Optimal Pebbling Number

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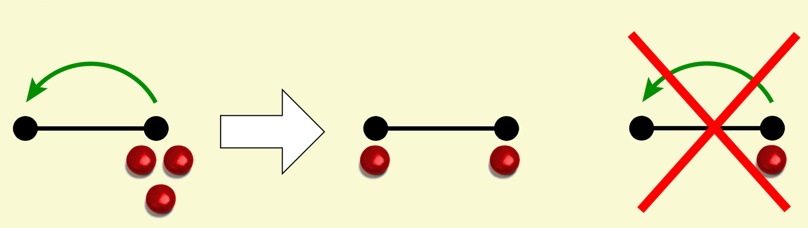
Introduction

Graph pebbling is a game on graphs, which is first used by Chung in 1989 to solve a problem in number theory[1]. It became an actively researched area. The field has huge literature, for a comprehensive list of results see the survey paper of Hurlbert[2].
In this poster we focus on the optimal pebbling problem, which is an optimization version of graph pebbling. The interesting quantity in optimal pebbling is the optimal pebbling number, denoted by π_{opt}. The determination of whether $\pi_{\mathrm{opt}}(G) \leq k$, like interesting combinatorial problems, is NP-complete [3]. It is not known that the same question can be decided in polynomial time for trees or it is also NP-complete.
We give a short introduction to pebbling by definitions and some previously known results. After that, we present our result, which is a linear time method for calculating the optimal pebbling number of spider graphs.

Definitions

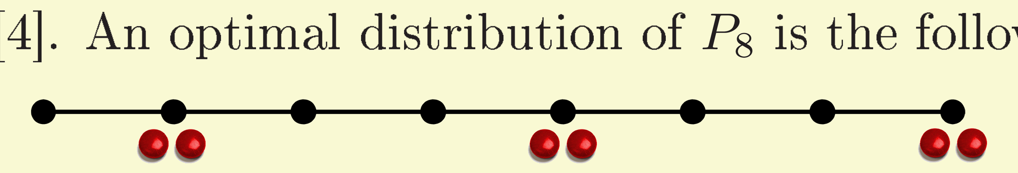
- A **Pebbling distribution** P on graph G is a $V(G) \to \mathbb{N}$ function. We consider $P(v)$ as a number of pebbles placed at vertex v.
- A **pebbling move** removes two pebbles from a vertex, throw away a pebble and put the remaining one at an adjacent vertex. Negative number of pebbles at a vertex is not allowed!



- A vertex v of G is **reachable** under P if there is a sequence of pebbling moves S, such that v has a pebble after we apply the pebbling moves contained in S.
- P is a **solvable distribution** of G if each vertex of G is reachable under P.
- The **optimal pebbling number** of G is the smallest number $\pi_{\mathrm{opt}}(G)$, such that there is a solvable distribution with $\pi_{\mathrm{opt}}(G)$ pebbles.
- If a pebbling distribution of G contains $\pi_{\mathrm{opt}}(G)$ pebbles, then we call it as **optimal distribution**.

Known Optimal Pebbling Numbers

- The optimal pebbling number of the n vertex path is $\lceil \frac{2}{3}n \rceil$[4]. An optimal distribution of P_8 is the following:



- $\pi_{\mathrm{opt}}(C_n) = \pi_{\mathrm{opt}}(P_n)$ [4].
- The optimal pebbling number of the complete m-ary tree with height h is 2^h if $m \geq 3$. It can be determined in polynomial time when $m = 2$[5].
- There is a linear time algorithm which calculates the optimal pebbling number of an arbitrary caterpillar.[6].

Open Questions

Is there a polynomial time algorithm which determines the optimal pebbling number of an arbitrary tree or is this problem NP-hard?
Is there an other subclass of trees where the optimal pebbling problem is easy?

Acknowledgements

This research was partially supported by National Research, Development and Innovation Office NKFIH, grant K108947.

Spider Graphs

A spider is a tree whose all but one vertices has degree at most two. The vertex with highest degree, which is at least three, is called as the body. The removal of the body breaks the spider to several components, each of them is a path. We call these paths as the legs of the spider. Let L be the multiset containing the order of the legs. We denote a spider graph with S_L.

Constructing three Solvable Distributions of S_L

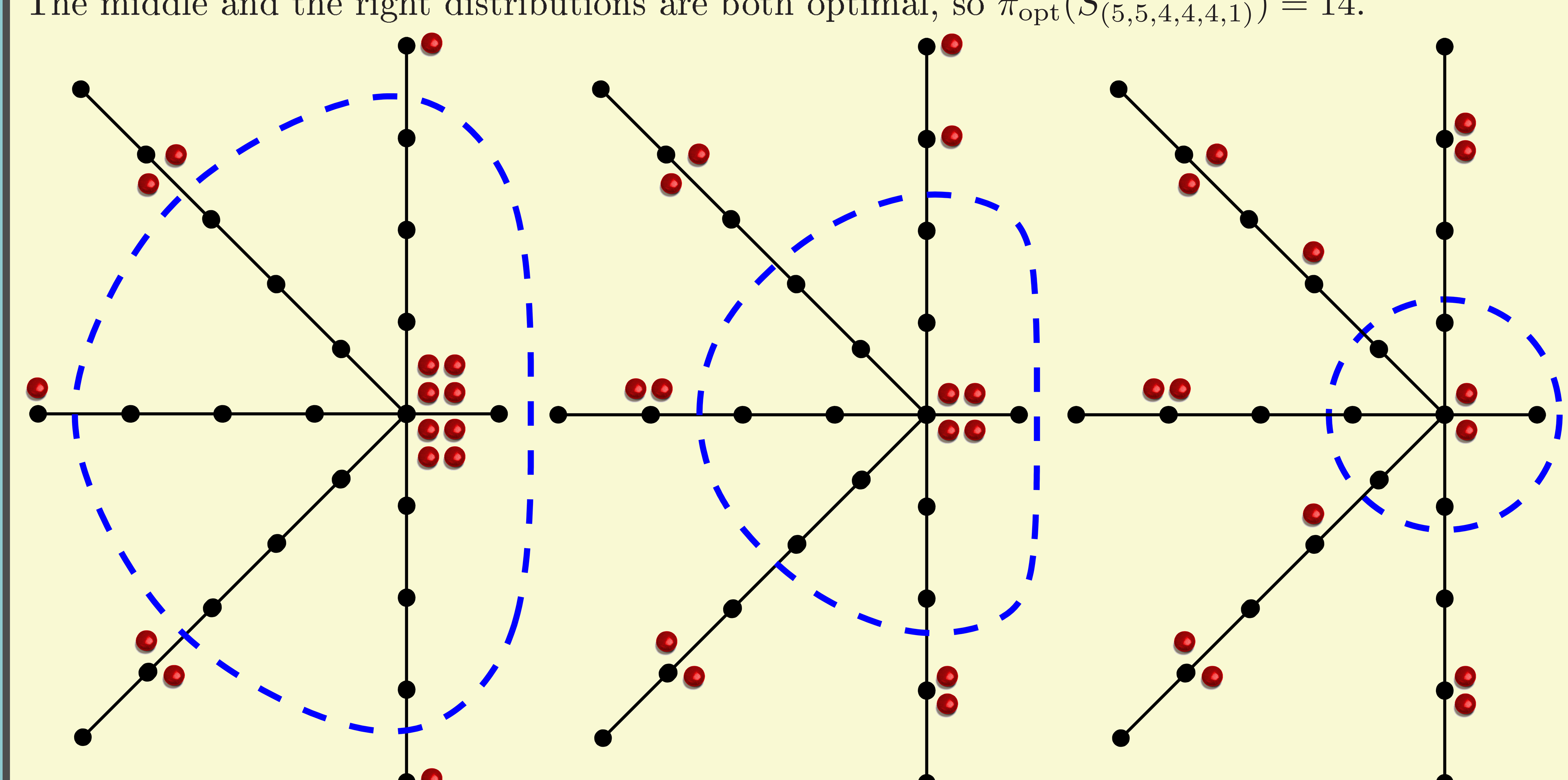
We place 2^k pebbles at the body, where k is an integer. This pile of pebbles guarantees that the distance-k closed neighborhood of the body is reachable.
If the body has 2^k pebbles and we double it, then the distance-$k+1$ open neighborhood becomes reachable. The size of this set is $M_L(k+1)$, where M_L is the multiplicity function of multiset L. We double the number of pebbles on the body, until the number of additional pebbles placed is less than $M_L(k+1)$.Using this we place $2^{B(L)}$ pebbles at the body, where $B(L)$ is defined below:

$$B(L) = \min\{k \in \mathbb{N} | 2^k \geq M_L(k+1)\}$$

The not reachable vertices, which are not contained in the distance-k closed neighborhood of the body, induce paths. We handle these paths one by one. We place pebbles at the vertices of such a path according to its optimal distribution.
This method creates a solvable distribution P. We create two other distributions P', P'' in the same way, but now we place $2^{B(L)-1}$ and $2^{B(L)-2)}$ pebbles at the body, respectively. Then we handle the remaining paths like in the construction of P.

Example

The construction gives three different solvable pebbling distributions of $S_{(5,5,4,4,4,1)}$. $B((5,5,4,4,4,1)) = 3$, hence the body contains 8, 4, or 2 pebbles. The vertices contained in the blue "circle" are reachable using only the pebbles placed at the body.
The middle and the right distributions are both optimal, so $\pi_{\mathrm{opt}}(S_{(5,5,4,4,4,1)}) = 14$.



15 pebbles 14 pebbles 14 pebbles

Theorem

The distribution which contains the least number of pebbles among the three constructed ones is an optimal distribution of S_L. Hence the following formula holds:

$$\pi_{\mathrm{opt}}(S_L) = \min_{j \in \{0,1,2\}} \left(2^{B(L)-j} + \sum_{k=B(L)-j+1}^{\infty} M(k) \left\lceil \frac{2}{3}(k - B(L) + j) \right\rceil \right)$$

References

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