The Problem of Covering Solids by Spheres of Different Radii

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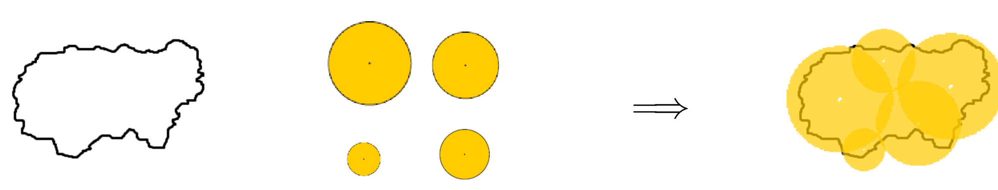
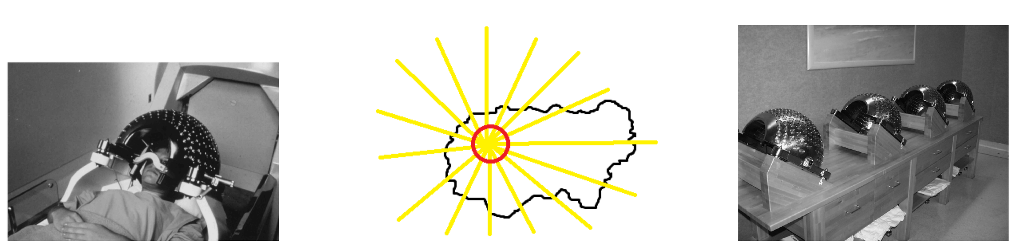
Abstract

We present a mathematical programming model for the problem of covering solids by spheres of different radii. Given a set of spheres, possibly with different diameters, and a solid, the goal is to locate the spheres in such a way their union forms a coverage for this solid, using the smallest possible number of spheres of this set.



This problem has an application in the radiosurgical treatment planning known as Gamma Knife and can be formulated as a nonconvex optimization problem with quadratic constraints and a linear objective function. We also present an approach based on a graph structure, where the maximum weight clique is the optimal solution to an approximation of the original model, aiming to find good solutions in reasonable times.



Modeling

The **Covering Problem** can be defined as follows. Given:

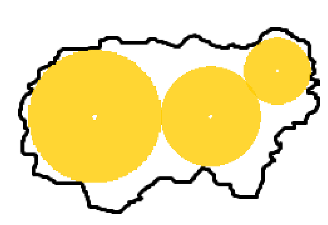
- a compact set $T \subset \mathbb{R}^3$,
- a finite set $R \subset \mathbb{R}_+$ of radii,
- a set N indexing the spheres and
- a function $\rho : N \to R$,

we have to find a set of spheres
$\{ B(x(i), \rho(i)) \mid i \in N \}$
of minimum cardinality and covering all the points in T.

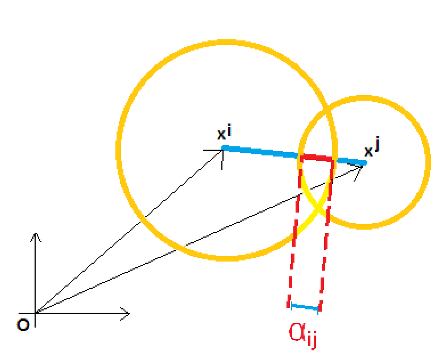
In [1], the authors formulated the problem as a nonlinear nonconvex mixed-integer infinite programming problem.

The model we propose use characteristics from the **Packing Problem** [2], such as

- the goal is to maximize the occupied volume;
- overlappings are not allowed; and
- the spheres must be totally inside the container.



Proposed Model for the Covering Problem:



$$\max \sum_{i=1}^{n} c_i y_i \qquad (1)$$

$$\|x^i - x^j\|^2 \geq (r_i + r_j - \alpha_{ij})^2 (y_i + y_j - 1), \quad \forall\, i<j \qquad (2)$$

$$x^i \in T, \quad \forall i \qquad (3)$$

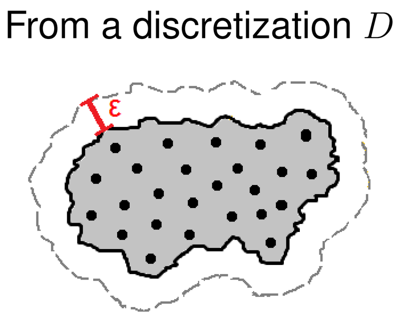
$$y \in \{0,1\}^n \qquad (4)$$

It is a non-convex MINLP (mixed integer nonlinear programming problem), for which there is no standard algorithm for finding the solution.

Parameters Existence Theorem [3]:
There are $\{\alpha_{ij} \geq 0\}_{1 \leq i < j \leq n}$ and $\{c_i \geq 0\}_{1 \leq i \leq n}$ for which an optimal solution of the proposed model is also an optimal solution of the covering problem.

Graph Approach

From a discretization D of T,



we construct a graph $G(V,E)$ defined by

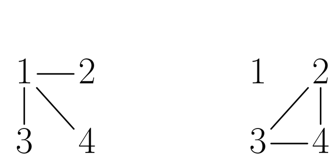
- $V = \{(p,r) \mid p \in D, r \in R\}$
- $E = \left\{\overline{(p,r)(q,s)} \mid \|p-q\| \geq (r+s-\alpha_{rs})\right\}$

In this graph, every clique represents a feasible solution to the proposed model (1)-(4). So we aim to find the Maximum Weight Clique in this graph.

$$\max \sum_{i=1}^{|V|} c_i y_i \qquad (5)$$

$$s.t. \quad y_i + y_j \leq 1, \quad \forall (i,j) \notin E \qquad (6)$$

$$y \in \{0,1\}^{|V|} \qquad (7)$$

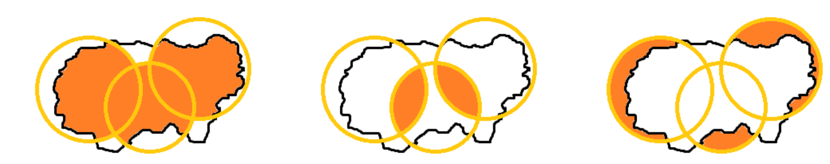


$$\begin{matrix} y_1 + y_2 + y_3 \leq 1 \\ y_1, y_2, y_3 \in \{0,1\} \end{matrix} \Leftrightarrow \begin{cases} y_1 + y_2 \leq 1 \\ y_1 + y_3 \leq 1 \\ y_2 + y_3 \leq 1 \\ y_1, y_2, y_3 \in \{0,1\} \end{cases}$$

The linear problem (5)-(7) was tackled by a Branch-and-Cut algorithm. Only the objective function (5) and the constraints (7) are solved in the root node. As solutions violate constraints (6), called independent sets, some of these constraints (found heuristically) are added to the problem in the form of cuts.

Results

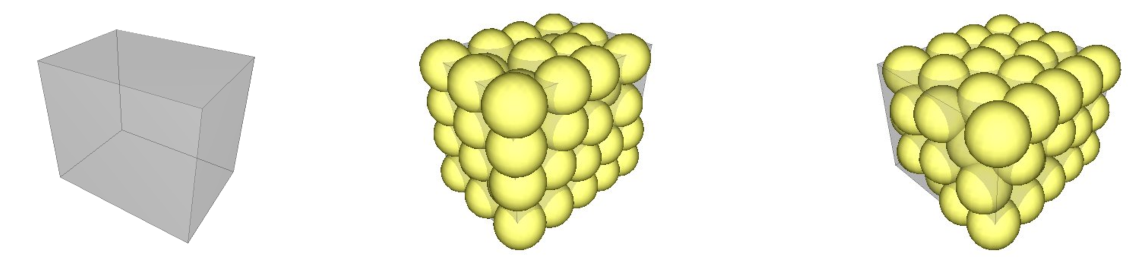
Qualifying the solutions:



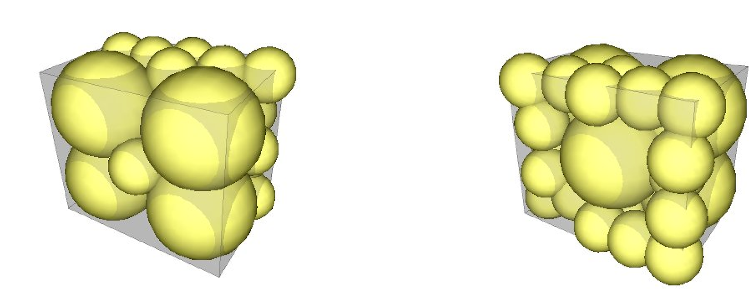
Data used in an example test:

- parallelepiped dimensions (mm): 14 x 12 x 10;
- spheres radii (mm): 2, 4, 7 and 9;
- $\epsilon = 1$, $\Delta = 1$
- $c_i = r_i^3 \quad / \quad c_4 = 20, c_2 = 1$
- $\alpha_{ij} = 0.5 \cdot min\{r_i, r_j\}$
- $t_{max} = 3600s$

	Cliquer	XPRESS	B&C
z^*	224	544	592
$\|S\|$	?	61	74
t	3400*	1957*	747*
cov	?	89.85	92.62
miscov	?	12.82	12.34
overlap	?	19.27	27.41



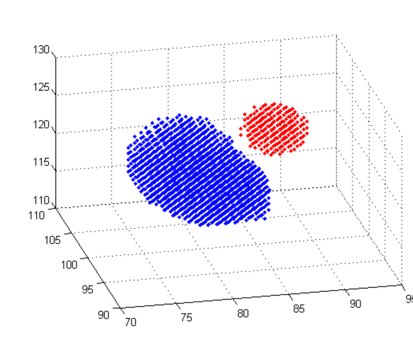
	B&C
z^*	472
$\|S\|$	24
t	6
cov	88.11
miscov	9.18
overlap	10.92



Comparing our results against some results from the literature, where R_x, R_y and R_z are the dimensions of an ellipsoid:

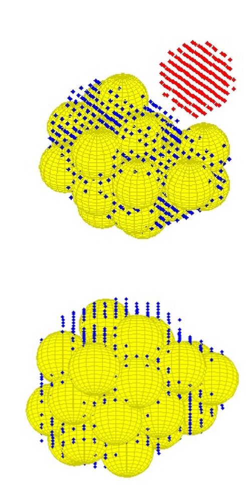
Instance	R_x	R_y	R_z	ε	Δ
bt1	12	8	6	4	5
bt2	12	8	6	4	3
bt3	12	8	6	4	2
bt4	10	10	10	6	3

	Liberti et al. [1]			*Branch and Cut*		
Instance	Time	*cov*	*#S*	Tempo	*cov*	*#S*
bt1	1407	0.874	2	0	0.8912	2
bt2	51168*	0.984	5	0	0.9626	4
bt3	60058*	1	2	1	0.962	8
bt4	58068*	0.907	3	1	0.9812	6

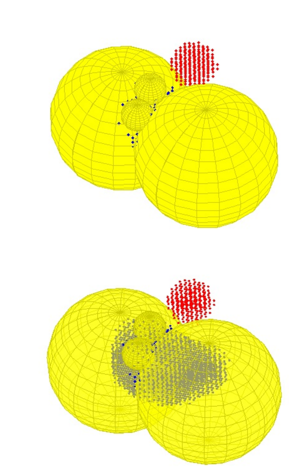


A real test case provided by *Elekta* [4], a human care company pioneering clinical solutions for treating cancer and brain disorders. In blue, the tumor and, in red, an organ at risk.

	B&C
z^*	208
$\|S\|$	26
t	7*
cov	82
miscov	?
overlap	13.8
ε	1



	B&C
z^*	1474
$\|S\|$	4
t	183
cov	99
miscov	?
overlap	20.25
ε	-



Conclusions

A model for the problem of covering solids by spheres of different radii is presented as a mixed-integer nonlinear programming problem, whose continuous relaxation is nonconvex. In the state-of-the-art, there is no standard algorithm for finding the solution to this class of problems. We presented another approach, using Graph Theory, consisting in finding the maximum weight clique in a conveniently constructed graph. This is a linear programming problem we solved using the Branch and Cut algorithm, capable of finding good solutions in a very reasonable time.

References

[1] L. Liberti, N. Maculan & Y. Zhang. “Optimal configuration of gamma ray machine radiosurgery units: the sphere covering subproblem”. *Optimization Letters*, vol. 3, pp. 109-121, 2009.

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[4] Elekta Instrument AB Stockholm. Available: *https://www.elekta.com/* [Accessed 6 July 2016].

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