The Sum of Signless Laplacian Eigenvalues of a Graph

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Introduction

Let $G = (V, E)$ be a finite simple graph on n vertices and $e(G)$ edges. The sequence degree of G is denoted by

$$d(G) = \{d_1(G) \geq d_2(G) \geq \cdots \geq d_n(G)\}.$$

Let A be the adjacency matrix of G and D be the diagonal matrix of the row-sums of A, i.e., the degrees of G. Consider the following matrix representations of a graph:

Laplacian Matrix $L(G)$

$$L(G) = D - A.$$
$$Spec(L(G)) = \{\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)\}.$$

Signless Laplacian Matrix $Q(G)$

$$Q(G) = A + D.$$
$$Spec(Q(G)) = \{q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)\}.$$

Consider $M(G)$ as a matrix representation of a graph G of order n and let k be a natural number such that $1 \leq k \leq n$. A general question related to G and $M(G)$ can be raised: *"How large can the sum of the k largest eigenvalues of $M(G)$ be?"* In this work we study how to solve this problem for $L(G)$ and $Q(G)$.

Grone's Inequality for Laplacian

In 1994, Grone and Merris, [7], proved that $\mu_1(G) \geq d_1(G) + 1$ with equality if and only if there exists a vertex of G with degree $n - 1$.

Based on Schur's Inequality, Grone, in [6], proved a more general bound for the sum of Laplacian eigenvalues related with the sum of the vertex degrees.

$$S_k(G) = \sum_{i=1}^{k} \mu_i(G) \geq 1 + \sum_{i=1}^{k} d_i(G)$$

Grone-Merris Inequality

Grone and Merris, [7], conjectured that for a graph G with n vertices, the following holds:

$$S_k(G) = \sum_{i=1}^{k} \mu_i(G) \leq \sum_{i=1}^{k} \overline{d}_i(G), \text{ for } k = 1, \ldots, n.$$

where $\overline{d}_i(G) = |\{v \in V : d_v \geq i\}|$.

Recently, it was proved by Hua Bai, in [3].

Brouwer's Conjecture for Laplacian

As a variation on the Grone–Merris conjecture, Brouwer, [1], conjectured that for a graph G with n vertices, the following holds:

$$S_k(G) = \sum_{i=1}^{k} \mu_i(G) \leq e(G) + \binom{k+1}{2}, \text{ for } k = 1, \ldots, n.$$

The conjecture is known to be true for:
- ▶ $k = 1,\ 2,\ n - 1$ and n.
- ▶ trees, unicyclic and bicyclic graphs.
- ▶ regular graphs.
- ▶ split graphs (graphs whose vertex set can be partitioned into a clique and an independent set).
- ▶ cographs (graphs with no path on 4 vertices as an induced subgraph).
- ▶ graphs with at most 10 vertices.

Grone's Inequality for signless Laplacian

The signless Laplacian version of Grone's inequality could be stated as:

Grone's Inequality for signless Laplacian

Let G be a simple connected graph on $n \geq 3$ vertices. Then

$$S_k^+ = \sum_{i=1}^{k} q_i(G) \geq 1 + \sum_{i=1}^{k} d_i(G).$$

Motivated by Grone's inequality, we studied the signless Laplacian version of the inequality.
- ▶ The case $k = 1$ has been proved in the literature.
- ▶ de Lima and Oliveira, [5], proved that inequality is true for $k = 2$. Equality holds if and only if G is one of the following graphs: the complete graph K_3 or a star S_n.
- ▶ For $k \geq 3$, de Lima and Oliveira showed a counterexample such that inequality is not true when G is the S_n plus one edge.

Brouwer's Conjecture for signless Laplacian

Motivated by the Brouwer's conjecture, F. Ashraf et al., [2], proposed a signless Laplacian version for this inequality.

Brouwer's Conjecture for signless Laplacian

$$S_k^+ = \sum_{i=1}^{k} q_i(G) \leq e(G) + \binom{k+1}{2}, \text{ for } k = 1, \ldots, n.$$

In [2], the authors proved that conjecture is true for:
- ▶ $k = 1, 2, n - 1$ and n.
- ▶ regular graphs.
- ▶ graphs with at most 10 vertices.

Yang and You, [4], showed that the conjecture is true for:
- ▶ connected graphs with sufficiently large k.
- ▶ unicyclic, bicyclic graphs.
- ▶ tricyclic graphs when $k \neq 3$.

Future Work

- ▶ State a signless Laplacian version of the Grone-Merris inequality.
- ▶ Prove the signless Laplacian version of Brouwer's Conjecture for other classes of graphs. In particular, to split graphs and cographs.
- ▶ Find classes of graphs in wich the signless Laplacian version of Grone's inequality is true.

References

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