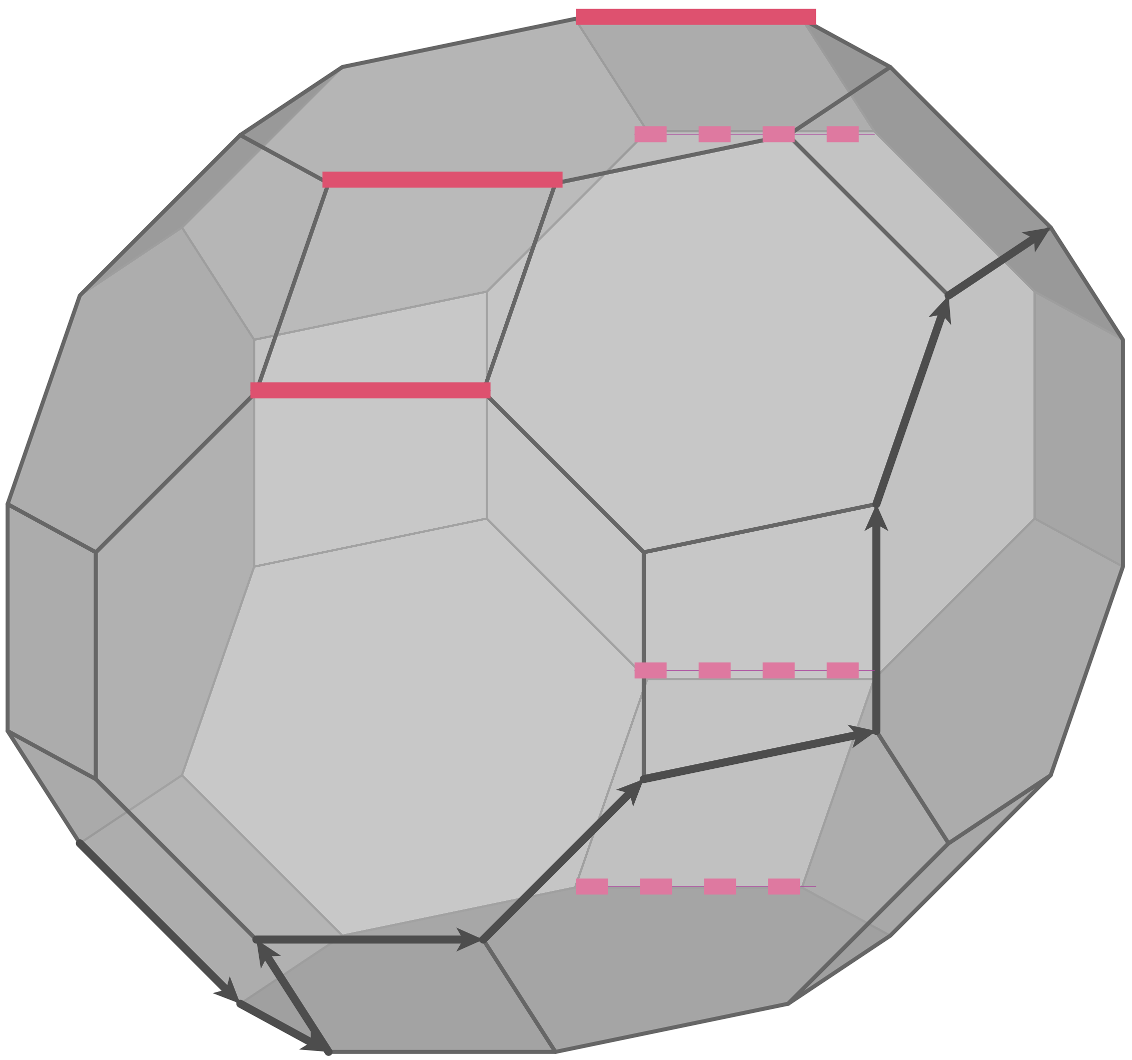

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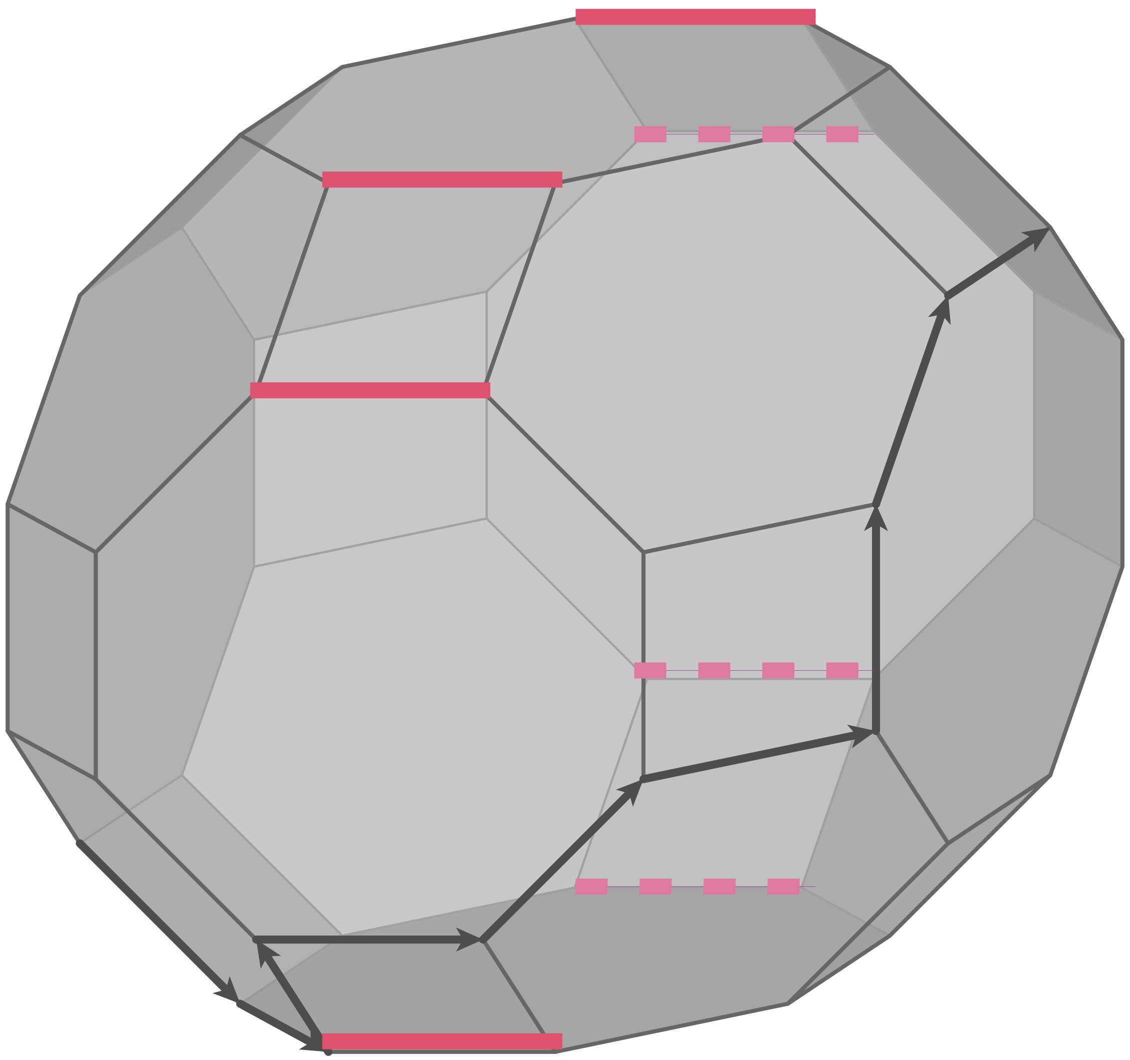

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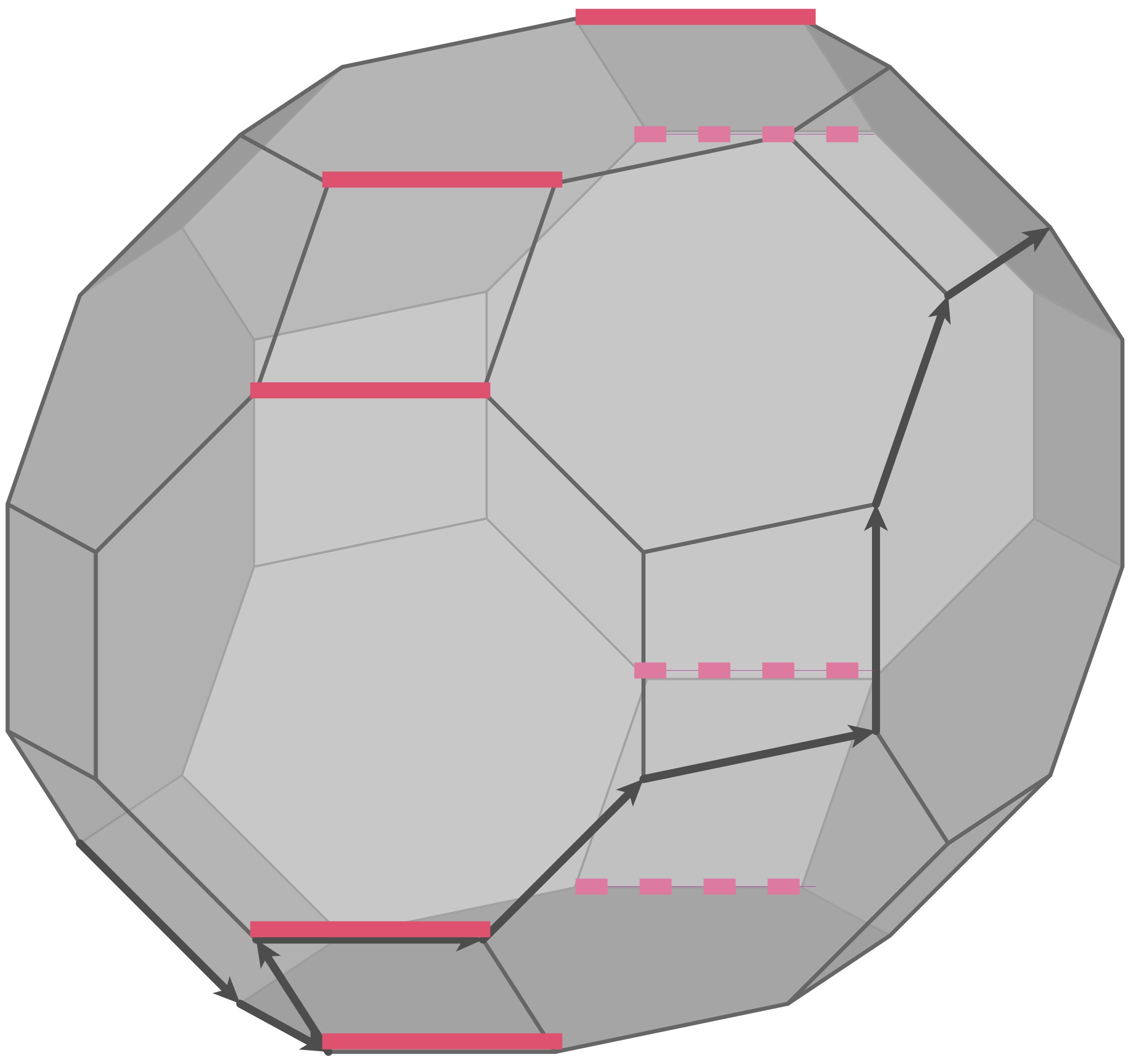

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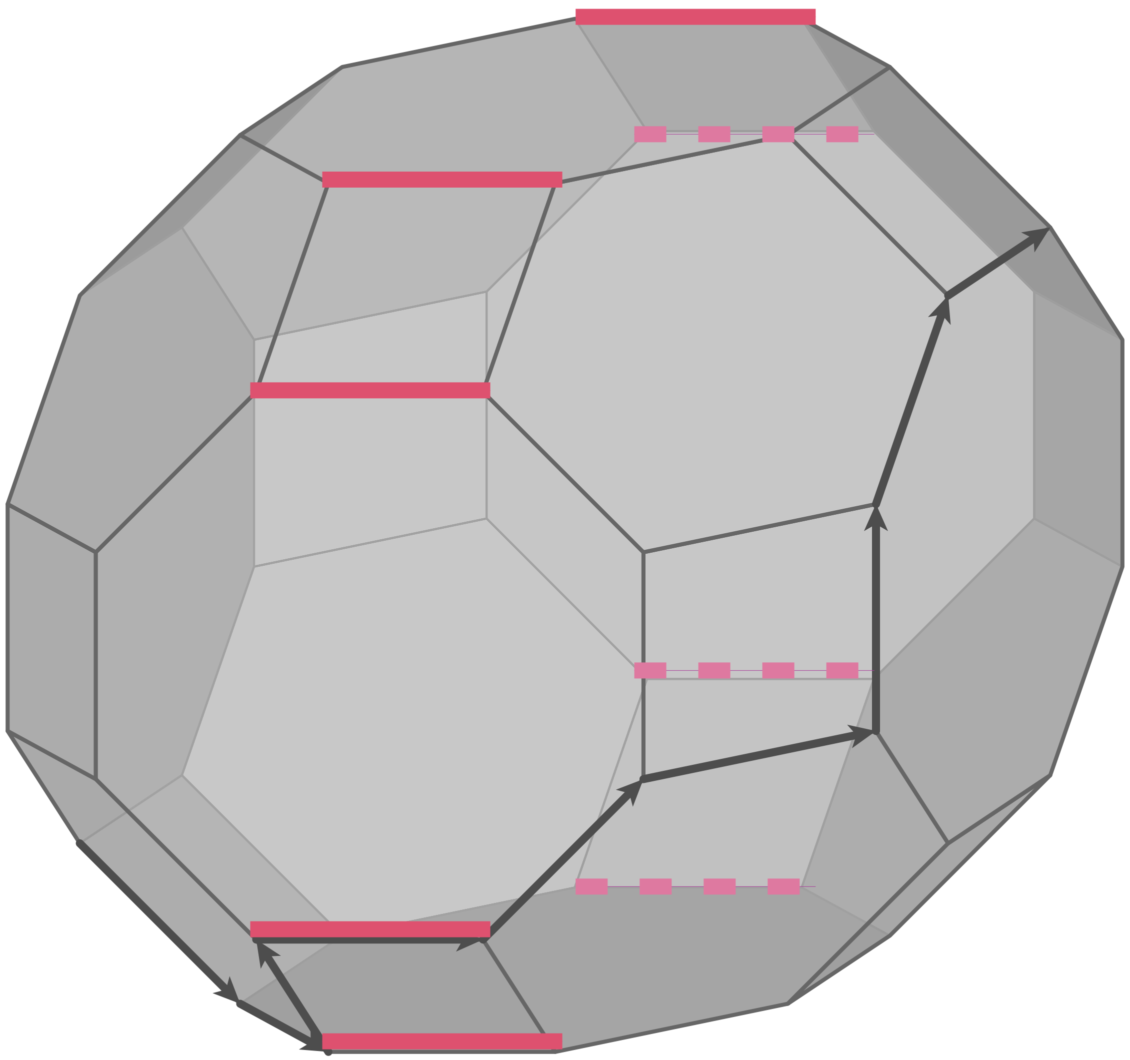
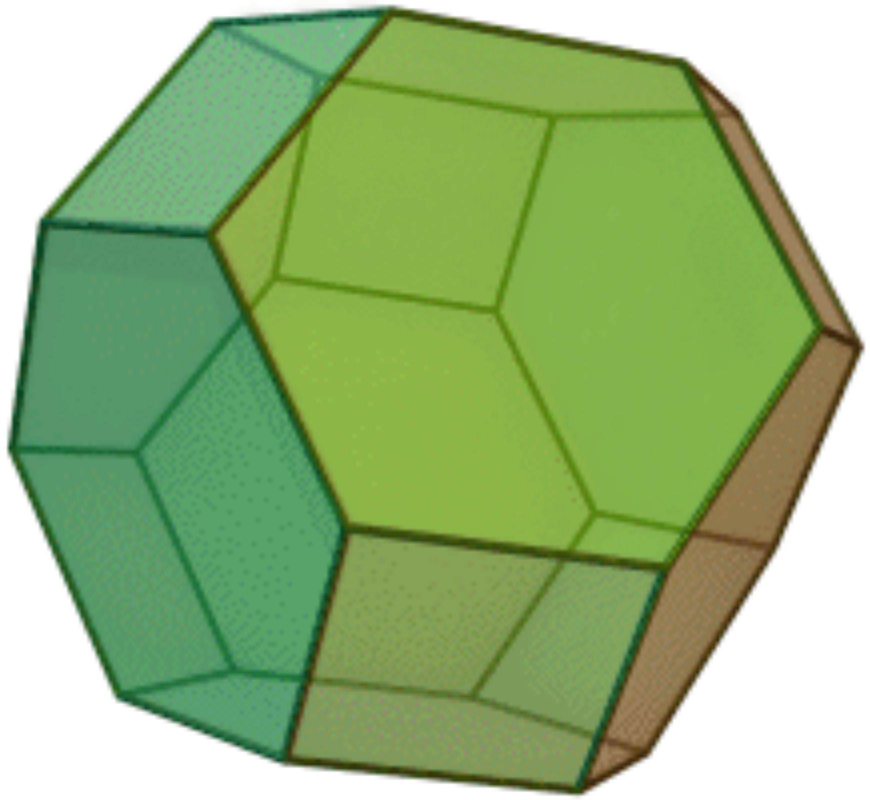


This polytope therefore does not tile $\mathbb{R}^3$ by translations, since it violates condition (3) of the Venkov-McMullen Theorem.

Example. Does this one tile by translations?



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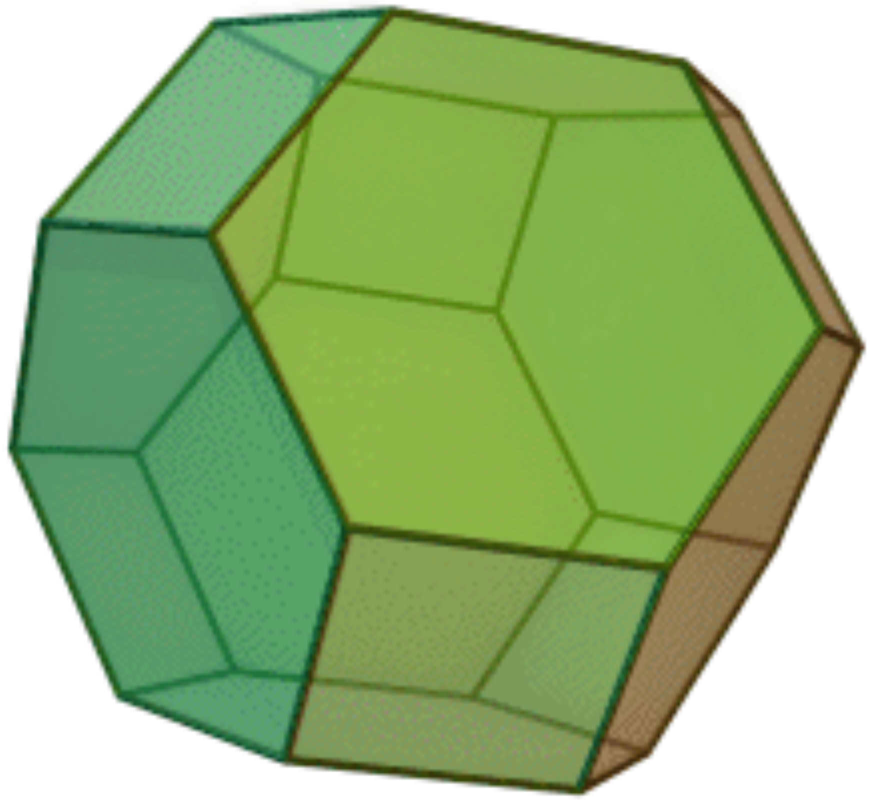

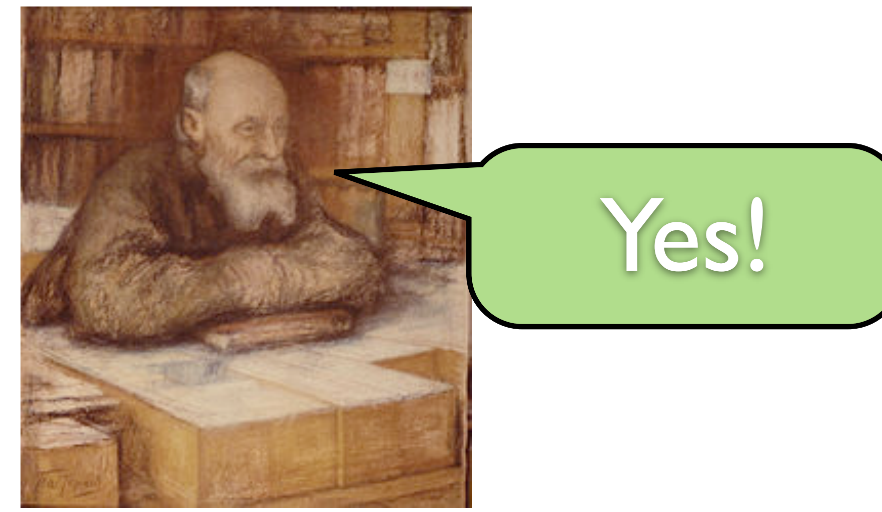

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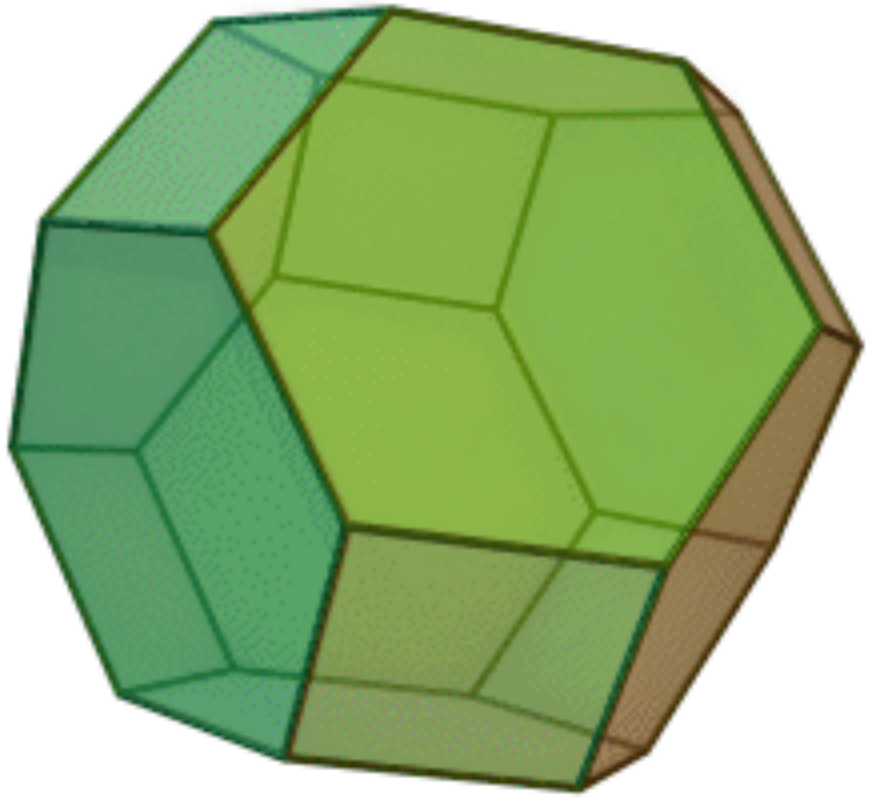
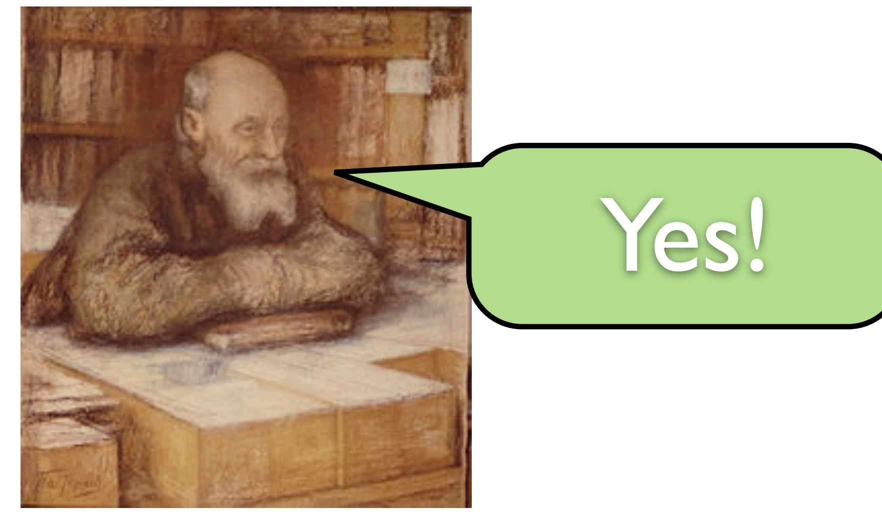
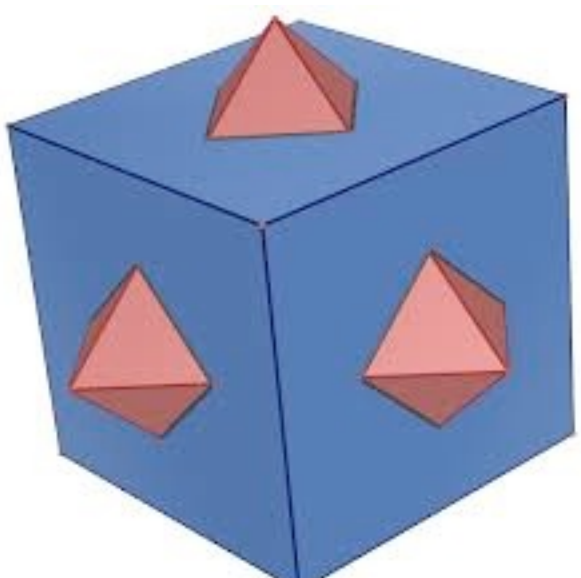






Another construction for this Fedorov solid is obtained by truncating the octahedron.

Yet another construction for it is obtained by considering it as a Permutahedron in $\mathbb{R}^4$

Part II. Multi-tilings (k-tilings)

Tiling with multiplicities

A natural generalization of a tiling is a tiling with multiplicity k. (also called a k-tiling, or a multi-tiling)

Definition.

We say that a polytope P tiles $\mathbb{R}^d$ with a discrete set of translation vectors Λ if

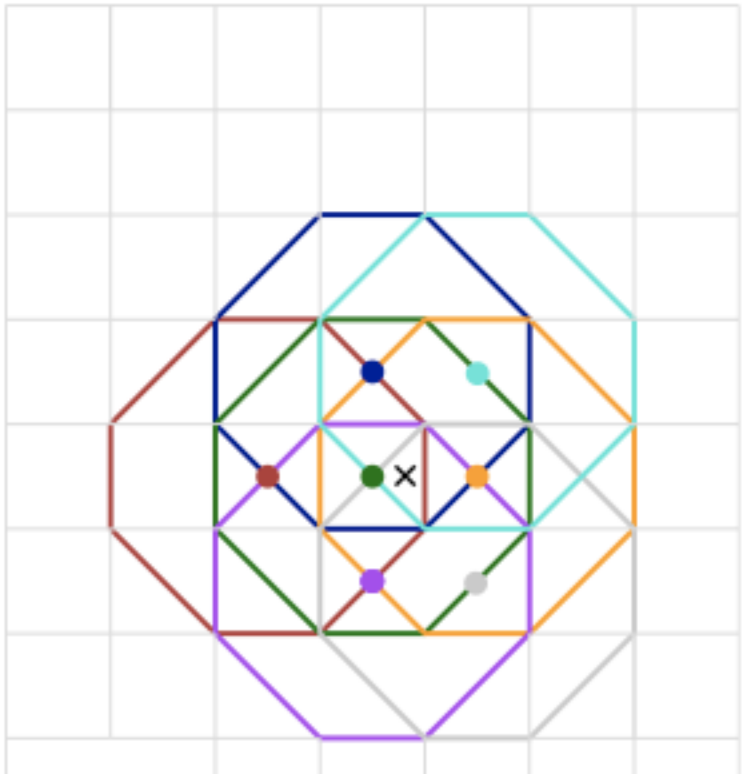
$$\sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

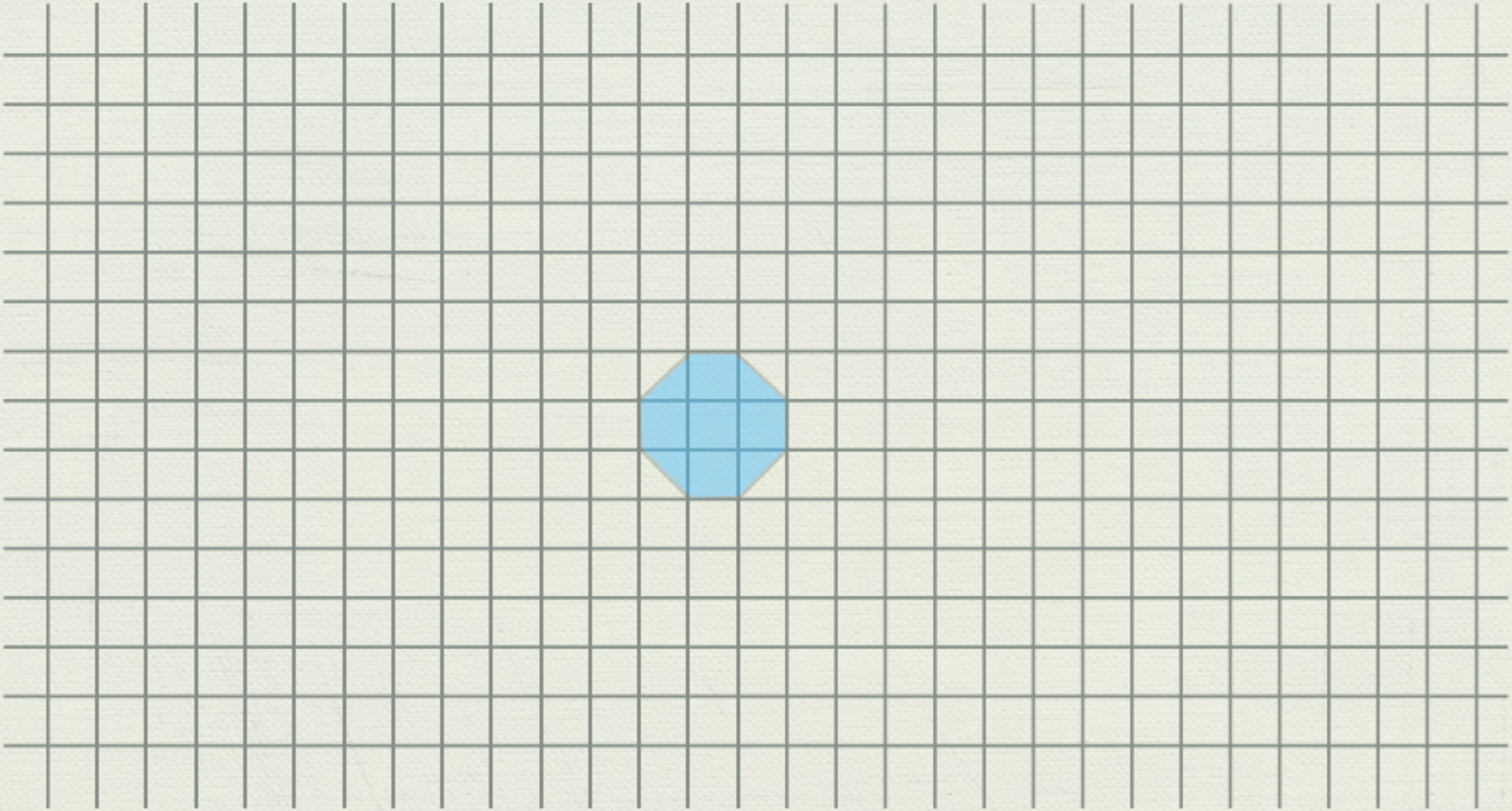
for all $v \notin \partial P + \Lambda$.

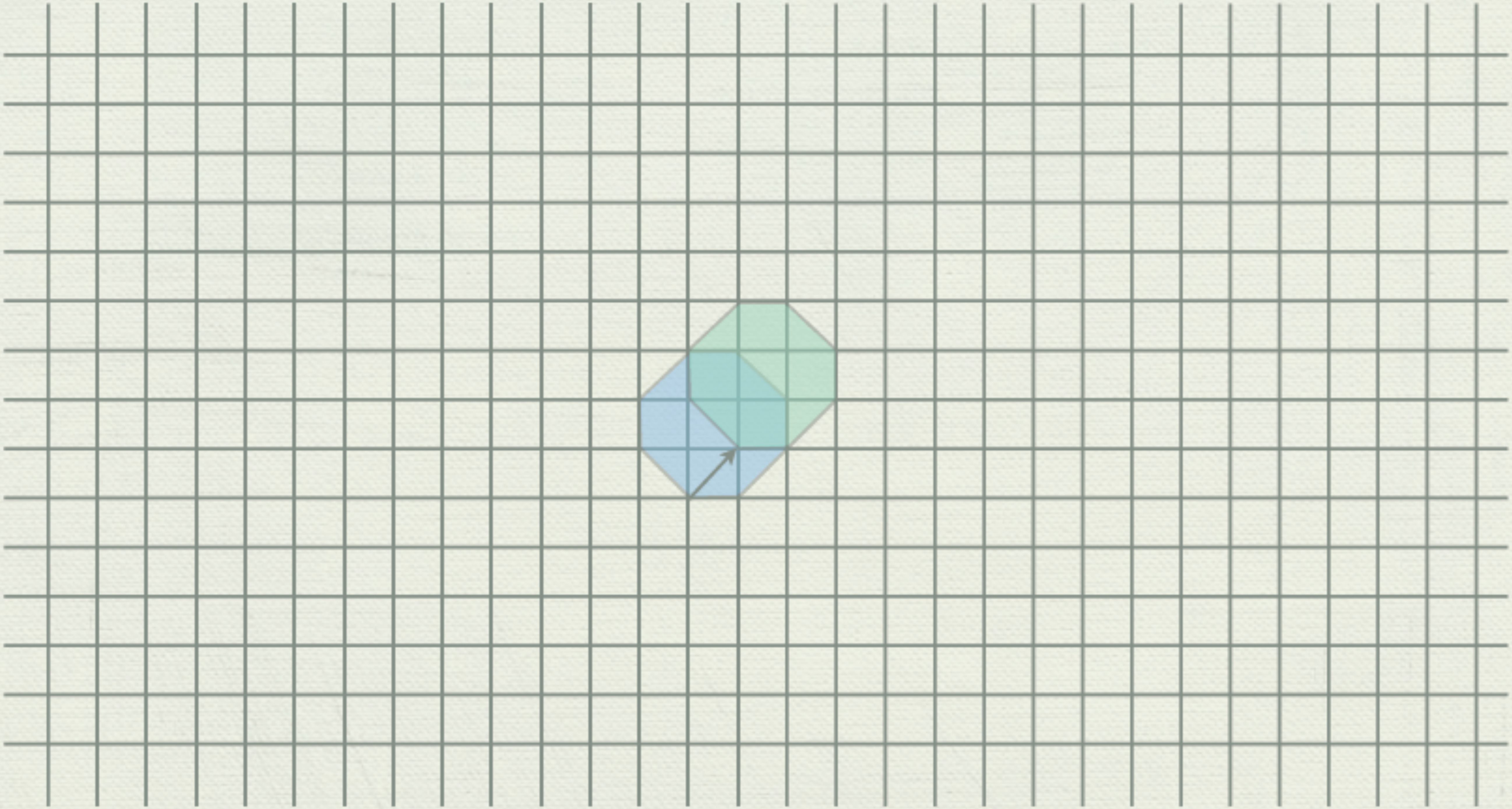
Example. The integer octagon

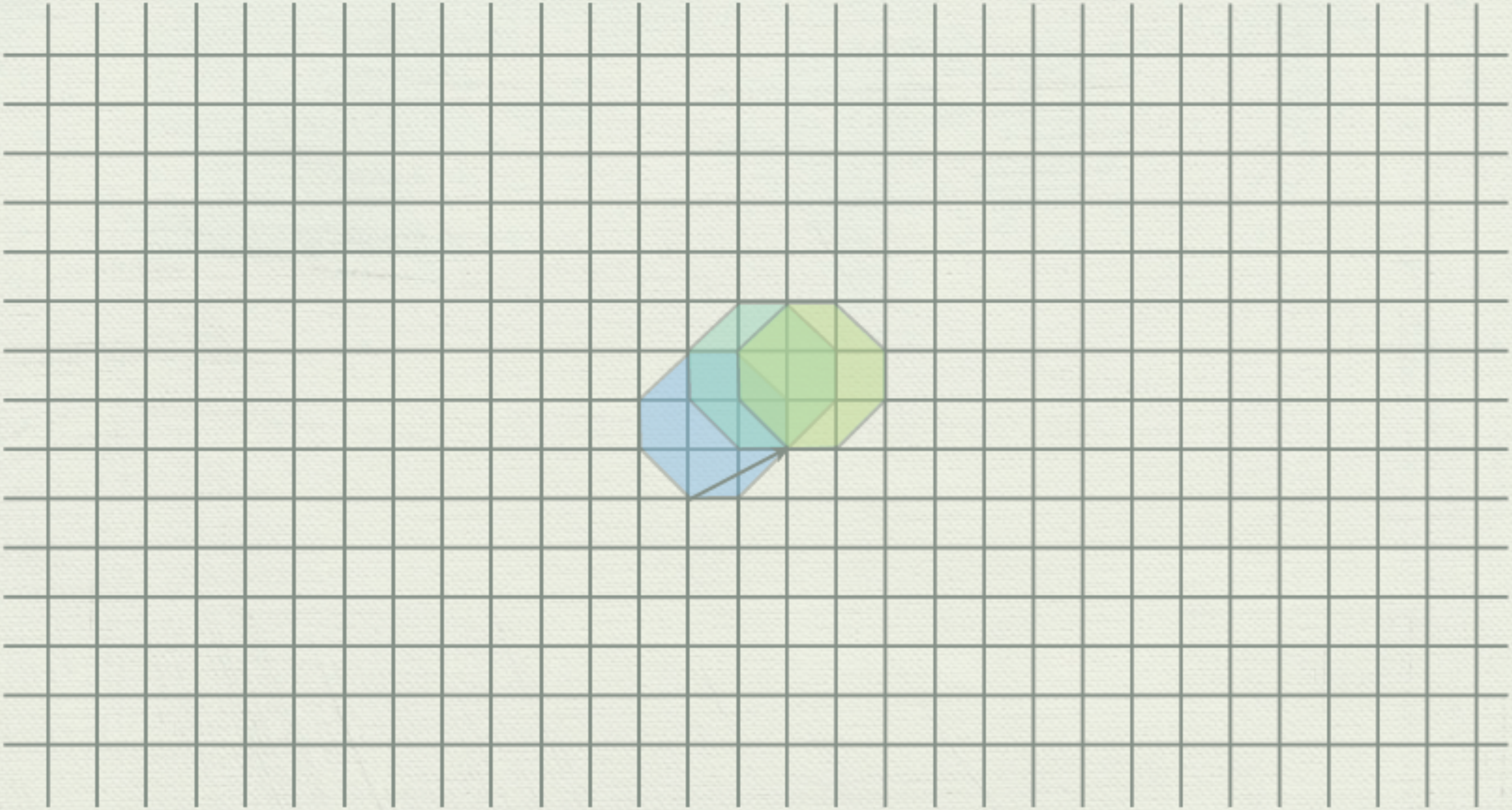
As we already know, it **does not tile** $\mathbb{R}^2$.

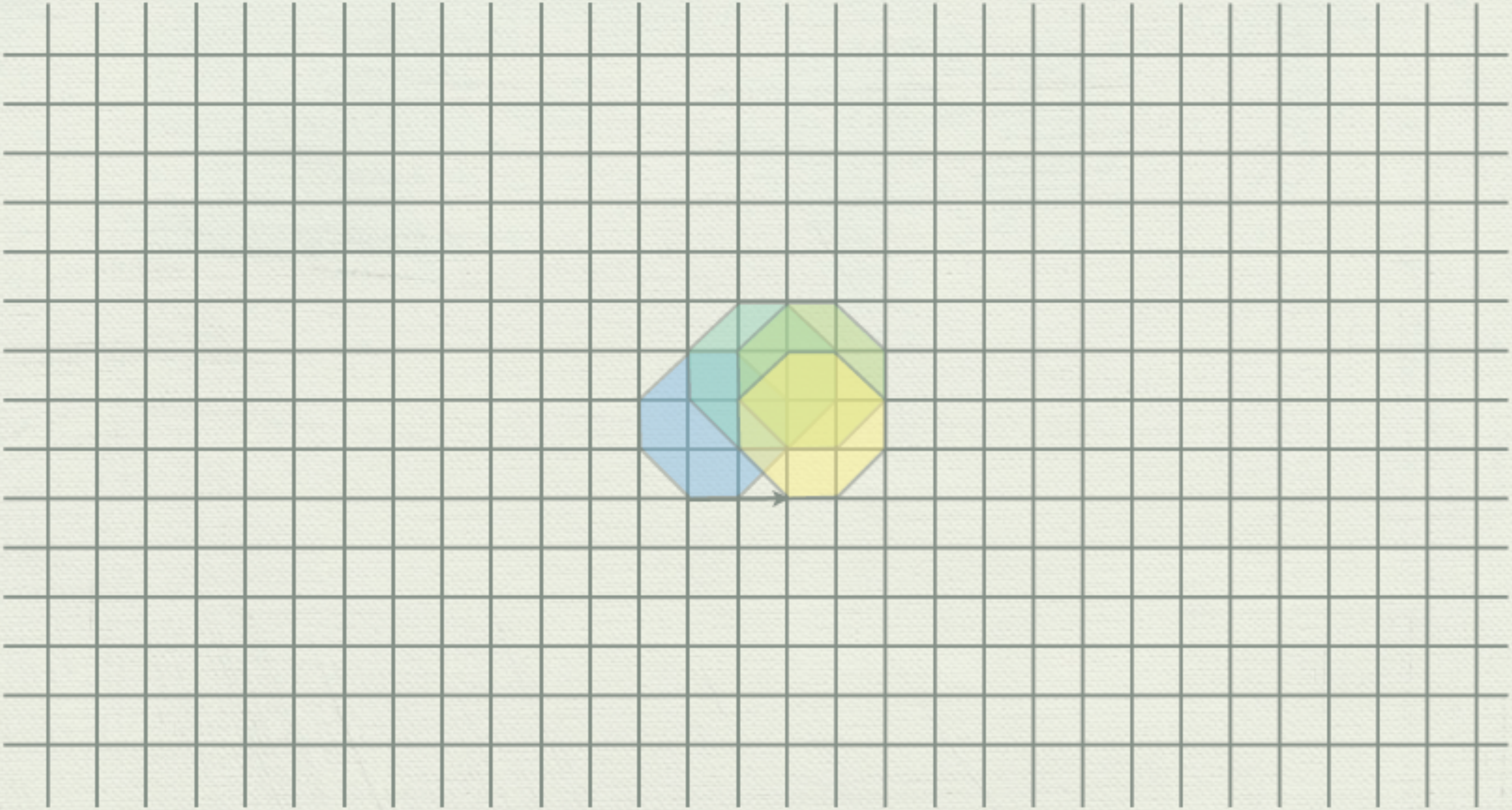
However, it **does** tile $\mathbb{R}^2$ with multiplicity 7, and with $\Lambda = \mathbb{Z}^2$!

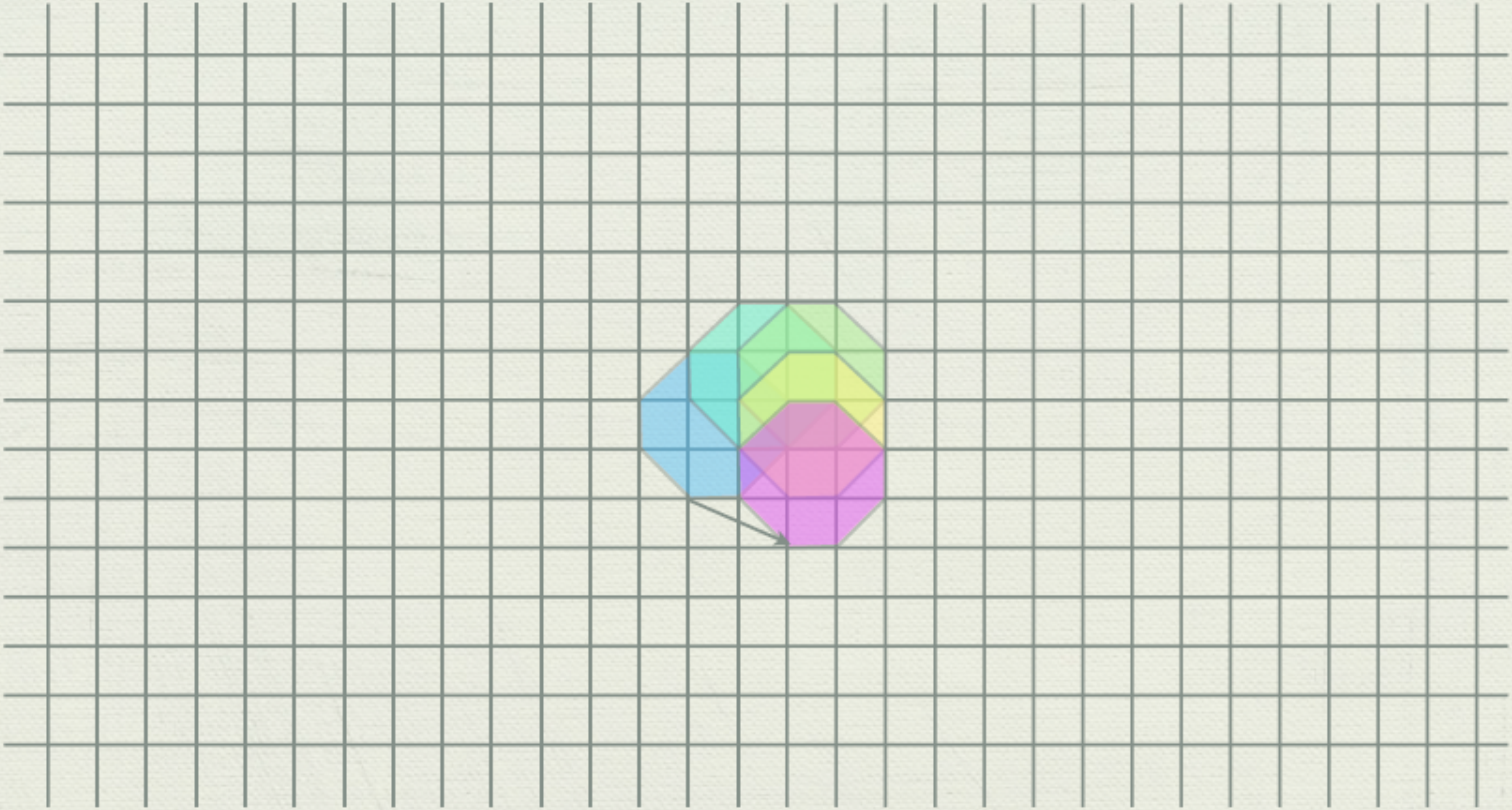


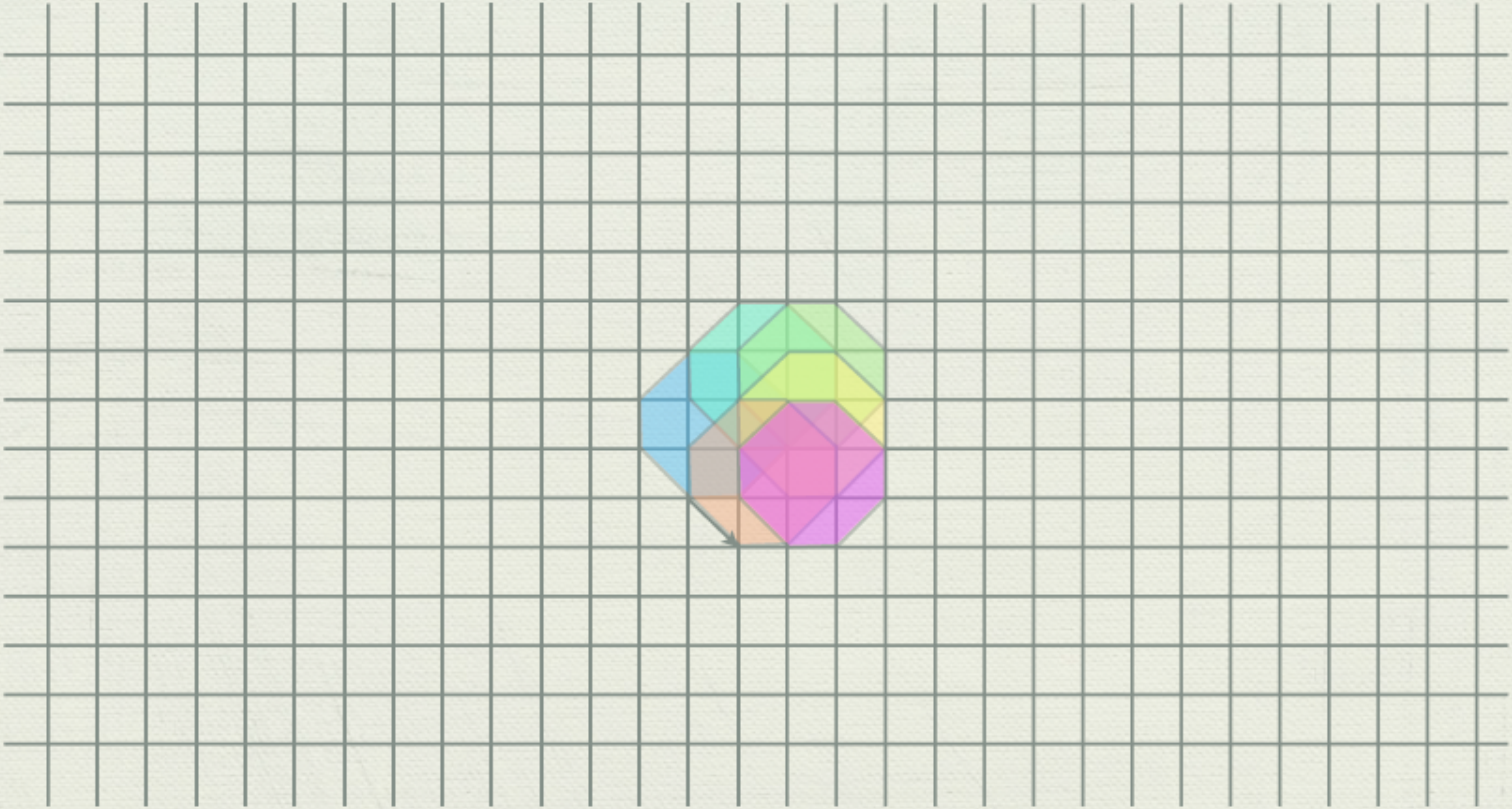


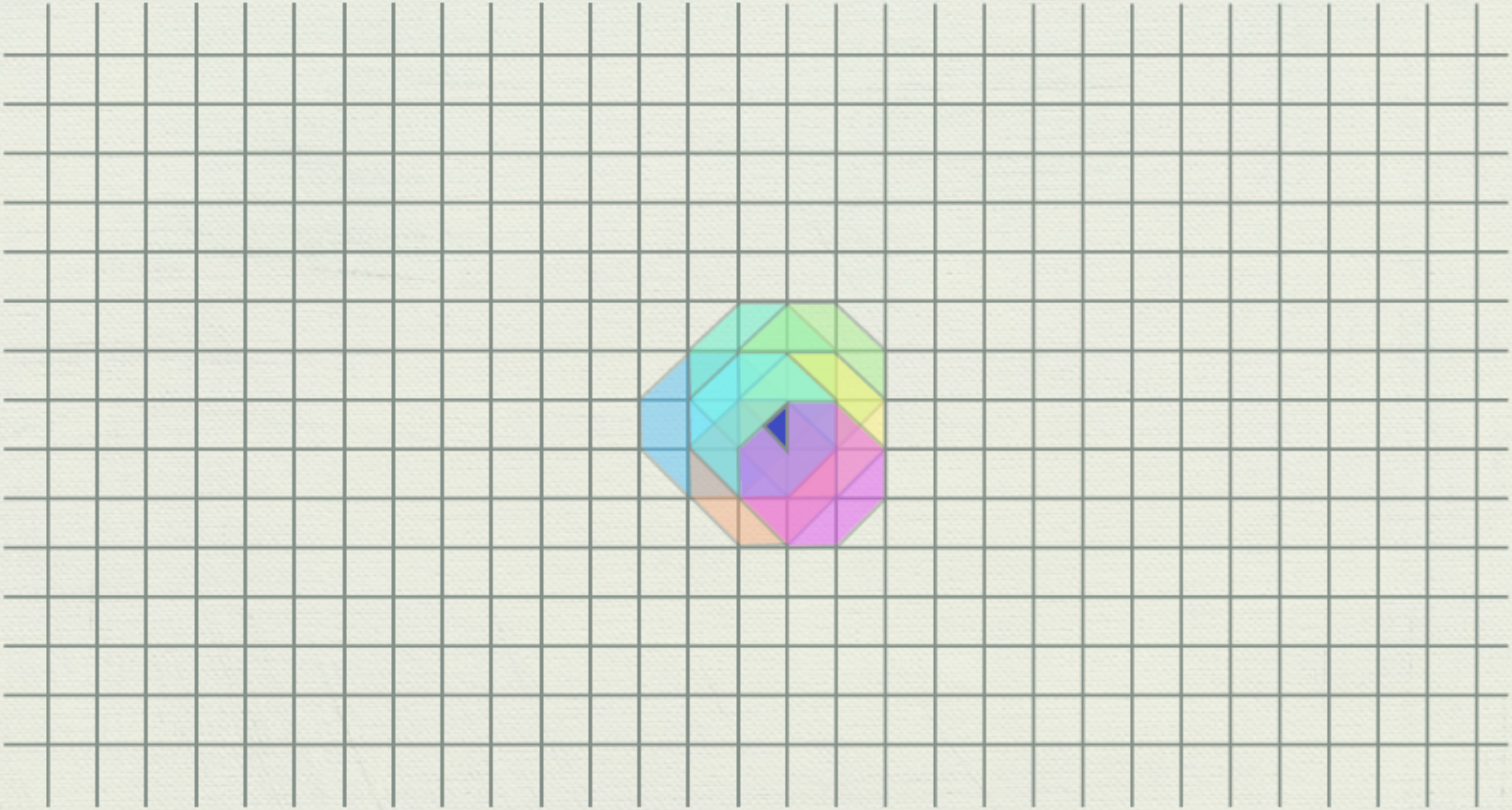












2-dimensional results for k-tilings

1994: Bolle gave a nice combinatorial characterization of all **lattice** k-tilings of $\mathbb{R}^2$ in terms of distances between vertices of a polygon.

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2013: Dmitry Shiryaev has recently shown (Ph.d thesis) that in $\mathbb{R}^2$ every k-tiler must in fact tile with one lattice (i.e. must be periodic).

Exercise. Find an infinite collection of multi-tiling polygons with cardinality equal to that of $\mathbb{R}$.

A structure theorem for d-dimensional polytopes that multi-tile

Theorem.(Gravin, R., Shiryaev, Combinatorica, 2012)

Suppose a polytope P multiply-tiles $\mathbb{R}^d$ with a discrete multiset $\mathcal{L}$. Then P is symmetric, and each facet of P is also symmetric.

A partial converse

Suppose that a polytope P enjoys the following properties:

1. P is symmetric
2. Each facet of P is also symmetric
3. P is a rational polytope (all vertices are rational points).

Then P multi-tiles with the integer lattice $\mathbb{Z}^d$.

Exercise. (Barvinok) If a polytope P is symmetric, and has symmetric facets, then show that the sum of the solid angles at all integer points (relative to P) equals its volume.

Technique: counting Λ-points inside P

Suppose P k-tiles $\mathbb{R}^d$ with the set of translation vectors Λ.

Then for every general position of -P, there are exactly k points of Λ in the interior of -P.

Easy proof:

$$\sum_{\lambda \in \Lambda} 1_{-P+v}(\lambda) = \sum_{\lambda \in \Lambda} 1_{P+\lambda}(v) = k,$$

because $\lambda \in -P + v$ if and only if $v \in P + \lambda$.

("standing at v and looking at λ" versus "standing at λ and looking at v)

Solid angles (volumes of spherical polytopes) play an equivalent role, too!

The previous simple observation has an interesting extension. Let $\omega_P(x)$ be the proportion of P which intersects a small sphere centered at $x \in \mathbb{R}^d$. It's also called a solid angle at x, relative to P.

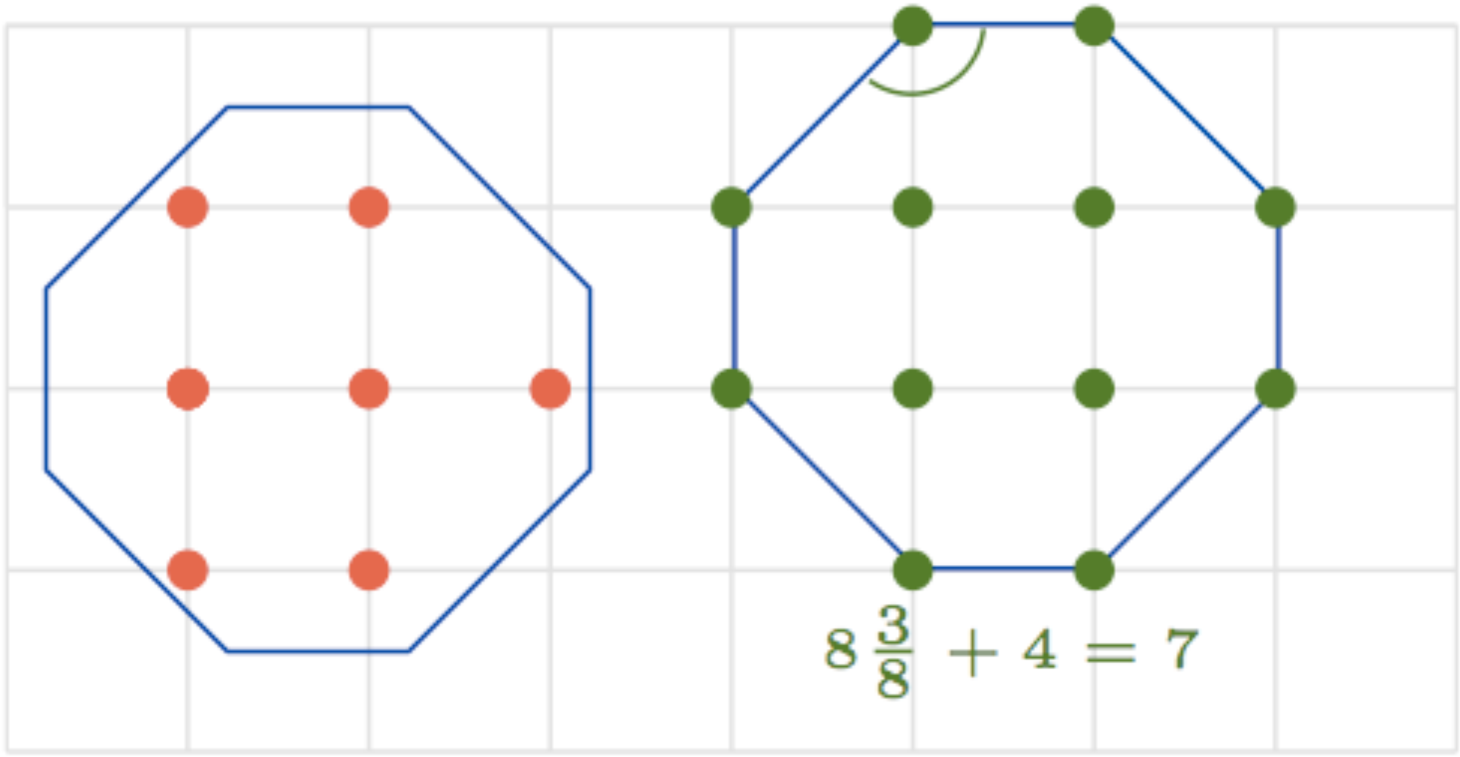
Theorem. (2013, Gravin, R, Shiryaev) A polytope P k-tiles $\mathbb{R}^d$ with the discrete set of translations Λ if and only if

$$\sum_{\lambda \in \Lambda} \omega_{P+v}(\lambda) = k,$$

for **all** $v \in \mathbb{R}^d$.

(k is necessarily equal to the volume of P)

Example. An integer octagon that 7-tiles $\mathbb{R}^2$.



Part III. Harmonic analysis approach/ideas

A structure theorem for the 3-dimensional set of translation vectors

Theorem. (Gravin, Kolountzakis, R, Shiryaev, 2013)

Suppose that a polytope P multi-tiles with a discrete multiset $\mathcal{L}$, and suppose that P is not a two-flat zonotope.
Then $\mathcal{L}$ is a finite union of translated lattices.

A structure theorem for the 3-dimensional set of translation vectors

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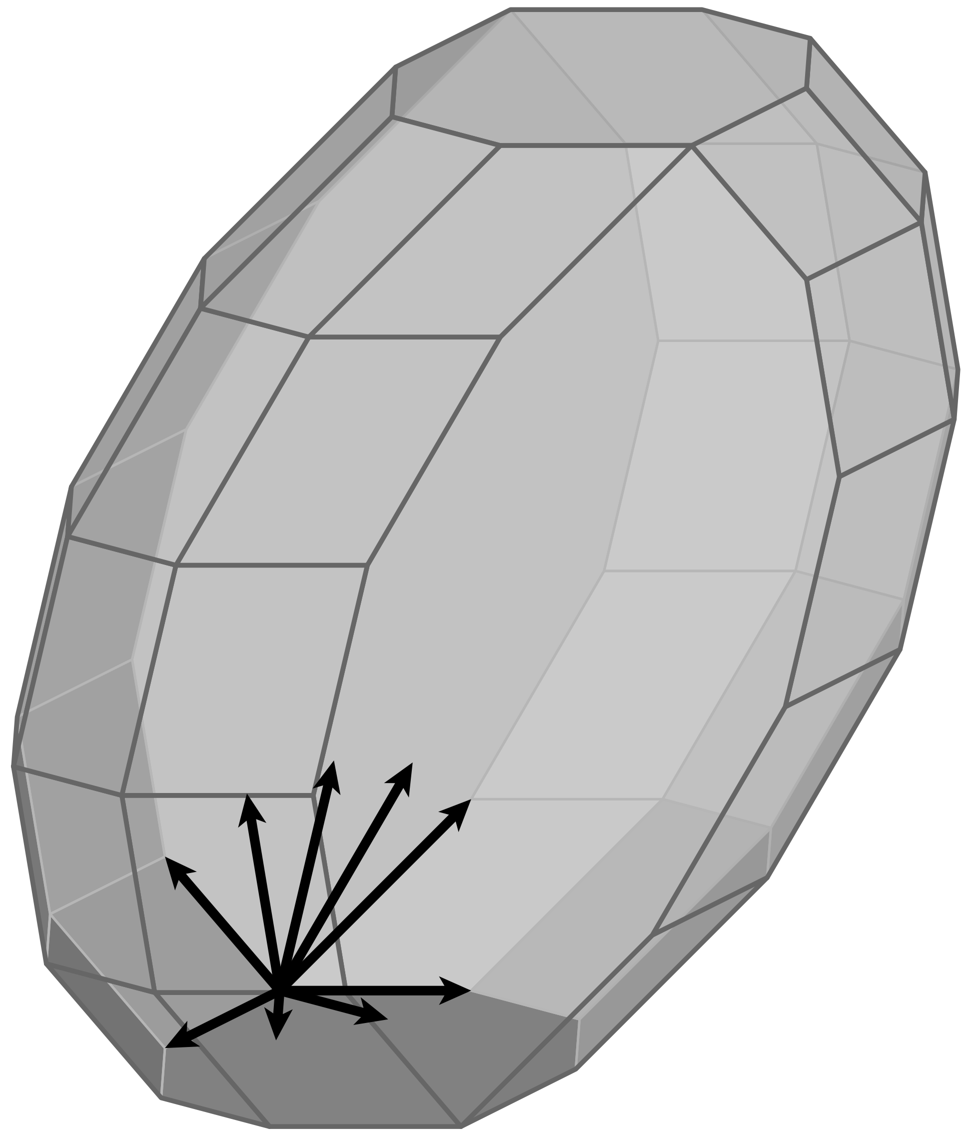
Suppose that a polytope P multi-tiles with a discrete multiset $\mathcal{L}$, and suppose that P is not a two-flat zonotope. Then $\mathcal{L}$ is a finite union of translated lattices.

(Proof uses the idempotent theorem in Fourier analysis, due to Meyer and later developed by Paul Cohen.)

Example. A two-flat zonotope with 9 generators



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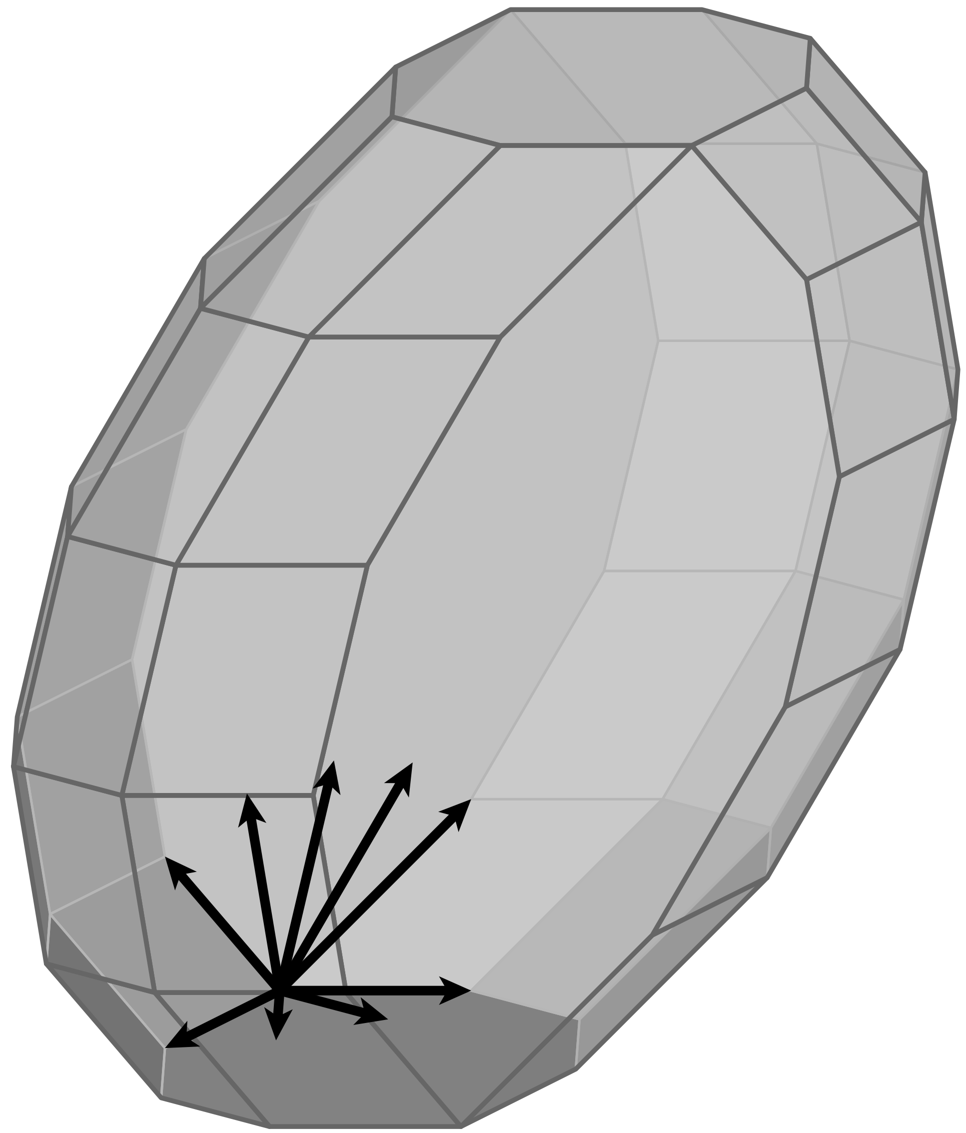


We discovered it by playing with the formulas for the Fourier transform of polytopes.

The Harmonic Analysis approach

We recall that for any integer k, the polytope P k-tiles $\mathbb{R}^d$ with a lattice L, if:

$$\sum_{\lambda \in L} 1_P(\lambda - v) = k,$$

for all $v \notin \partial P + L$. We notice now that the left-hand side is a periodic function of v, where the period is a fundamental parallelepiped of L. It therefore has a Fourier expansion on this domain. The Poisson summation formula now gives us this Fourier expansion, as follows.

More generally, Poisson summation (and its proof) is even better stated as follows:

$$\sum_{n\in L} f(n+x) = \frac{1}{|\det L|} \sum_{m\in L^*} \hat{f}(n) e^{2\pi i\langle m,x\rangle},$$

where $x \in \mathbb{R}^d$, and L^* is the dual lattice to L,

defined by $L^* := \{x \in \mathbb{R}^d \mid \langle x, n\rangle \in \mathbb{Z},\ \text{for all } n \in L\}$.

The Harmonic Analysis approach

Thus, by Poisson summation, we have

$$k = \sum_{\lambda \in L} 1_P(\lambda - v) = \frac{1}{|\det L|} \sum_{m \in L^*} \hat{1}_P(m) e^{-2\pi i \langle v, m \rangle}.$$

Now we use the fact that Fourier series expansions are unique, so all the nonzero Fourier coefficients on the right must vanish, because k is constant. We have thus been naturally lead to the proof of another fascinating equivalence for any k-tiling.

The Harmonic Analysis approach

So we have proved this (easy) Lemma, but it already shows a nice approach using Harmonic analysis.

Lemma

A convex polytope P k-tiles $\mathbb{R}^d$ by translations with the lattice L if and only if both of the following conditions are true:

- $\hat{1}_P(m) = 0$, *for all nonzero vectors* $m \in L^*$, *the dual lattice of* L.
- $k = \frac{Vol(P)}{|\det(L)|}$.

The Harmonic Analysis approach

We expand on the second part of this Lemma. When we compute $\hat{1}_P(0)$, we get:

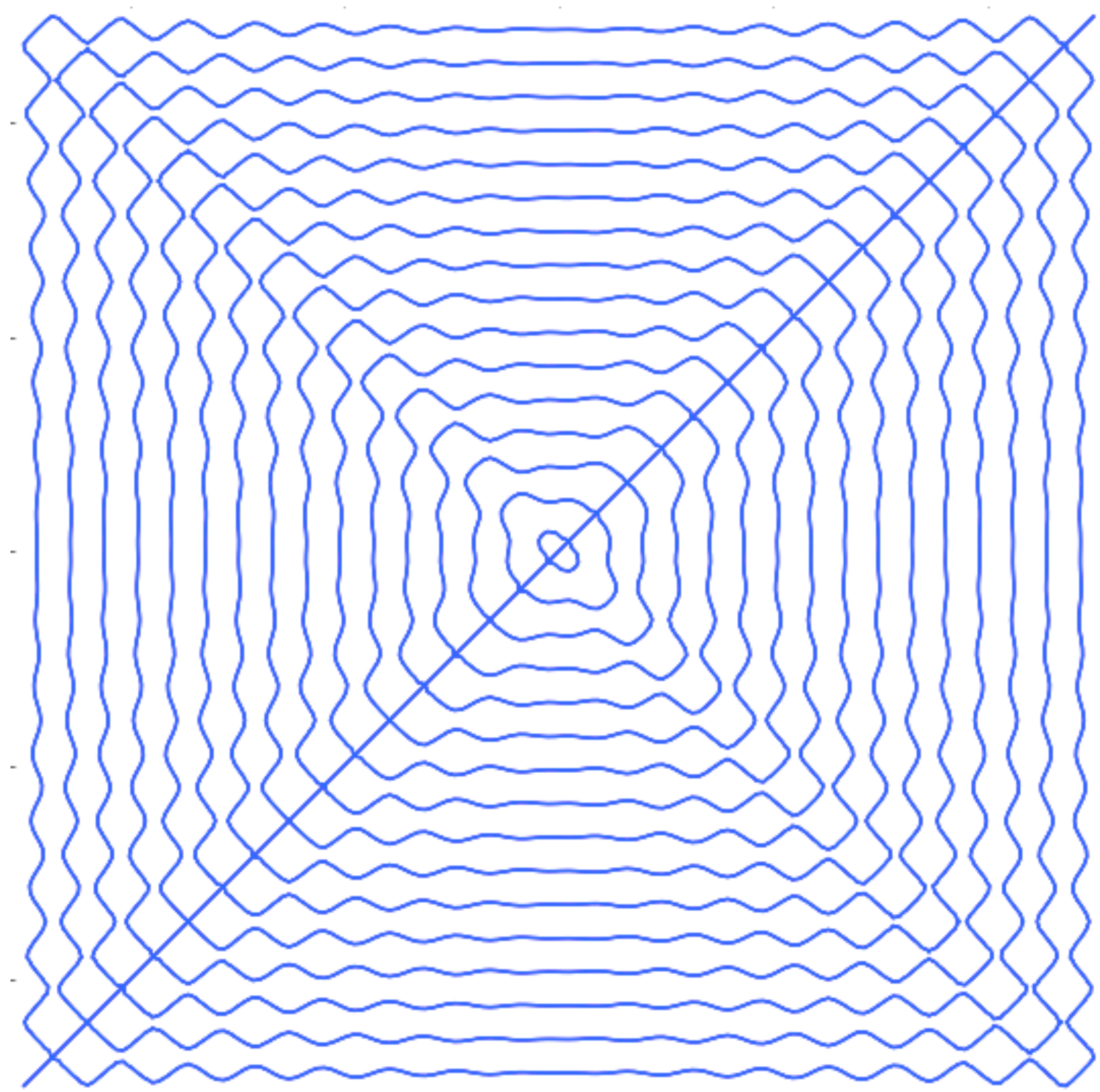
$$\hat{1}_P(0) = \int_{R^d} 1_P(x)e^{(2\pi i)(0)}dx = \int_{1_P} dx = Vol(P).$$

Therefore, comparing the constant terms of both sides of the Poisson summation formula, we get $k = \frac{Vol(P)}{|\det(L)|}$.

Example.

The real zero set of the Fourier transform of the square [0,1]^2
(w.r.t. uniform measure)

$$x\ cos(x) = y\ cos(y)$$



Harmonic Analysis approach

Thus, we can study the vanishing of the Fourier transform of a polytope, namely $\hat{1}_P(m) = 0$.

The vanishing of Fourier transforms of convex bodies in general has been studied, in the context of the Fuglede conjecture, by Alex Iosevich, Mihalis Kolountzakis, Mate Matolci, Izabella Laba, Terry Tao, and others.

Harmonic Analysis approach

Some other open questions:

1. Give an analogue of the Venkov-McMullen converse for k-tilings.

2. Given k, describe all polytopes that k-tile. What is the smallest non-trivial k that is possible in dimension d?

Harmonic Analysis approach

Some other open questions:

1. Give an analogue of the Venkov-McMullen converse for k-tilings.

2. Given k, describe all polytopes that k-tile. What is the smallest non-trivial k that is possible in dimension d?

3. Find the number of vertices of a k-tiler.

4. Most importantly for us: Using the vanishing set (as a subset of $\mathbb{R}^d$) of the Fourier transform $\hat{1}_P(m)$, classify all k-tiling polytopes. Focus on $d = 2$ first.

5. Study the real zero set of $\hat{1}_P(m)$, and develop a theory.

Harmonic Analysis approach

Something that we keep seeing is that it's very fruitful to simultaneously think about the Fourier analysis and the Discrete/Combinatorial geometry.

Exercises.

1. Find an infinite collection of multi-tiling polygons with cardinality equal to that of $\mathbb{R}$.

2. (Barvinok) suppose that an integer polytope P is symmetric, and has symmetric facets as well. Then the sum of all solid angles (a discrete volume) at all integer points equals its volume.

3. Plot the zero set of the Fourier transform of a symmetric hexagon; find patterns among the collection of curves that you see.

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