The Kidney Exchange Problem

Problem definition

Let P be the set of PDPs, N be the set of NDDs.
If the version of KEP concerns about cycles as well as chains is called Cardinality Constrained Cycles and Chains Problem (CCCCP):

* We have $G=(V,E)$, for $V=N\cup P$ and $E=\{(i,j)\mid i\in V, j\in P, i\neq j\}$. Notice that $\{(i,j)\mid i,j\in N, i\neq j\}=\varnothing$

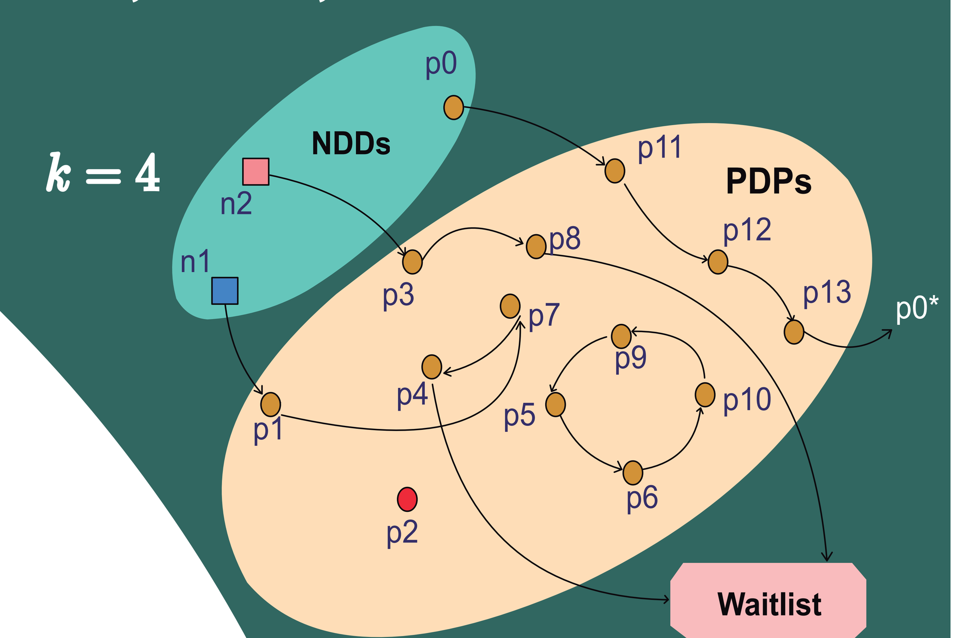
If the version of KEP only concerns about cycles is called Cardinality Constrained Multi-cycle Problem (CCMcP):

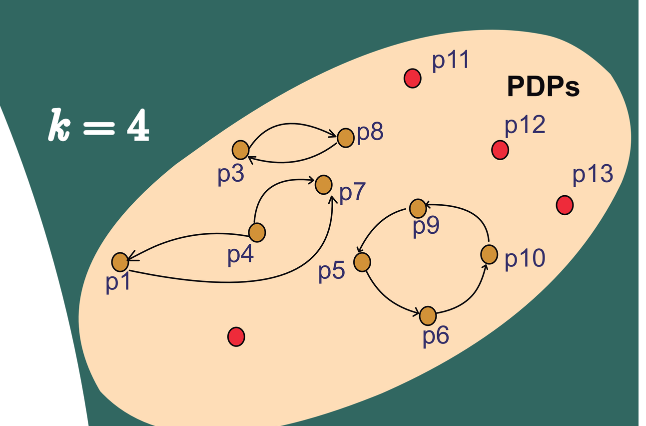
* We have $G=(V,E)$, for $V=P$ and $E=\{(i,j)\mid i,j\in P, i\neq j\}$

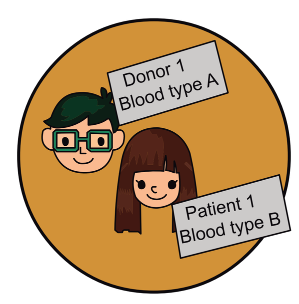
There is an edge between two nodes i and j if the donor of node i is compatible with the patient of node j.

A weight w_{ij} is associated to each arc $(i,j)\in E$. These weights are used to capture some priority (i.e., importance, urgency) for conducting the corresponding matches.

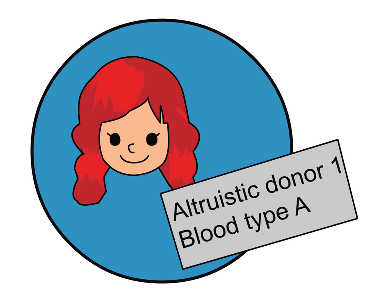
Constraints:
* PDPs can be part of either a chain or a cycle, but not both.
* NDDs can only form one and only chain.
* Cycles must meet cardinality constraints. Cycle lenght is at most $k\in\mathbb{N}$, chains may or may not be constrained.



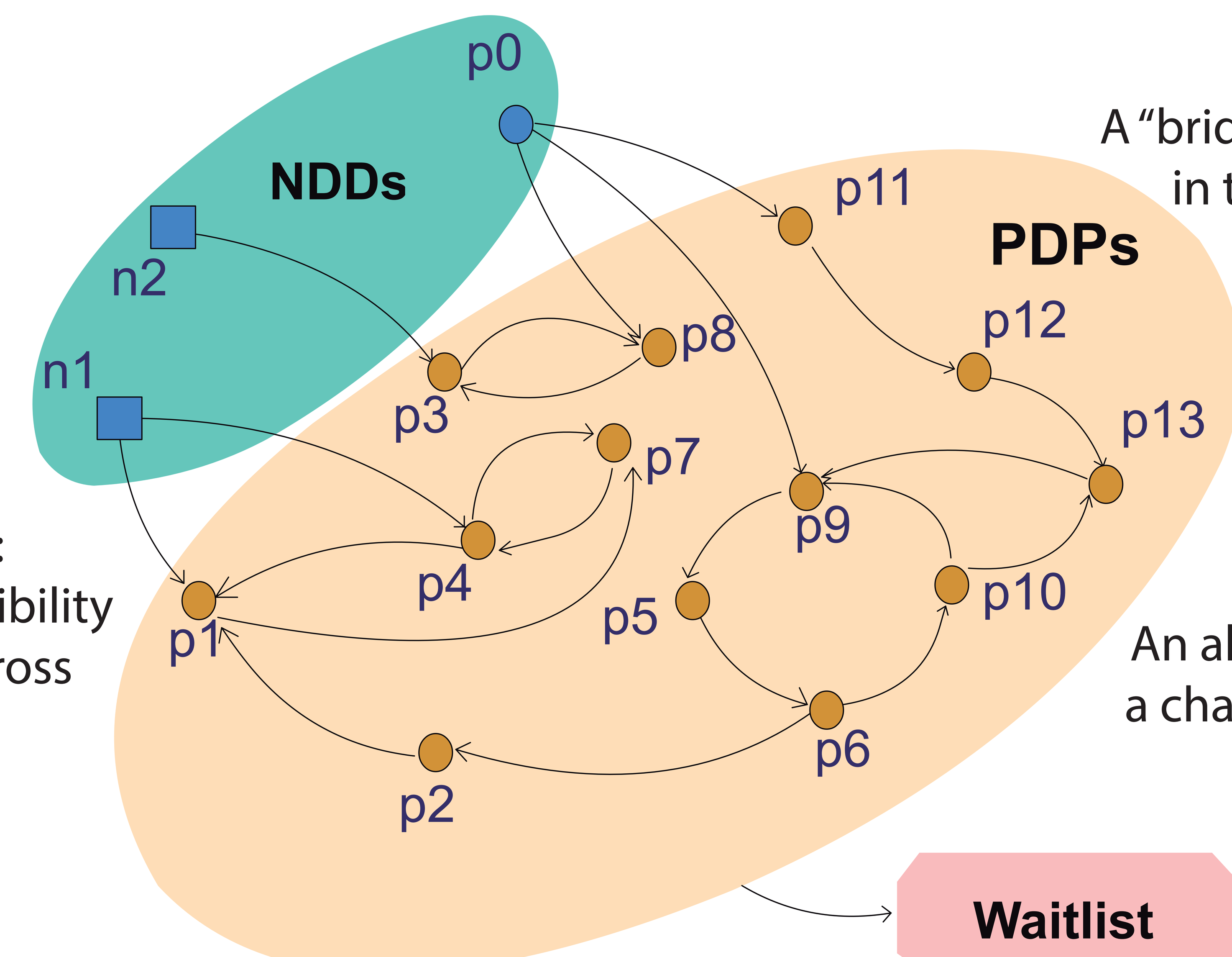




Incompatible Patient-Donor Pair (PDP)



Non-Directed Donor (NDD)

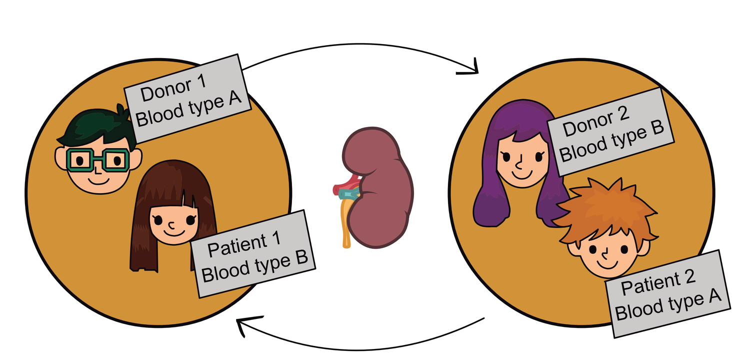


A "bridge donor" is a donor in the last incompatible pair forming a chain in a previous solution.

The reasons why a willing donor's kidney may no be compatible:
* Blood type incompatibility
* Positive serological cross match.

An altruistic NDDs can iniciate a chain. Also, a "bridge donor" can do it.

Suppose we have two PDPs, the first donor's kidney is compatible with the second patient and vice versa (2-way exchange).



2k operating rooms and 2k surgical teams are required for a k-way exchange. Where k is the maximal cycle size.

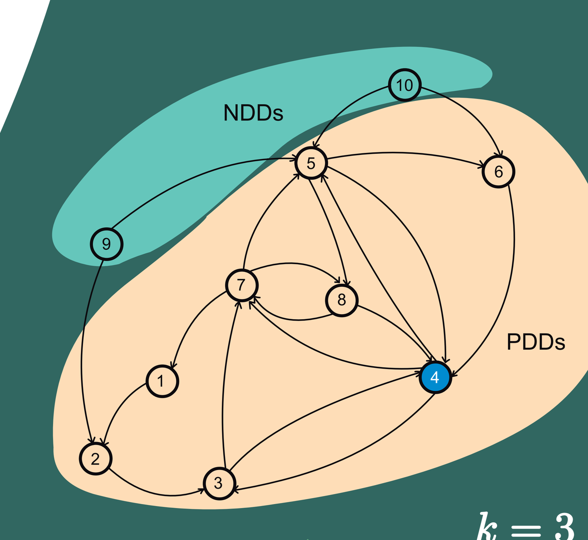
Model for a CCCCP

(Adapted from Mak-Hau (2015))

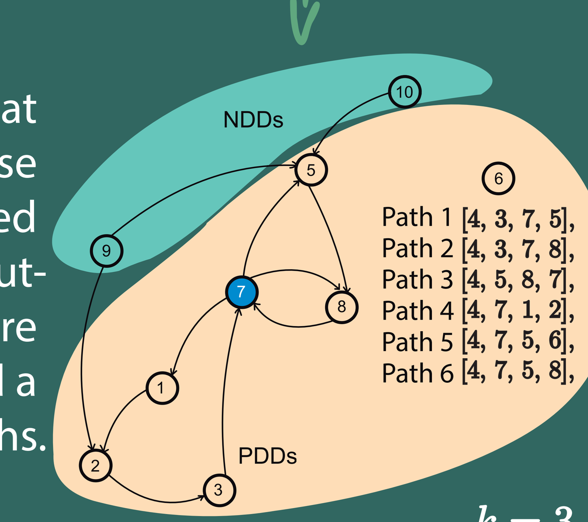
* $E' = E\cup\{(i,j)\mid i\in V, j\in P\cup\{\tau\}\}$, with $E=\{(i,j)\mid i,j\in P\}$.
* u_{ij}, for each $(i,j)\in E$, with $u_{ij}=1$ indicating arc (i,j) is used in a cycle.
* $y_{ij}=1$ indicating the arc $(i,j)\in E'$ forms part of a chain.

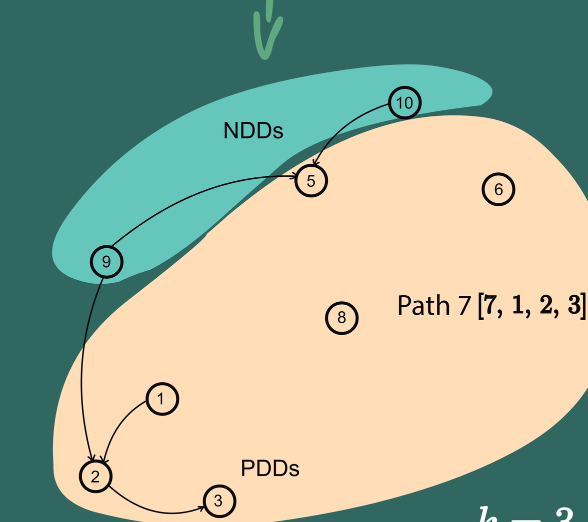
Let,
* t_i for $i\in V$, a continuous variable as "time stamp" for vertex j should it be part of a chain.
* $\pi=(i_1,\ldots,i_{k+1})$ as a minimal infeasible path in a strongly connected component.
* Π be the index set of all π.

Roth et al. (2007) proposed $\pi=(i_1,\ldots,i_{k+1})$ to be a minimal infeasible path. For the example below, we would add 35 constraints. As we propose, we only need 7 constraints, instead.



For finding paths that begin at node i, we choose in every strongly connected component the node with highest out-degree, hoping this path can eliminate more than one infeasible path. Different choices yield a different number of paths.





Objective function: $$max \sum_{(i,j)\in E'} w_{ij}y_{ij} + \sum_{(i,j)\in E} w_{ij}u_{ij}$$

S.t.

$$\sum_{j\in P\cup\tau:(i,j)\in E'} y_{ij} = \sum_{j\in V:(j,i)\in E'} y_{ji} \qquad \forall\, i\in P$$

$$\sum_{j\in P:(i,j)\in E} u_{ij} = \sum_{j\in P:(j,i)\in E} u_{ji} \qquad \forall\, i\in P$$

$$\sum_{j\in P\cup\tau:(i,j)\in E'} y_{ij} \leq 1 \qquad \forall\, i\in N$$

$$\sum_{j\in P\cup\tau:(i,j)\in E'} y_{ij} + \sum_{j\in P:(i,j)\in E} u_{ij} \leq 1 \qquad \forall\, i\in P$$

$$\sum_{(i,j)\in\pi} u_{ij} <= K-1 \qquad \forall\, \pi\in\Pi$$

$$t_i - t_j + |P|y_{ji} + (|P|+2)y_{ij} <= |P|+1 \qquad \forall\, i\in V,\ j\in P\cup\tau$$

$$t_i = 0 \qquad \forall\, i\in N$$

$$t_i >= 0 \qquad \forall\, i\in P\cup\tau$$

$$t_\tau <= |P|+1$$

$$u_{ij}\in\{0,1\} \qquad \forall\,(i,j)\in E$$

$$y_{ij}\in\{0,1\} \qquad \forall\,(i,j)\in E'$$

Results for test instances

NDDs	PDPs	Density	Objective	Running Time (s)	Gap
7	198	12%	14,115	163,907	0
3	202	11%	11,02	0,75	0
6	215	13%	9,01	19,922	0
6	261	13%	14,01	65,469	0
6	263	12%	9,005	19,359	0
5	284	12%	9,205	25,422	0
6	312	13%	20,025	1.200+	25%
6	324	12%	10,105	77,109	0
6	328	12%	8,105	26,437	0
6	330	12%	10,005	175,234	0

*Roth, A. E., Sünmez, T., & Utku Ünver, M. (2007). Efficient kidney exchange: Coincidence of wants in markets with compatibility-based preferences. American Economic Review, 97(3), 828–851.
*Mak-Hau, V. (2015). On the kidney exchange problem: cardinality constrained cycle and chain problems on directed graphs: a survey of integer programming approaches. Journal of Combinatorial Optimization.

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