Harmonic analysis on polytopes and cones

São Paulo school of advanced science on algorithms, combinatorics, and optimization

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7. (more applied) Moments of polytopes, and the inverse moment problem for polytopes

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10. Cone theta functions, as generating functions of discrete volumes

11. Many, many open problems, often combinatorial in nature

Lecture 1.

Motivation: to study discrete analogues of differential-geometric ideas.

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It's much more useful to discretize the Fourier-Laplace transform:

$$\hat{1}_P(z) := \int_{\mathbb{R}^d} 1_P(x)e^{2\pi i\langle z,x\rangle}dx,$$

by using $$\sigma_P(z) := \sum_{n \in \mathbb{Z}^d} 1_P(n)\, e^{2\pi i \langle z,n \rangle},$$

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Answer: the number of integer points in P.
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Answer: the volume of P.

What about *other* discrete volumes for P?

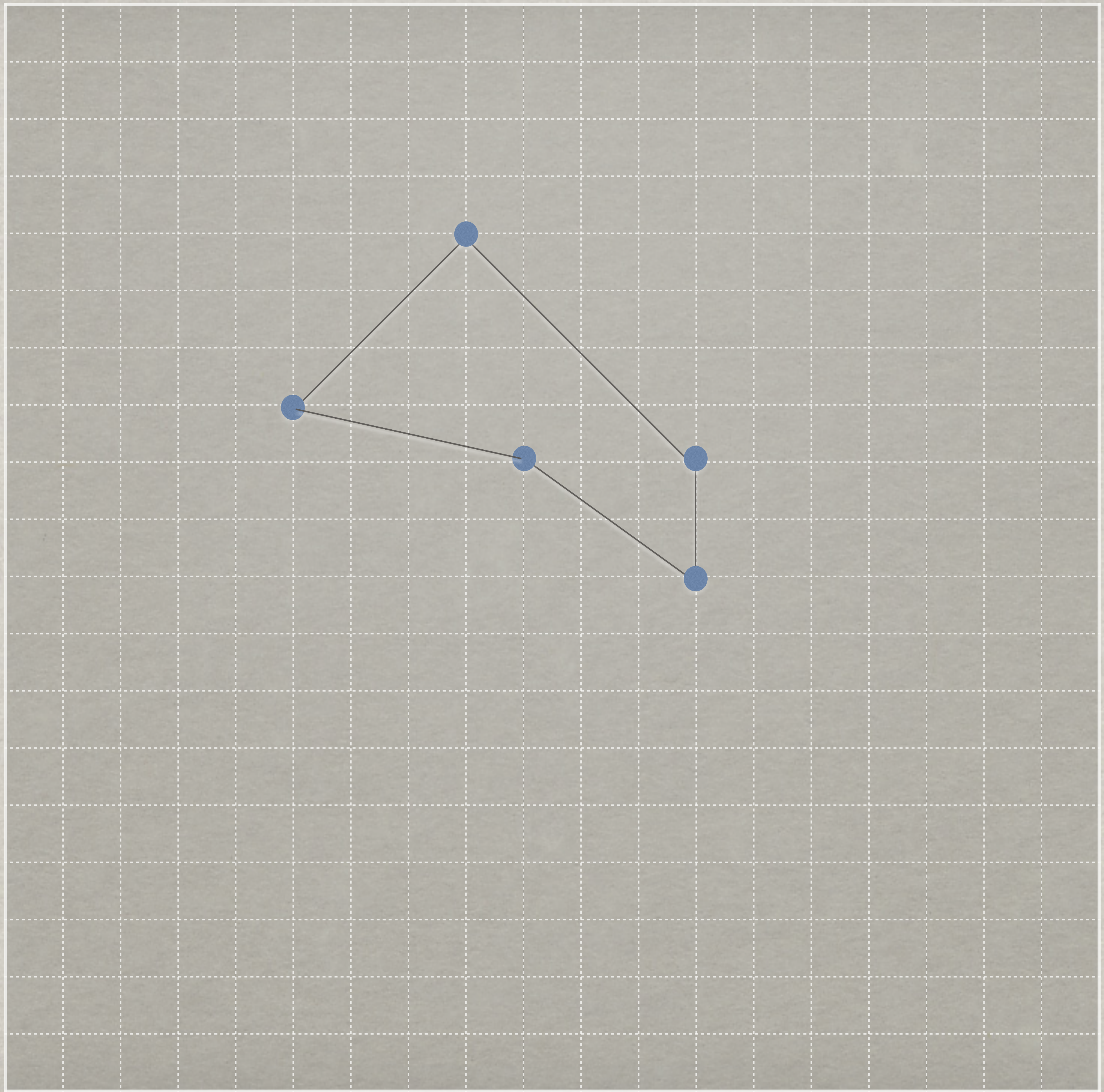
A note about notation: for algebraic geometers, the integer point transform is more familiar if we change notation using $x_k := e^{2\pi i z_k n_k}$, so that

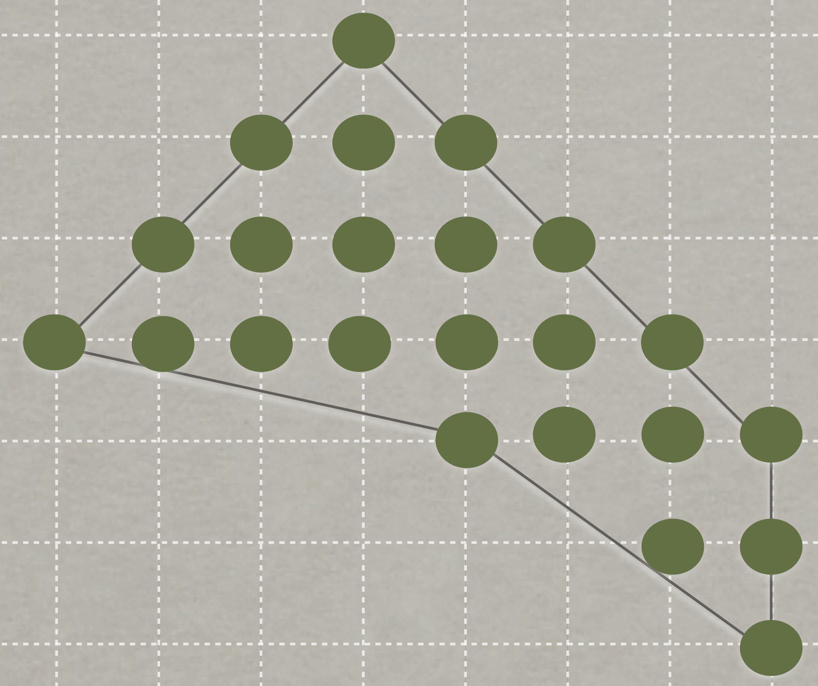
$$x^n := {x_1}^{n_1} \cdots {x_d}^{n_d} := e^{2\pi i (z_1 n_1 + \cdots + z_d n_d)} = e^{2\pi i \langle z, n \rangle}.$$

Using this standard multinomial notation, we see that the integer point transform is really a polynomial:

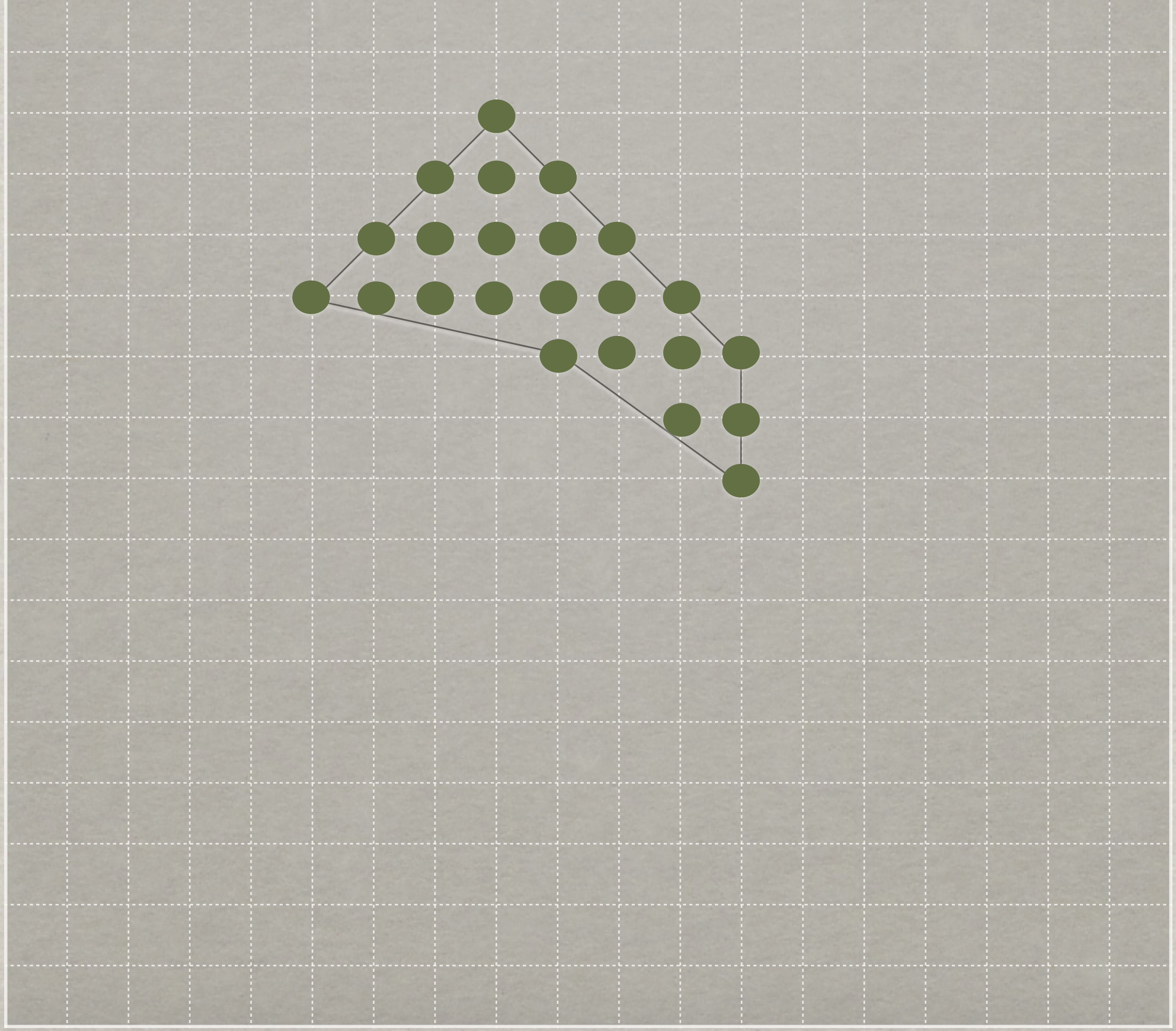
$$\sum_{n \in \mathbb{Z}^d} 1_P(n)\, x^n := \sum_{n \in \mathbb{Z}^d \cap P} x^n.$$

*note: Toric varieties can offer an algebraic study of such objects





The first discrete volume is $\sum_{n \in \mathbb{Z}^2} 1_P(n) := |P \cap \mathbb{Z}^2|$.



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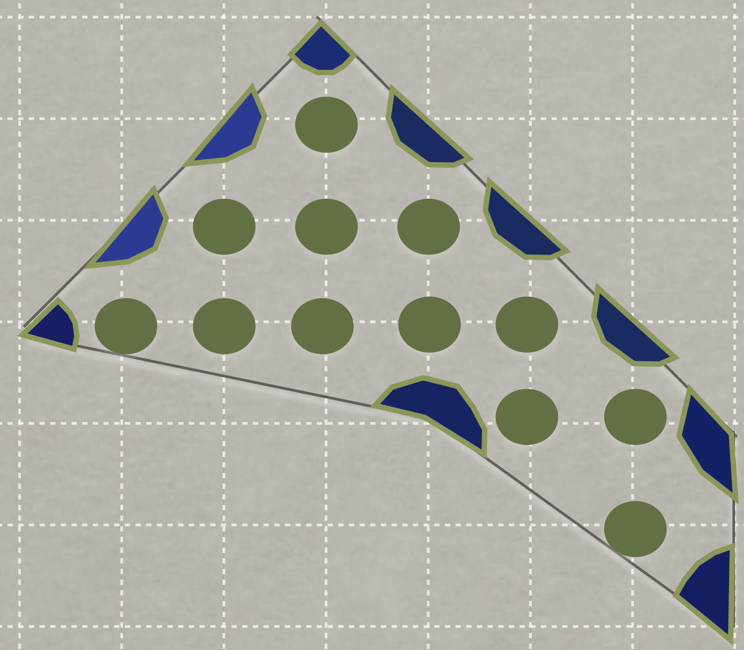

It turns out that this is a better approximation to the area of P.

In fact, for an integer polygon P, this approximation equals area(P)!

The solid angle, at any point $x \in \mathbb{R}^d$, of a d-dim'l polytope $P \subset \mathbb{R}^d$, is defined by:

$$\omega_P(x) := \frac{\mathrm{vol}(S_\epsilon(x) \cap P}{\mathrm{vol}(S_\epsilon(x)}$$

where $S_\epsilon(x)$ is a small sphere of radius ϵ, centered at x.

Notice: if $x \notin P$, then the solid angle at x (relative to P) is equal to zero: $\omega_P(x) = 0$.

If x lies in the interior of P, then $\omega_P(x) = 1$.

If x lies on the boundary of P, then $0 < \omega_P(x) < 1$.

Exercise 1.

Let P be an integer polygon. Prove that the sum of the local solid angles at all integer points equals the area of P.

In other words, show that

$$\sum_{n \in \mathbb{Z}^2} \omega_P(n) = \text{area}(P).$$

In many applications, it is of interest to compute the volume of a high-dimensional polytope, or to approximate it.

Here it is our goal to use the second approximation to the volume of of a polytope P, focusing on the Fourier-Laplace methods we will develop here.

What is the Poisson summation formula, and why is it so useful?

Theorem. (Poisson summation)

$$\sum_{n \in \mathbb{Z}^d} f(n) = \sum_{m \in \mathbb{Z}^d} \hat{f}(m),$$

for all functions f that are "sufficiently nice".

Initially, we might try using $f(x) := 1_P(x)$, the indicator function of a polytope P:

$$\sum_{n \in \mathbb{Z}^d} 1_P(n) = \sum_{m \in \mathbb{Z}^d} \hat{1}_P(m),$$

a combinatorial quantity

$|\mathbb{Z}^d \cap P|$

an analytic quantity

So if only we knew some more about the Fourier-Laplace transform of a polytope.....

The traditional space of functions for which one applies Poisson summation is called the “Schwartz class” of rapidly decreasing functions.

But in practice, Poisson summation holds for functions that “it really shouldn’t”.

Basic principles of the Fourier-Laplace transform of a polytope, with examples in low dimensions

Historically, some of the first examples of Fourier transforms include the Fourier transform of a unit interval:

$$\hat{1}_{[-\frac{1}{2},\frac{1}{2}]}(z) := \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i x z} dx$$

$$= \frac{\sin(\pi z)}{\pi z},$$

also known as the "sinc-function" to Engineers.

Adding in the missing computation, we have:

$$\hat{1}_{[-\frac{1}{2},\frac{1}{2}]}(z) := \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i x z}\, dx$$

$$= \frac{e^{2\pi i(1/2)z} - e^{2\pi i(-1/2)z}}{2\pi i z}$$

$$= \frac{\sin(\pi z)}{\pi z},$$

and as we will see, it's crucial to do this computation, since the 'middle' step here represents an incredible piece of geometry, especially in higher dimensions. (Brion's Theorem)

Something very interesting happens when we ask how this transform extends to higher dimensions.

It turns out that many combinatorial identities arise in the process, which allow us to reduce the Fourier transform of a polytope to the Fourier-Laplace transform of its tangent cones.

So whatever the answer is, it extends the sinc function to $\mathbb{R}^d$.

Let's consider now a separate computation, involving 1-dimensional 'tangent cones'.

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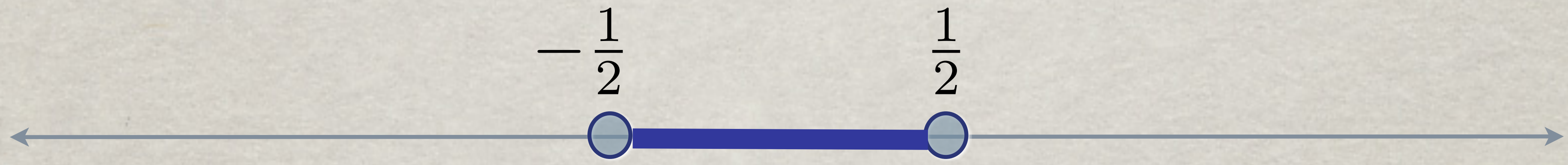

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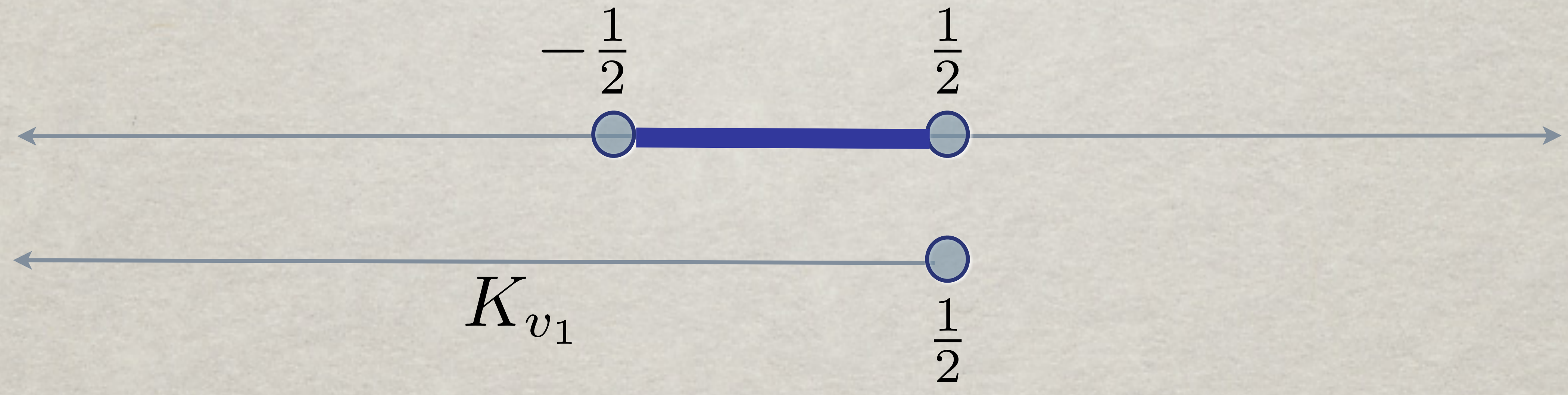

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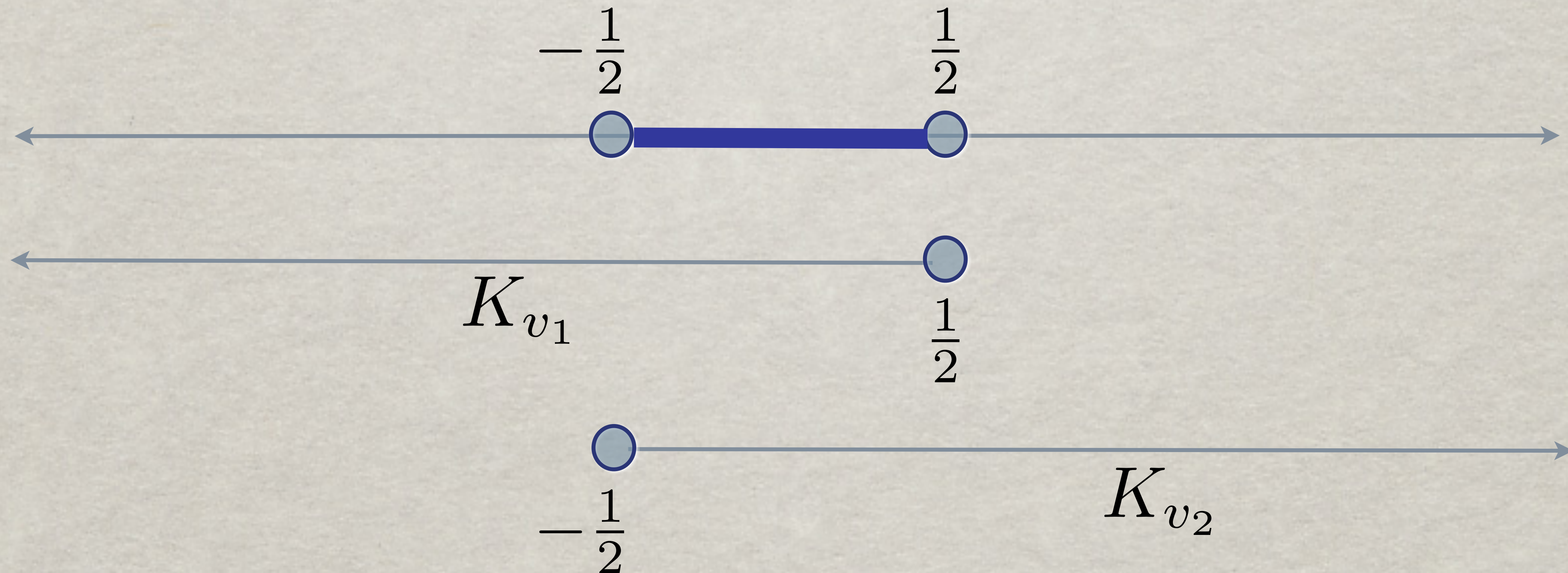

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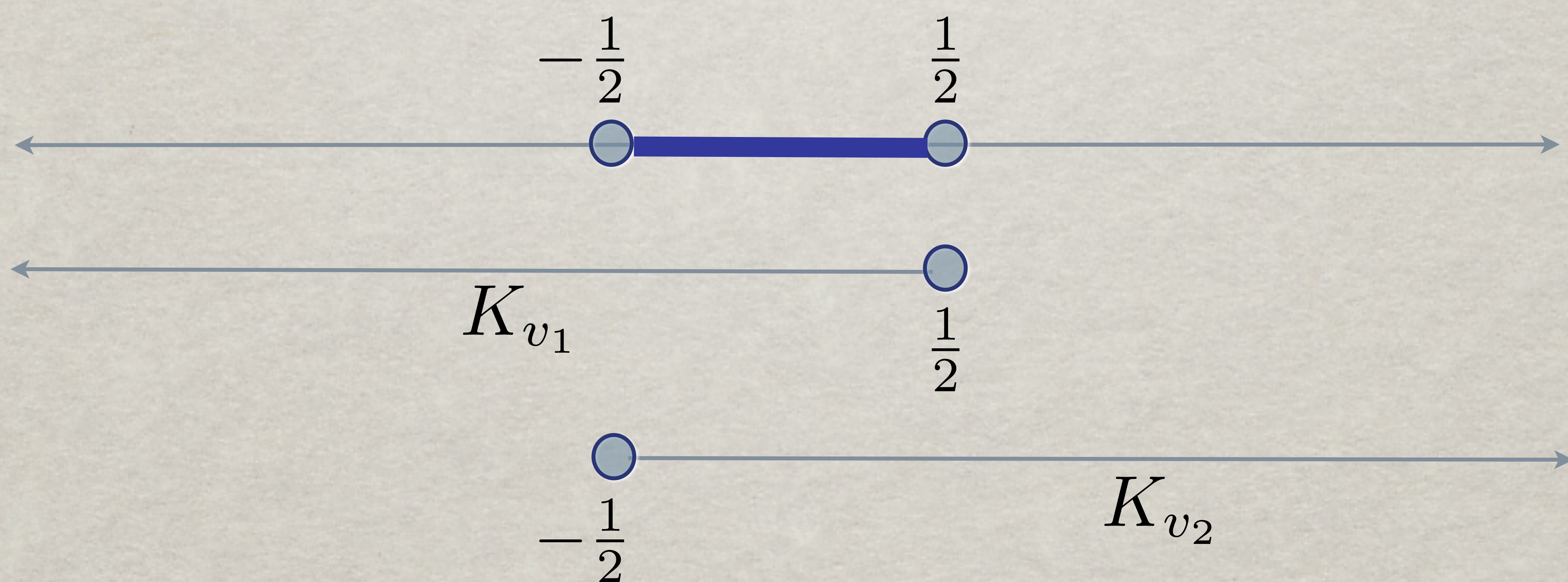


Here $v_1 := \frac{1}{2}, v_2 := -\frac{1}{2}$.

We compute the Fourier-Laplace transform of each vertex tangent cone separately.

$$1_{K_{v_1}}(z) := \int_{-\infty}^{\frac{1}{2}} e^{2\pi i x z} dx = \frac{e^{2\pi i \frac{1}{2} z}}{2\pi i z},$$

valid for $\Im(z) < 0$.

For the other vertex tangent cone, we have

$$1_{K_{v_2}}(z) := \int_{\frac{-1}{2}}^{\infty} e^{2\pi i x z} dx = -\frac{e^{2\pi i (\frac{-1}{2}) z}}{2\pi i z},$$

valid for $\Im(z) > 0$.

If we add the two contributions from the vertex tangent cones, we get the same answer that we got from the direct computation of $\hat{1}_P(z)$. This is "magic"! So at least intuitively, we have:

$$\hat{1}_{K_{v_1}}(z) + \hat{1}_{K_{v_2}}(z) = \hat{1}_P(z),$$

But...

"magical" things are just the tip of an interesting mountain.....

The two Fourier-Laplace transforms of the tangent cones have DISJOINT DOMAINS OF CONVERGENCE!

What to do?

The main point is that the identity above does not hold at the level of integrals, but turns out to be true for the analytic continuation of the integrals, which are luckily very concrete functions, called exponential rational functions.

Exercise 2.

In $\mathbb{R}^2$, compute the Fourier-Laplace transform of the indicator function of the triangle whose vertices are $v_1 := (0,0), v_2 := (a,0)$, and $v_3 := (0,b)$, with $a > 0, b > 0$.

In fact, show that

$$\int_{\Delta} e^{2\pi i \langle x,z\rangle} dx = \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{z_1 z_2} + \frac{-b\ e^{2\pi i a z_1}}{(-a z_1 + b z_2) z_1} + \frac{-a\ e^{2\pi i b z_2}}{(a z_1 - b z_2) z_2}\right),$$

valid for "generic" vectors $z := (z_1, z_2) \in \mathbb{C}$.

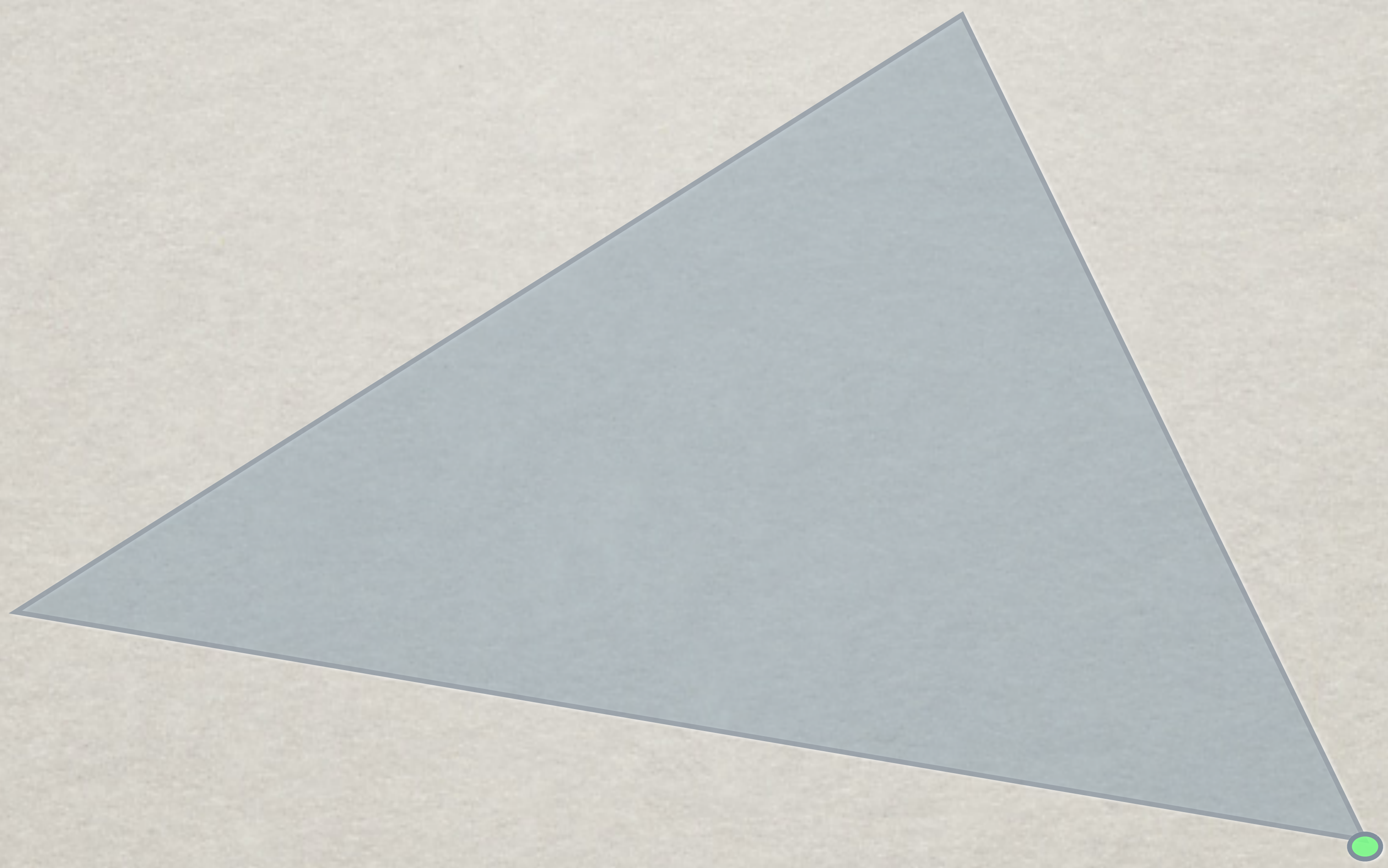
We can compare this computation with another computation, namely the Fourier-Laplace transform of each tangent cone of Δ.

So we now need the definition of a tangent cone of a polytope.

So, what are Tangent cones?

(A quick tutorial)

Example: If the face F is a vertex, what does the tangent cone at the vertex look like?



Face $= v$, a vertex

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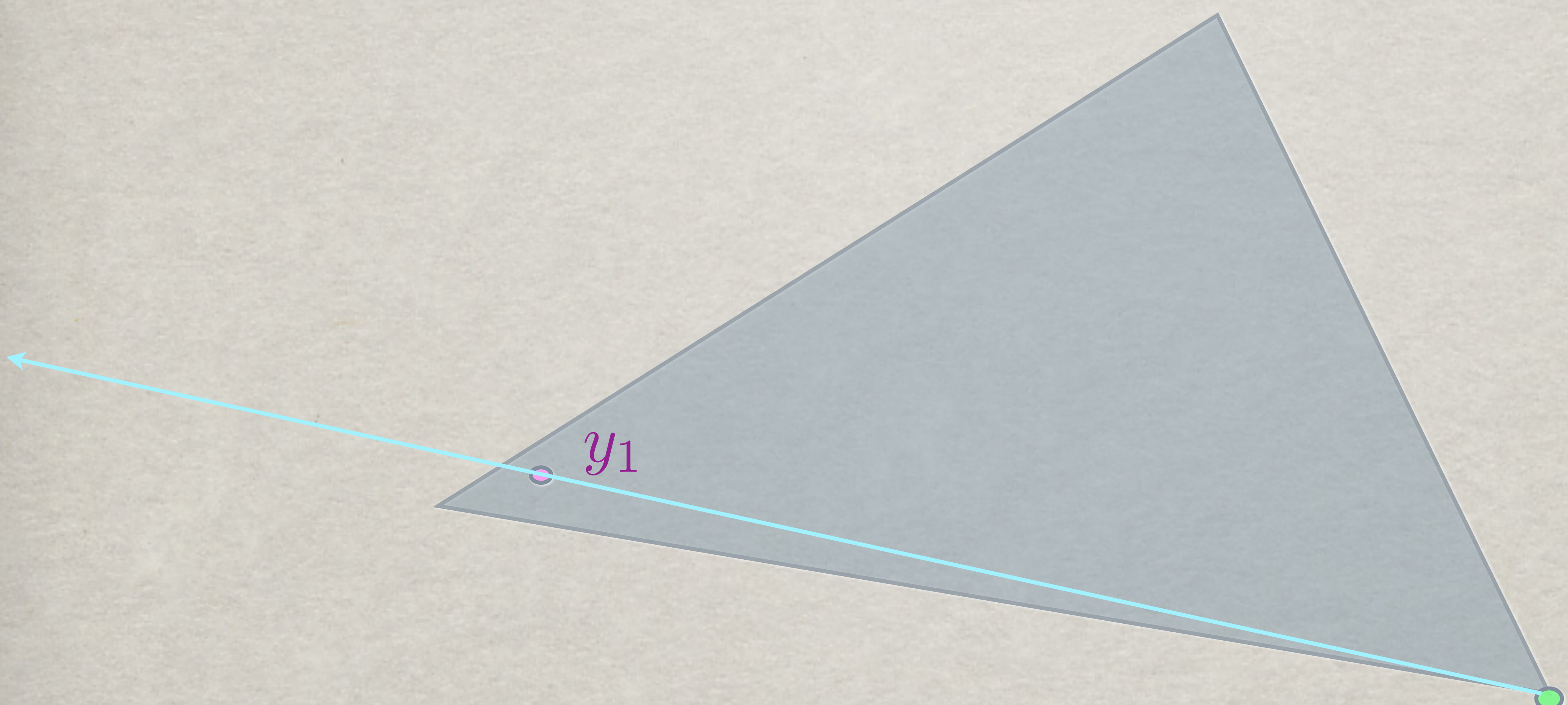
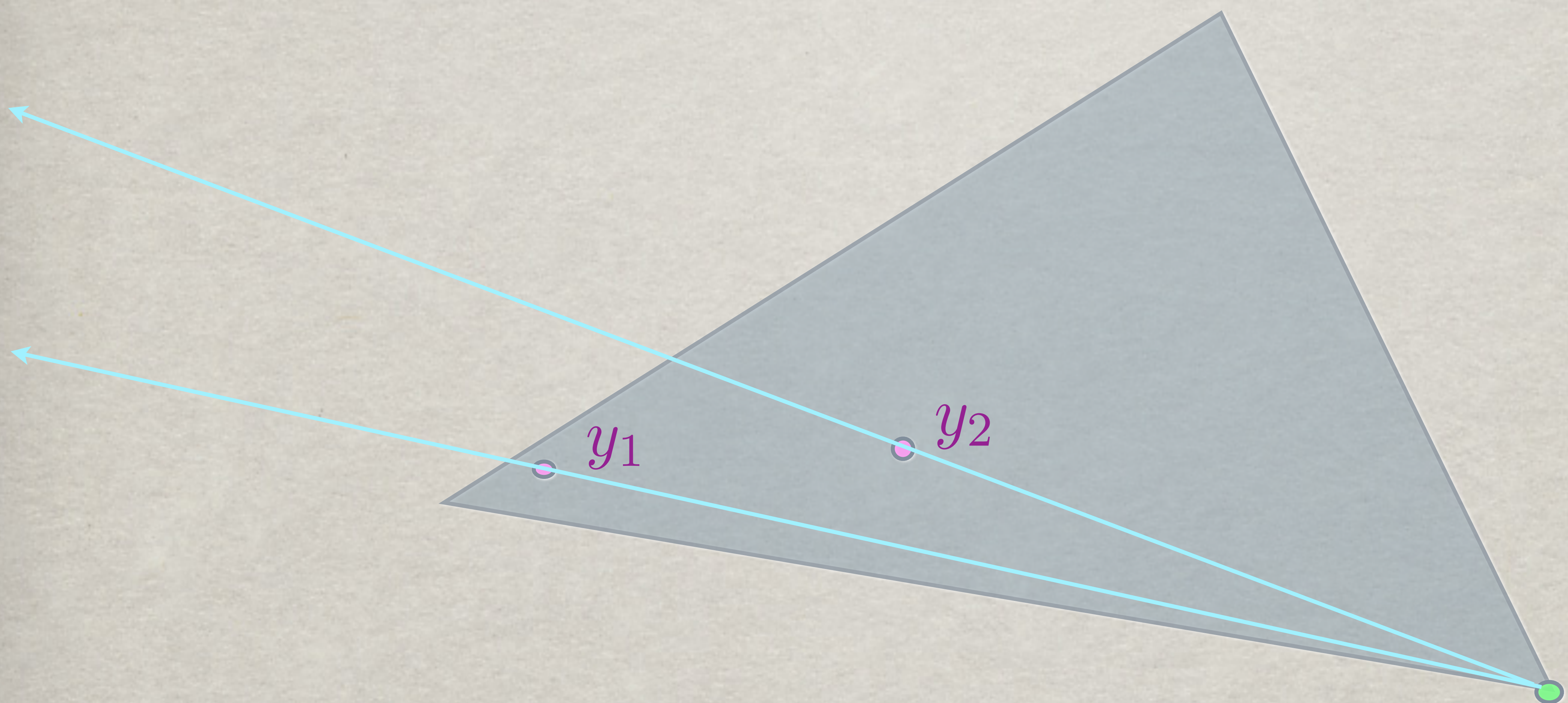
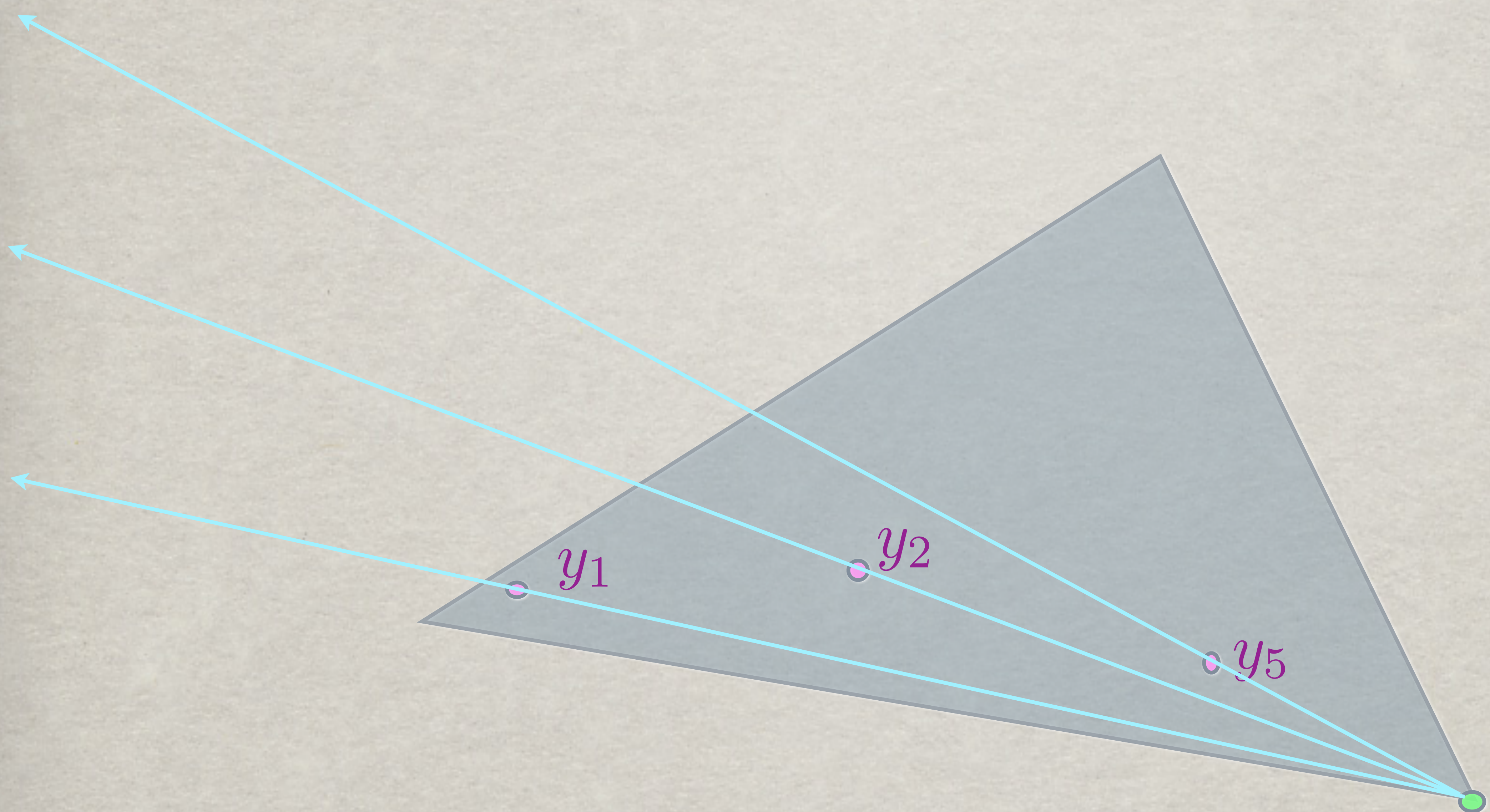


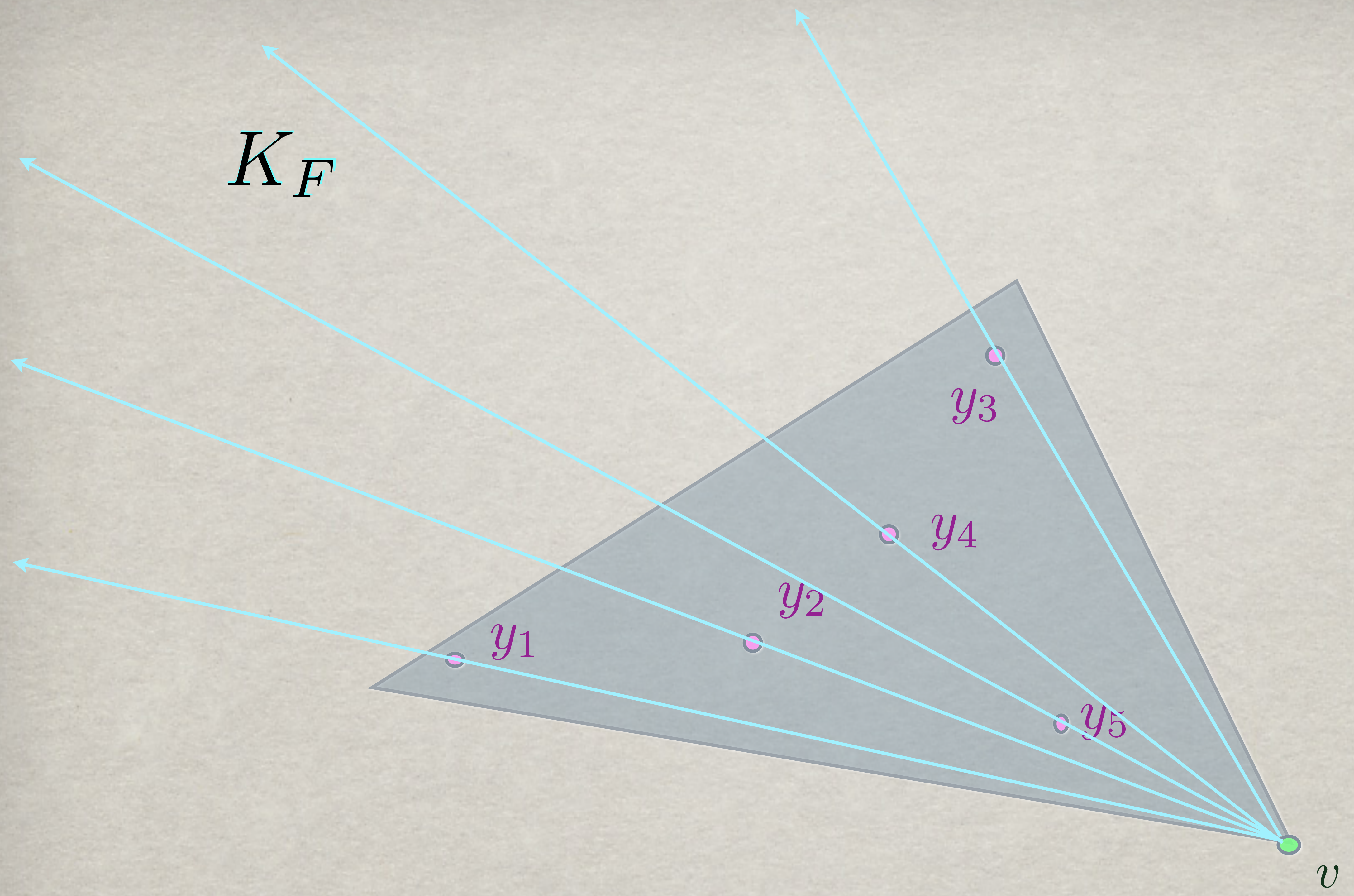
$$\text{Face} = v, \text{ a vertex}$$



Face $= v$, a vertex



Face $= v$, a vertex



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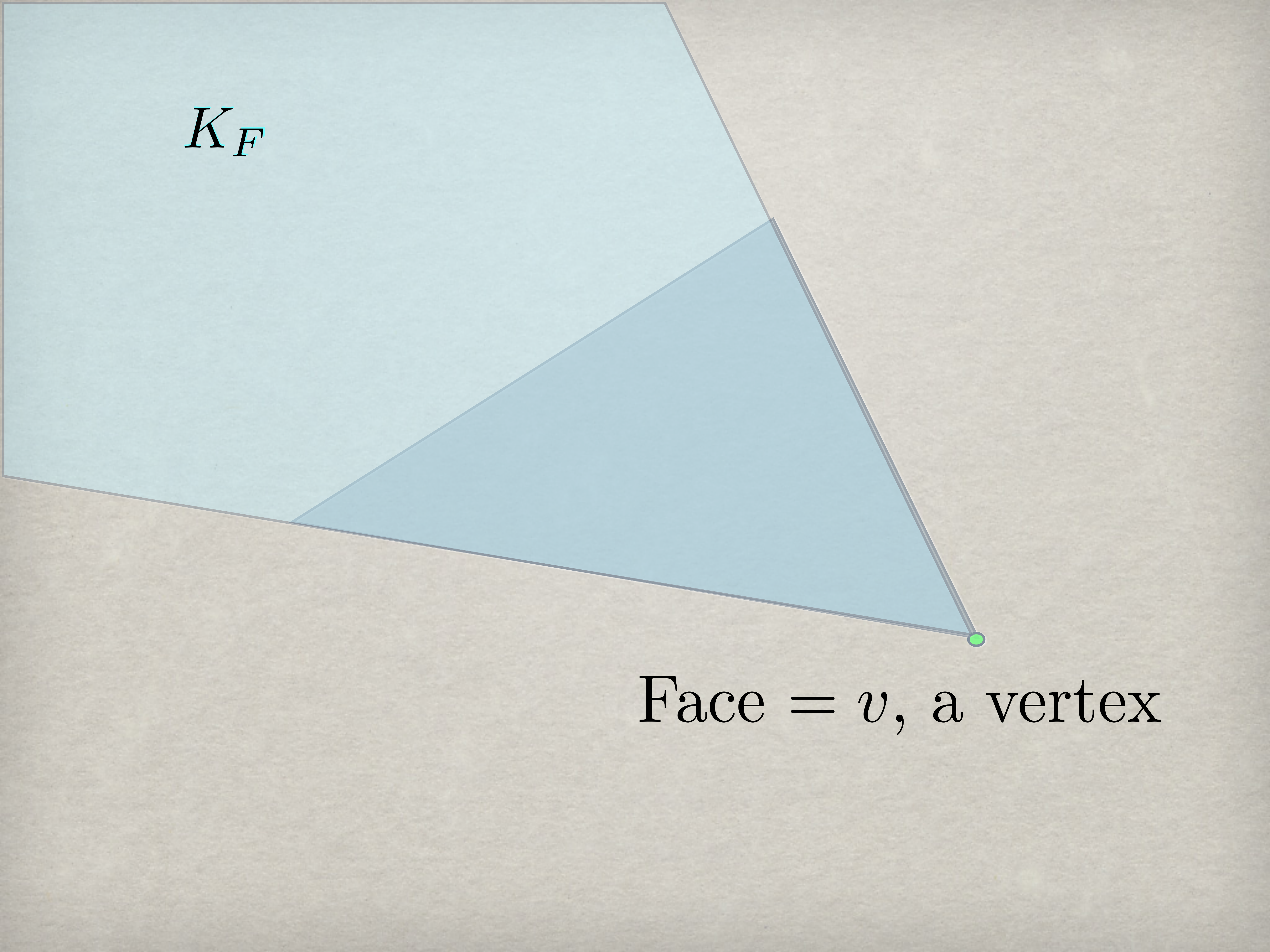

K_F
Face $= v$, a vertex

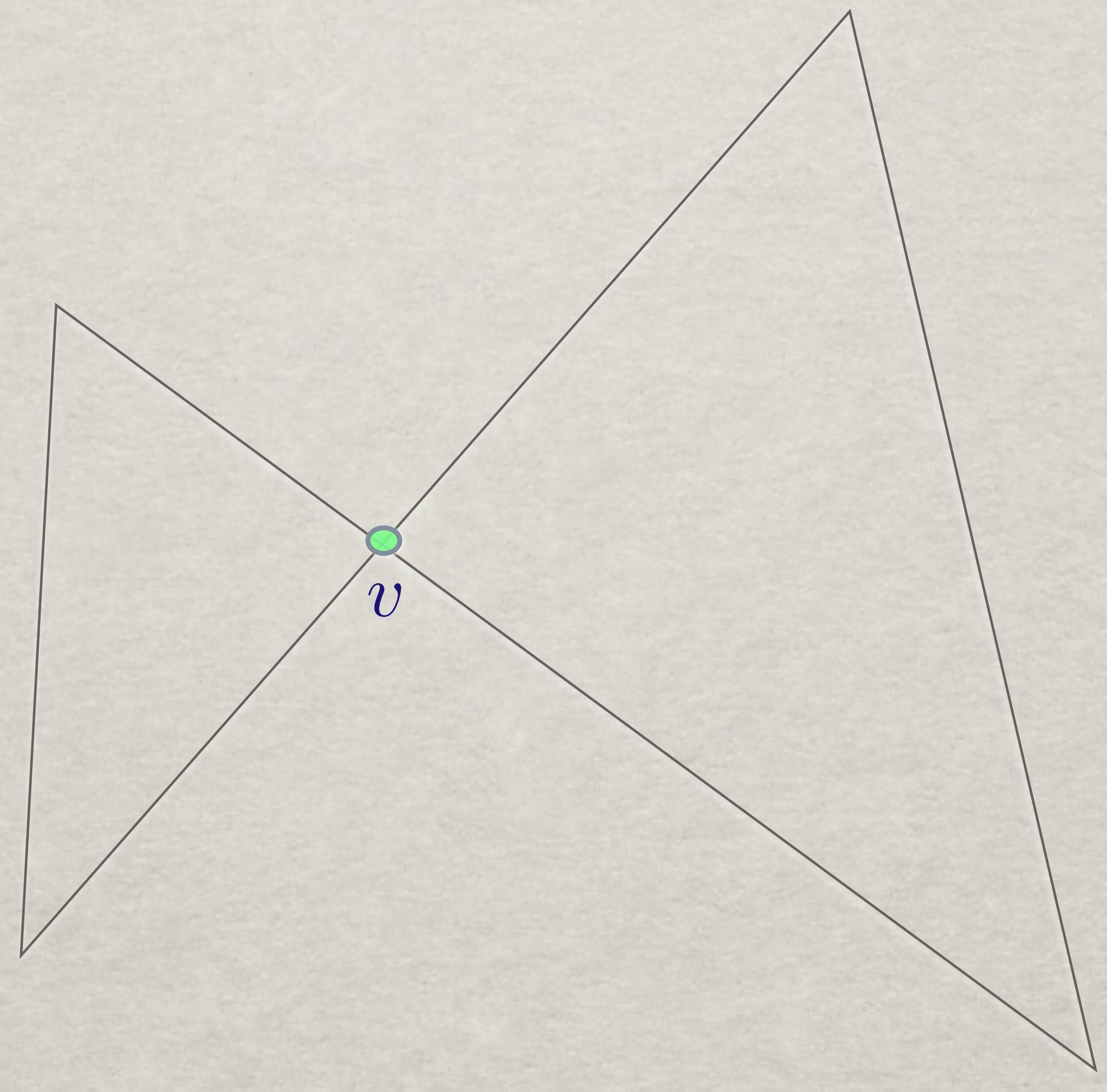
Definition. The tangent cone K_F of a face $F \subset P$ is defined by

$$K_F := \{v + x \mid v + \epsilon x \in P, \text{ for all sufficiently small } \epsilon > 0\}.$$

This definition is still valid for non-convex polytopes.

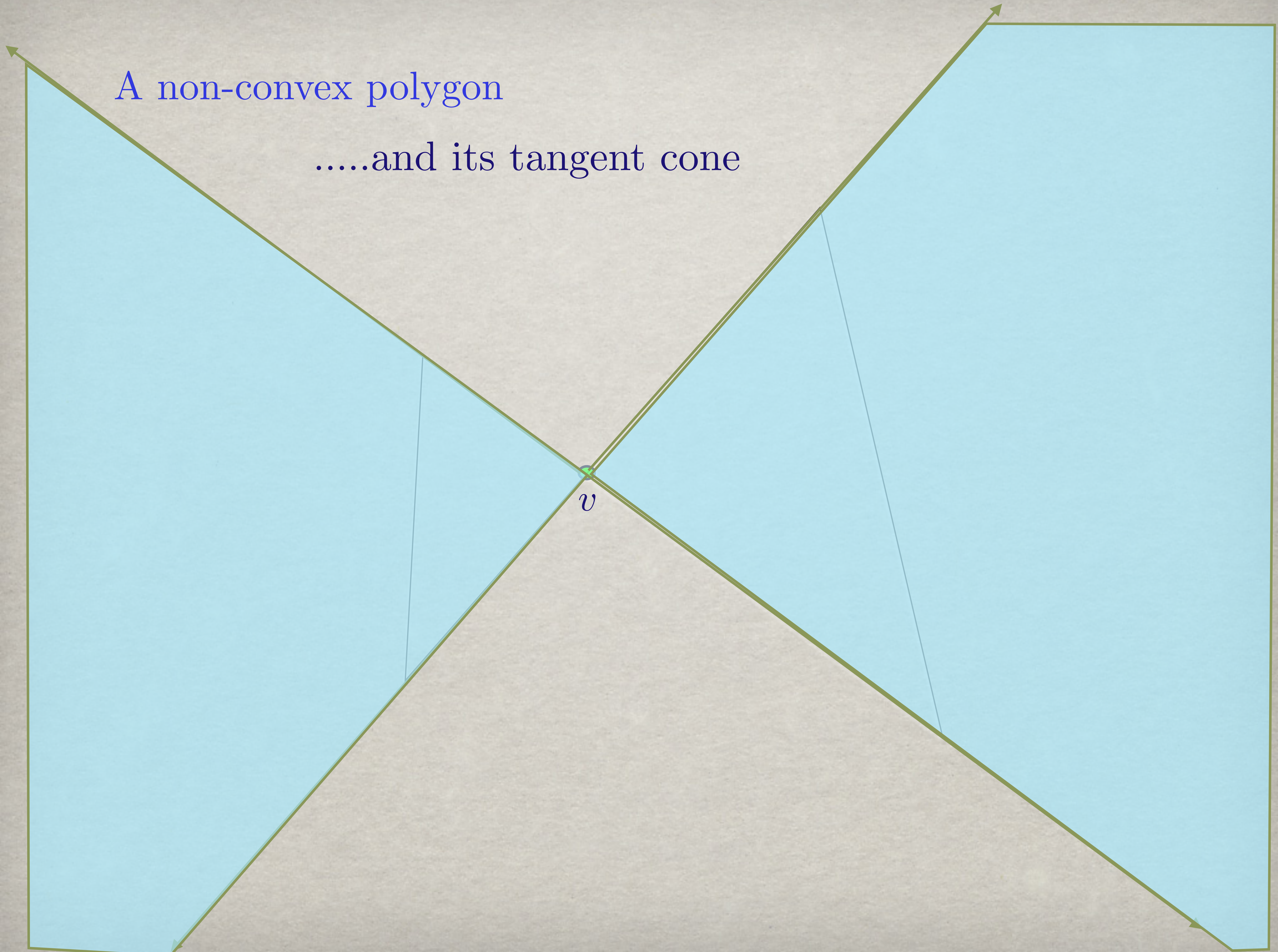
Intuitively, the tangent cone of any face F of P is the union of all rays that have a base point in F and 'point towards P'.

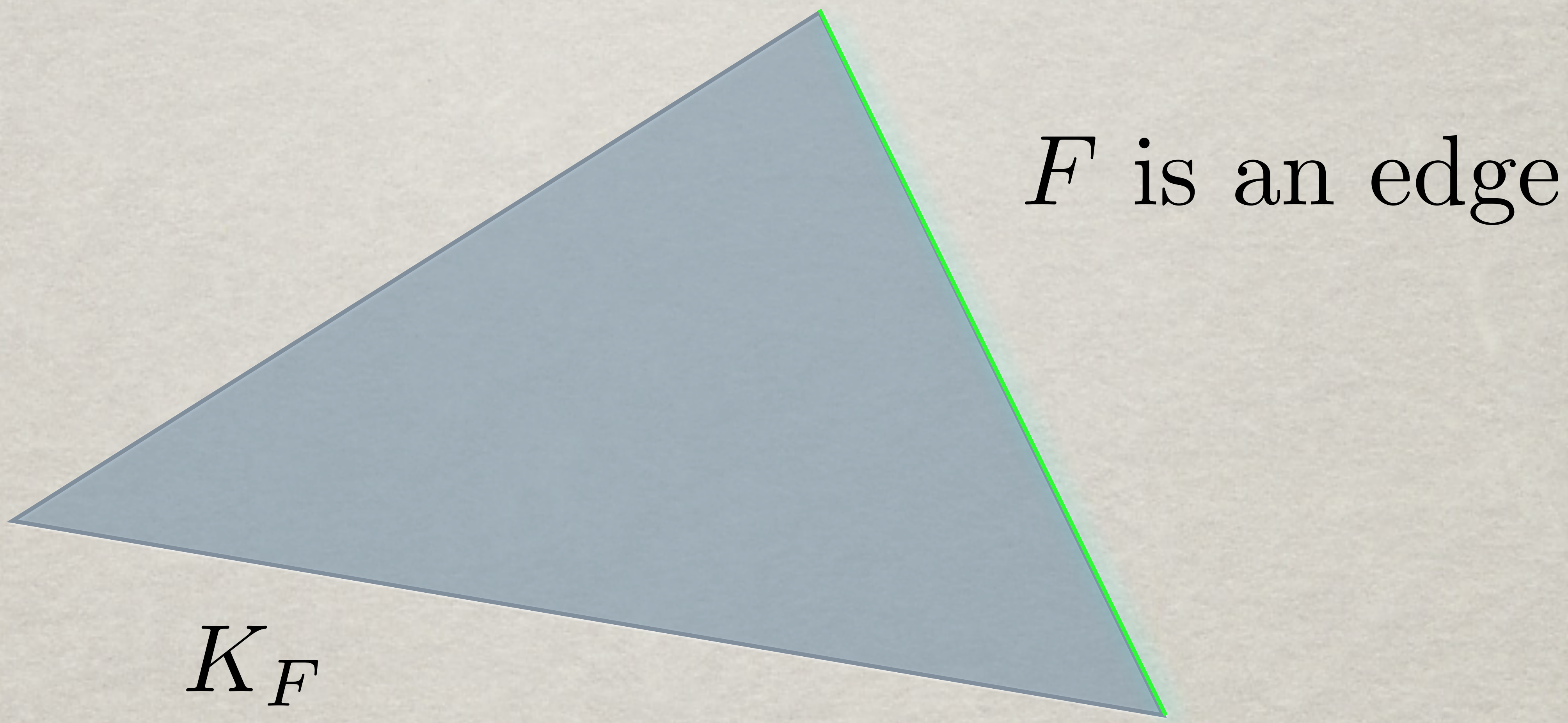
A non-convex polygon



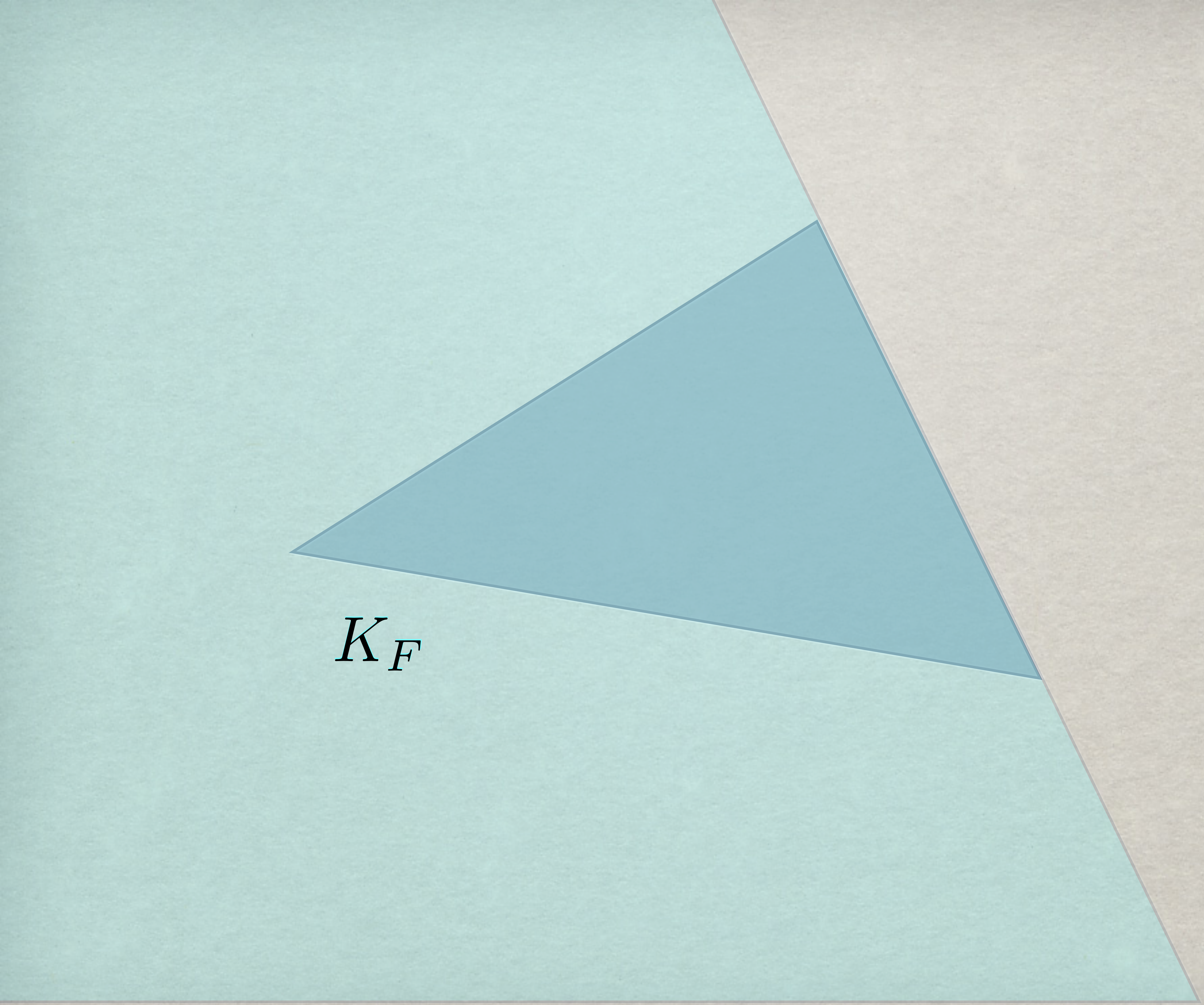
A non-convex polygon

.....and its tangent cone





Example. when the face is a 1-dimensional edge of a polygon, its tangent cone is a half-plane.


K_F

Exercise 3.

(Continuing our computations from Exercise 2) We had:

In $\mathbb{R}^2$, compute the Fourier-Laplace transform of the indicator function of the triangle whose vertices are $v_1 := (0,0), v_2 := (a,0)$, and $v_3 := (0,b)$, with $a > 0, b > 0$.

In fact, show that

$$\int_\Delta e^{2\pi i\langle x,z\rangle}dx = \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{z_1 z_2} + \frac{-b\ e^{2\pi i a z_1}}{(-az_1 + bz_2)z_1} + \frac{-a\ e^{2\pi i b z_2}}{(az_1 - bz_2)z_2}\right),$$

valid for "generic" vectors $z := (z_1, z_2) \in \mathbb{C}$.

Now, let's compute the Fourier-Laplace transforms of each of the three tangent cones of the triangle Δ.

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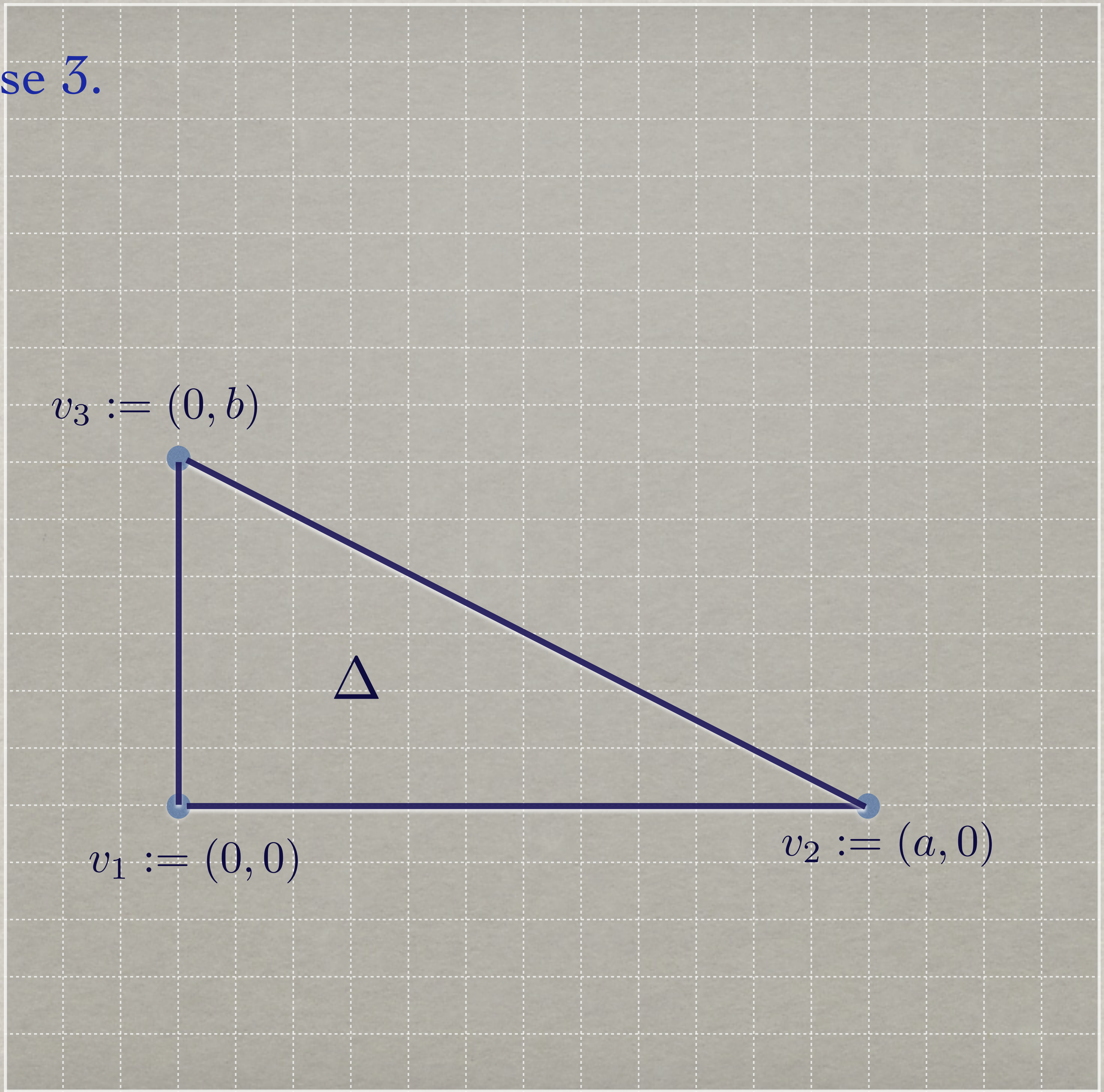

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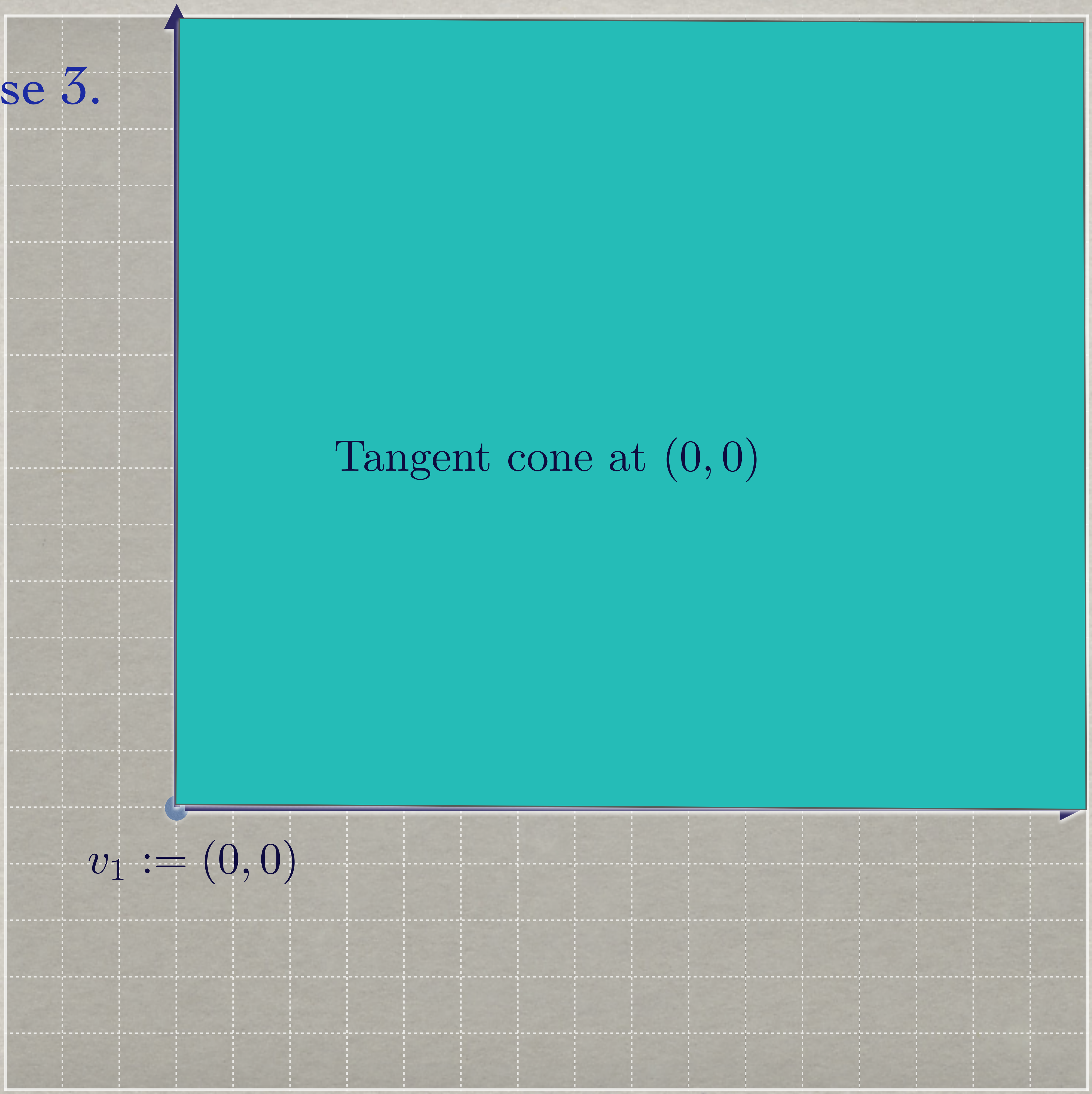

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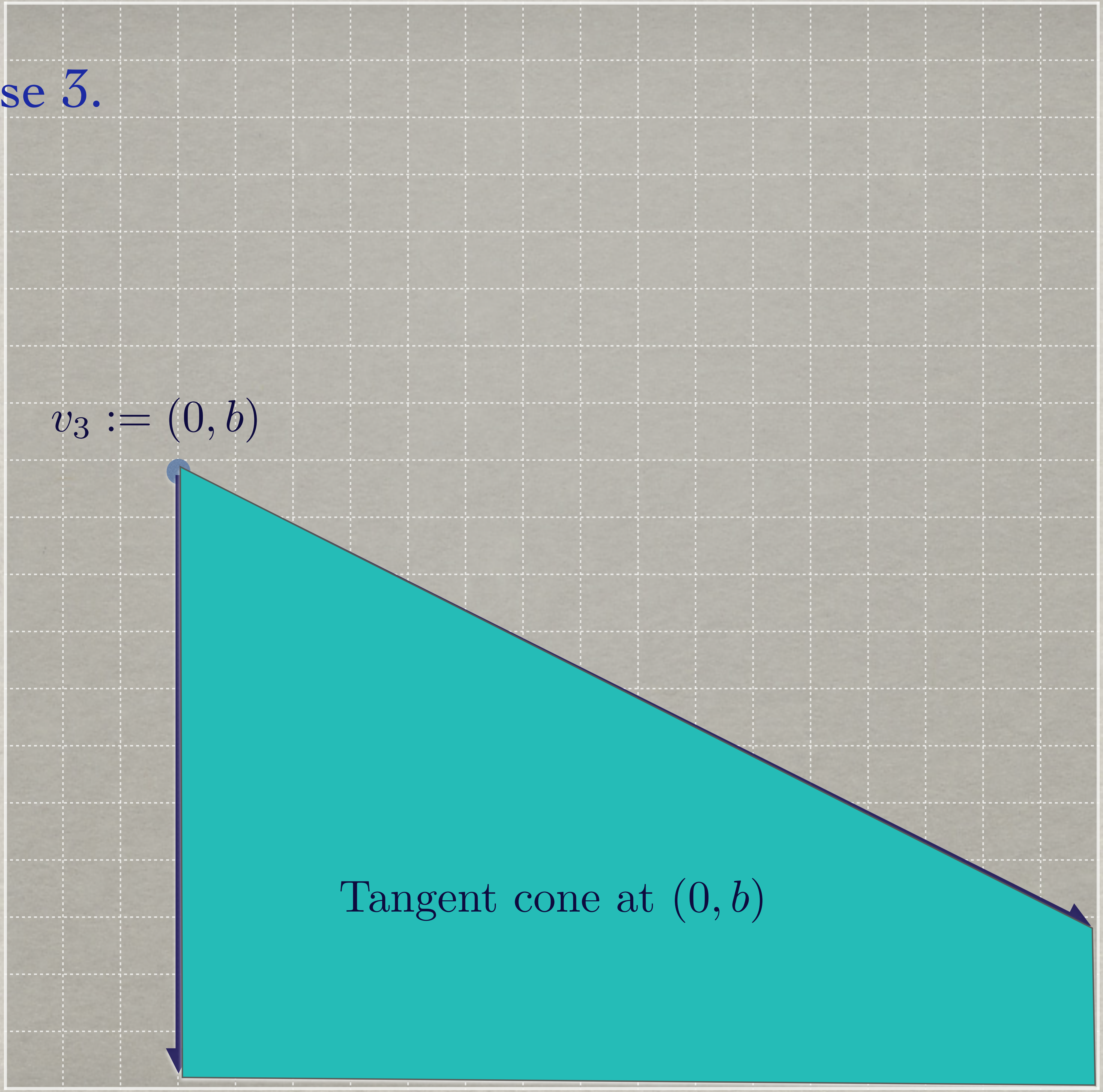


Exercise 3.

For the tangent cone at $(0,0)$, show that:

$$1_{K_{v_1}}(z) := \int_{K_{v_1}} e^{2\pi i \langle x,z \rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{1}{z_1 z_2}.$$

For the tangent cone at $(0,b)$, show that:

$$\hat{1}_{K_{v_3}}(z) := \int_{K_{v_3}} e^{2\pi i \langle x,z \rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{(ab)e^{2\pi i b z_2}}{(az_1 - bz_2)(-bz_2)}.$$

For the tangent cone at $(a,0)$, show that:

$$\hat{1}_{K_{v_2}}(z) := \int_{K_{v_2}} e^{2\pi i \langle x,z \rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{(ab)e^{2\pi i a z_1}}{(-az_1 + bz_2)(-az_1)}.$$

Exercise 3.

Finally, check that indeed we appear to have the identity:

$$\hat{1}_{K_{v_1}}(z) + \hat{1}_{K_{v_2}}(z) + \hat{1}_{K_{v_3}}(z) = \hat{1}_P(z),$$

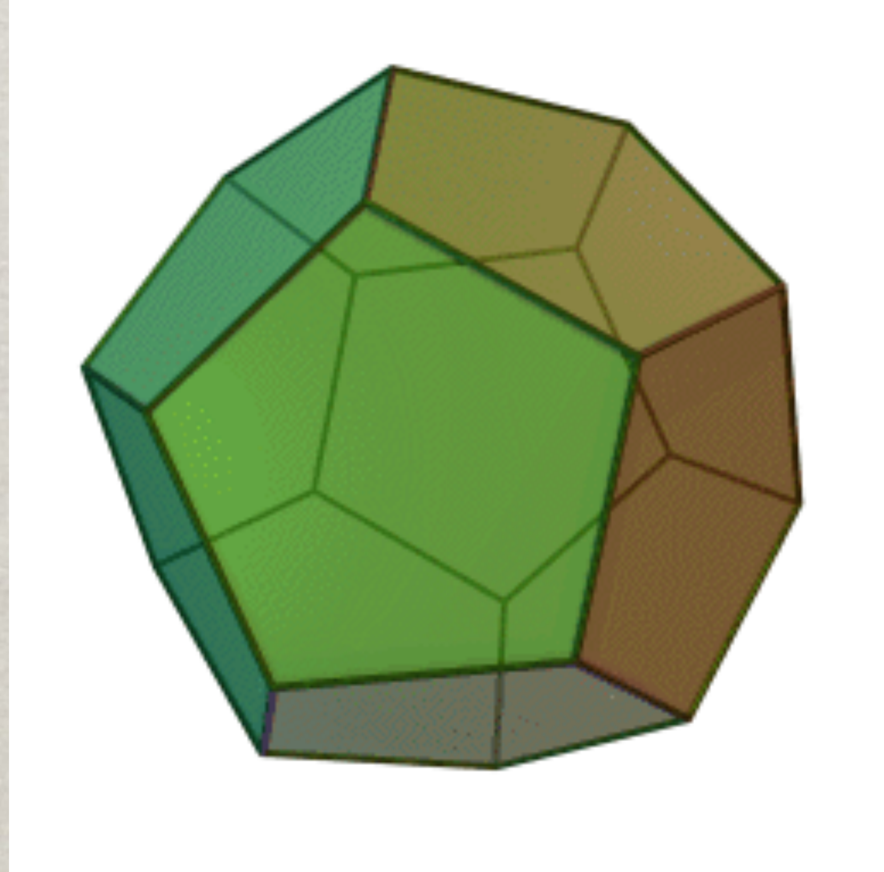
valid for all "generic" values of $z := (z_1, z_2) \in \mathbb{C}$.

Definition

A d-dimensional polytope enjoying the property that each of its vertices shares an edge with exactly d other vertices is called a simple polytope.

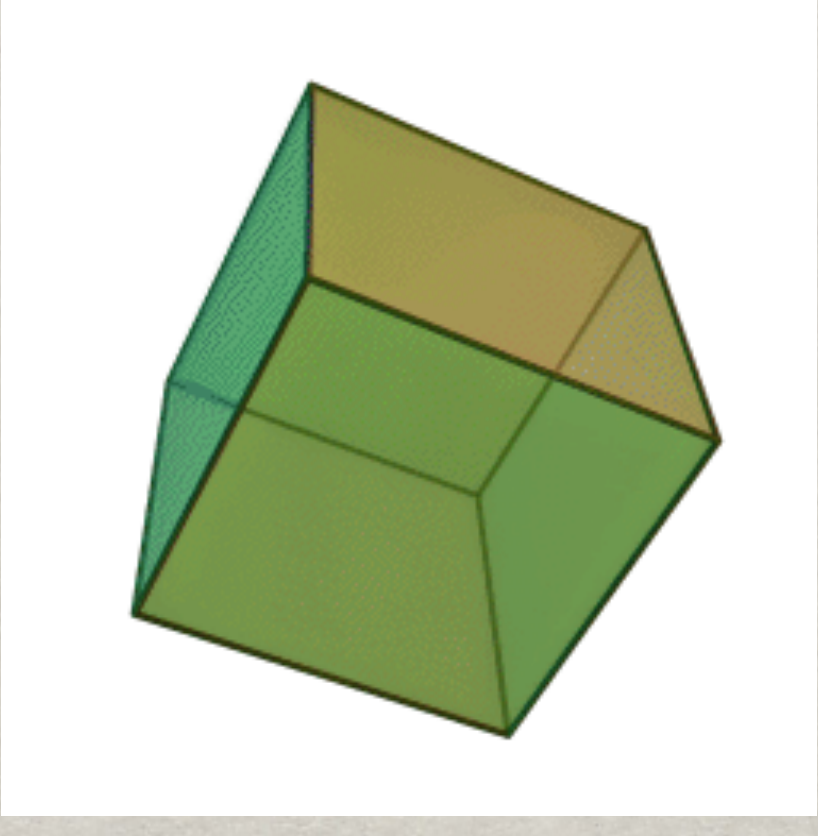
Example of a simple polytope:

The dodecahedron



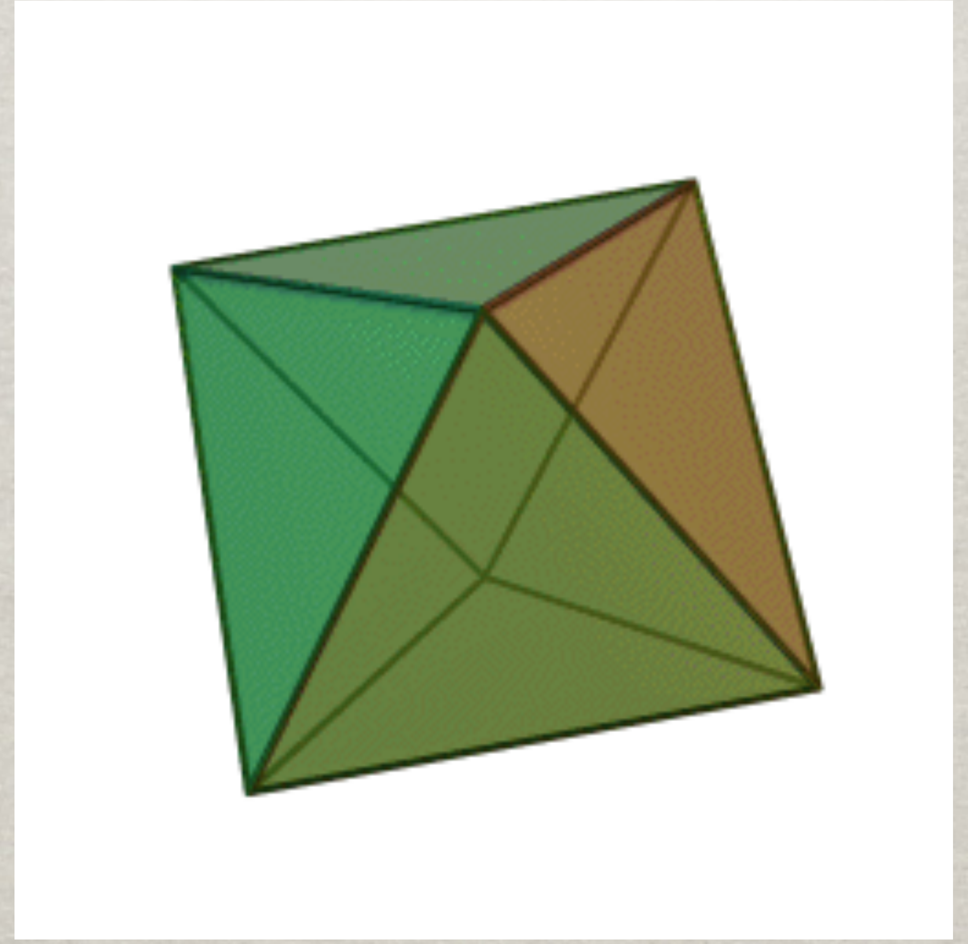
Example of a simple polytope:

The cube

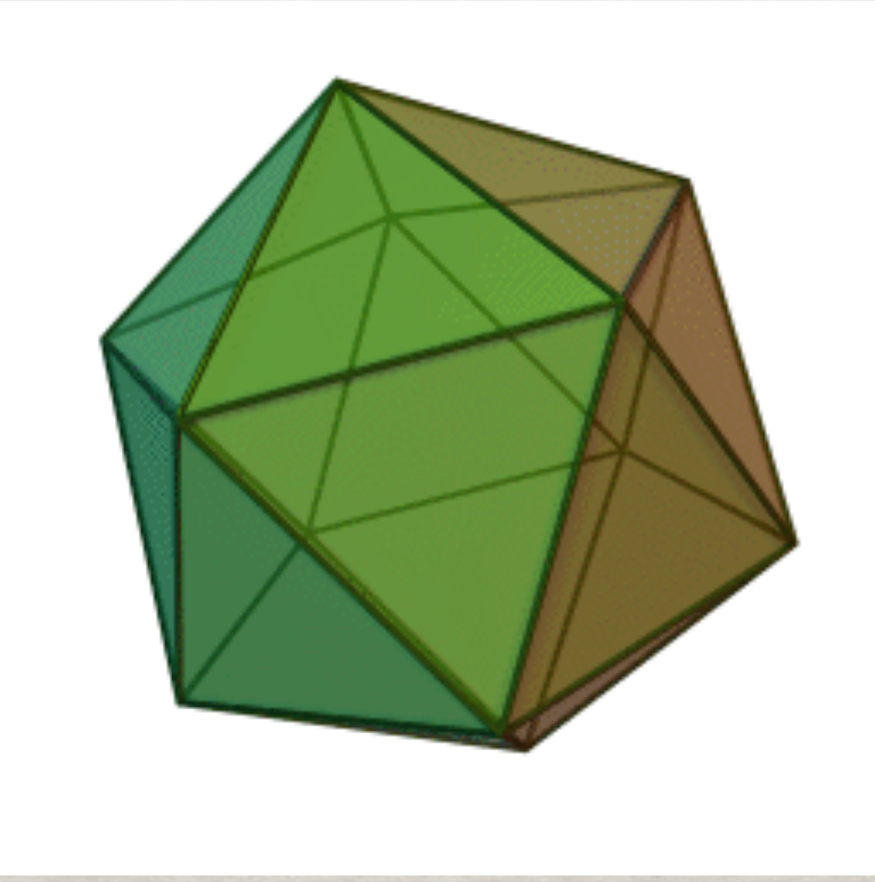


Example of a non-simple polytope:

the Octahedron



Example of a non-simple polytope:
the Icosahedron



Question (for later): can Fourier techniques tell us whether these polytopes tile space by translations?

Answer: yes.

Definition

We define the Fourier-Laplace transform of a polytope P by

$$\hat{1}_P(z) := \int_P e^{2\pi i\langle z,x\rangle}\, dx$$

where is any compact, but not necessarily convex, polytope.

For any convex cone K, its Fourier-Laplace transform converges, for some $z \in \mathbb{C}^d$:

$$\hat{1}_K(z) := \int_K e^{2\pi i\langle z,x\rangle}\, dx$$

Exercise 4.

Find the domain of convergence for the Fourier-Laplace transform of a cone K.

Definition

A cone $K \subset \mathbb{R}^d$ is the nonnegative real span of a finite number of vectors in $\mathbb{R}^d$.

A d-dim'l simplicial cone K has d extreme vectors emanating from its vertex. That is:

$$K = \{\lambda_1 w_1 + \cdots + \lambda_d w_d \mid all\ \lambda_j \geq 0\},$$

where we assume that the extreme vectors (called edges) $w_1, \ldots, w_d$ are linearly independent.

The following result of Brion was later extended by Barvinok, Lawrence, and Varchenko.

Theorem (Brion, 1988)

Given a convex, simple d-dim'l polytope P, with vertex set V, and known local tangent cone data at each vertex $v_j \in V$, we have:

$$\int_P e^{2\pi i \langle z,x \rangle} dx = \sum_{v \in V} \int_{K_v} e^{2\pi i \langle z,x \rangle} dx,$$

where K_v is the vertex tangent cone at the vertex $v \in P$.

There is a beautiful relationship between all of the tangent cones of P, given by the "Brianchon-Gram" relation, and it has an "Euler characteristic flavor". When P is convex it is given as follows.

Theorem. (Brianchon-Gram)

$$1_P = \sum_{F \subset P} (-1)^{\dim F} 1_{K_F}$$

where 1_{K_F} is the indicator function of the tangent cone to F.

Important. The Brianchon-gram identity for indicator functions allows us to transfer the computation of functions defined over the polytope P to the LOCAL computation of functions defined over each tangent cone of P.

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Cones are nice: semigroups, so they are 'almost' linear.

Polytopes are complex: highly non-linear.

Going back to the statement of Brion, we note that for a simplicial d-dim'l cone K, we have

(Exercise 5)

$$\hat{1}_{K_v}(z) = \frac{(-1)^d |\det K_v| e^{2\pi i \langle v, z\rangle}}{\prod_{j=1}^d \langle \omega_j, z\rangle},$$

so that we may rewrite the Brion identity as:

$$\int_P e^{2\pi i \langle z, x\rangle} dx = \sum_{v \in V} \int_{K_v} e^{2\pi i \langle z, x\rangle} dx,$$

$$\int_P e^{2\pi i \langle z, x\rangle} dx = \sum_{v \in \text{Vertices of P}} \frac{(-1)^d |\det K_v| e^{2\pi i \langle v, z\rangle}}{\prod_{j=1}^d \langle \omega_j, z\rangle}.$$

Thus we get an explicit formulation for the Fourier-Laplace transform of a simple, convex polytope.

Open problem: Is there a similar (or more complicated) formulation for the Fourier-Laplace transform of a *general* convex polytope?

(Many results would follow from such a formulation, for example computing quickly the volume of a high-dimensional polytope)

So what can we do with such formulas?

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Well, the most apparent application is the fast computation of volumes of simple polytopes:

$$\hat{1}_P(0) := \int_P e^{2\pi i * 0} dx = \int_P dx = vol(P).$$

In high dimensions???

Lattice point enumeration in polytopes: Discrete Volumes

Lattice point enumeration in polytopes: Discrete Volumes

The discrete volume of a polytope is defined by

$$L_P := \#\{\mathbb{Z}^d \cap P\}.$$

If we dilate P by a real number t first, and then count, we get:

$$L_P(t) := \#\{\mathbb{Z}^d \cap tP\}.$$

Lattice point enumeration in polytopes: Discrete Volumes

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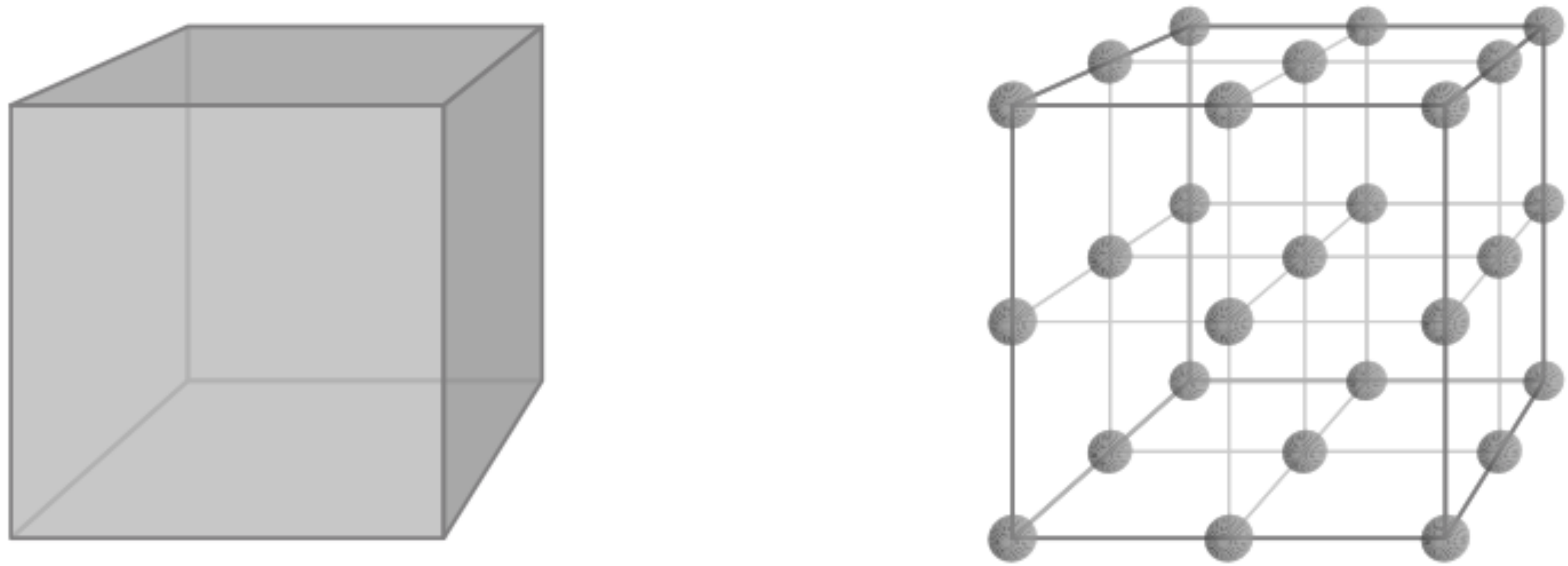
Theorem (Ehrhart, 1960's).

When t is restricted to nonnegative integer values, $L_P(t)$ is a polynomial in the discrete variable t.



Continuous and discrete volume.

M. Beck and S. Robins, Computing the continuous discretely: integer point enumeration in polytopes, Springer UTM series, 2'nd edition, 2015.

Theorem (Diaz and SR, 1997).

Suppose P^{closed} is a closed d-dimensional integer simplex in $\mathbb{R}^d$.

Suppose P^{open} is its relative d-dimensional interior, and let the vertices of P be given by $\{v_1, \ldots, v_{d+1}\}$.

Theorem (Diaz and SR, 1997).

Suppose P^{closed} is a closed d-dimensional integer simplex in $\mathbb{R}^d$.

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Let G be the finite abelian group defined via the Hermite-normal form of the matrix whose columns are given by the integer vertices of P.   Then:

$$\sum_{t=0}^{\infty} L_{P^{closed}}(t) e^{-2\pi s t} = \frac{1}{2^{d+1}|G|} \sum_{g \in G} \prod_{k=1}^{d+1} \left(1 + \coth \frac{\pi}{p_k} (s + i \langle v_k, g \rangle) \right)$$

$$\sum_{t=0}^{\infty} L_{P^{open}}(t) e^{-2\pi s t} = \frac{1}{2^{d+1}|G|} \sum_{g \in G} \prod_{k=1}^{d+1} \left(-1 + \coth \frac{\pi}{p_k} (s + i \langle v_k, g \rangle) \right).$$

Here $p_k := \prod_{i \le k} M_{i,i}$, where M is the integer matrix whose columns are the vertices of P.

Interesting corollaries have appeared since these results from the 90's, including applications to the Frobenius coin exchange problem, applications to Euler-Maclaurin summation formulae over polytopes, applications to the theory of Dedekind sums in Number Theory, etc.

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Exercises.

Exercise 1.

Let P be an integer polygon. Prove that the sum of the local solid angles at all integer points equals the area of P.

In other words, show that

$$\sum_{n \in \mathbb{Z}^2} \omega_P(n) = \text{area}(P).$$

Exercise 2.

In $\mathbb{R}^2$, compute the Fourier-Laplace transform of the indicator function of the triangle whose vertices are $v_1 := (0,0), v_2 := (a,0)$, and $v_3 := (0,b)$, with $a > 0, b > 0$.

In fact, show that

$$\int_\Delta e^{2\pi i \langle x, z \rangle} dx = \left(\frac{1}{2\pi i}\right)^2 \left(\frac{1}{z_1 z_2} + \frac{-b\ e^{2\pi i a z_1}}{(-a z_1 + b z_2) z_1} + \frac{-a\ e^{2\pi i b z_2}}{(a z_1 - b z_2) z_2}\right),$$

valid for "generic" vectors $z := (z_1, z_2) \in \mathbb{C}$.

Exercises.

Exercise 3.

For the tangent cone at $(0,0)$, show that:

$$1_{K_{v_1}}(z) := \int_{K_{v_1}} e^{2\pi i \langle x, z\rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{1}{z_1 z_2}.$$

For the tangent cone at $(0,b)$, show that:

$$\hat{1}_{K_{v_3}}(z) := \int_{K_{v_3}} e^{2\pi i \langle x, z\rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{(ab)e^{2\pi i b z_2}}{(az_1 - bz_2)(-bz_2)}.$$

For the tangent cone at $(a,0)$, show that:

$$\hat{1}_{K_{v_2}}(z) := \int_{K_{v_2}} e^{2\pi i \langle x, z\rangle}\, dx = \left(\frac{1}{2\pi i}\right)^2 \frac{(ab)e^{2\pi i a z_1}}{(-az_1 + bz_2)(-az_1)}.$$

Exercises.

Exercise 4.

Find the domain of convergence for the Fourier-Laplace transform of a cone K.

Exercise 5.

For a simplicial cone $K \subset \mathbb{R}^d$, with apex at v, and edges $\omega_1, \ldots, \omega_d$, show that

$$\hat{1}_{K_v}(z) = \frac{(-1)^d |\det K_v| e^{2\pi i \langle v, z \rangle}}{\prod_{j=1}^{d} \langle \omega_j, z \rangle},$$

valid for all complex points $z \in \mathbb{C}^d$ such that the denominator does not vanish.

Exercises.

Exercise 6.

Compute the Fourier-Laplace transform of
The cross polytope $\diamond := \{x \in \mathbb{R}^d \mid |x_1| + \cdots + |x_d| \leq 1\}$.

Exercise 7.

Show that the Fourier-Laplace transform $\hat{1}_S(z)$ (for any measurable subset $S \subset \mathbb{R}^d$) is real-valued if and only if S is symmetric about the origin.

We recall that a set is said to be symmetric about the origin if $x \in S$ whenever $-x \in S$, for all $x \in \mathbb{R}^d$.

Thank You