“K-theoretic” analog of Postnikov-Shapiro algebra distinguishes graphs

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Postnikov-Shapiro algebra counting subforests

1st Let Φ_G be the algebra over $\mathbb{K}$ generated by $\{\phi_e : \ e \in G\}$ with relations $\phi_e^2 = 0$, for any $e \in G$. Take any linear order of vertices of G. For $i = 0, \dots, n$, set

$$X_i = \sum_{e \in G} c_{i,e}\phi_e,$$

where $c_{i,e} = \pm 1$ for vertices incident to e (for the smaller vertex, $c_{i,e} = 1$, for the bigger vertex, is $c_{i,e} = -1$) and 0 otherwise. Denote by $\mathcal{C}_G$ the subalgebra of Φ_G generated by $X_1, \dots, X_n$.
2nd Let J_G be the ideal in $\mathcal{K}[x_1, \dots, x_n]$ generated by the polynomials

$$p_I = \left(\sum_{i \in I} x_i\right)^{D_I+1},$$

where I ranges over all nonempty subsets of vertices, and D_I is the total number of edges between I and its complementary. Define the algebra B_G as the quotient $\mathbb{K}[x_1, \dots, x_n]/J_G$.

Theorem(Postnikov, Shapiro) *The algebras B_G and C_G are isomorphic. Their total dimension as vector spaces over $\mathbb{K}$ is equal to the number of subforests in G.*

The Hilbert series of algebra C_G is given by

$$HS_{C_G}(t) = T_G\left(1 + t, \frac{1}{t}\right) \cdot t^{|G| - v(G) + c(G)},$$

where $c(G)$ is number of connected component of G and T_G is Tutte polynomial of G.

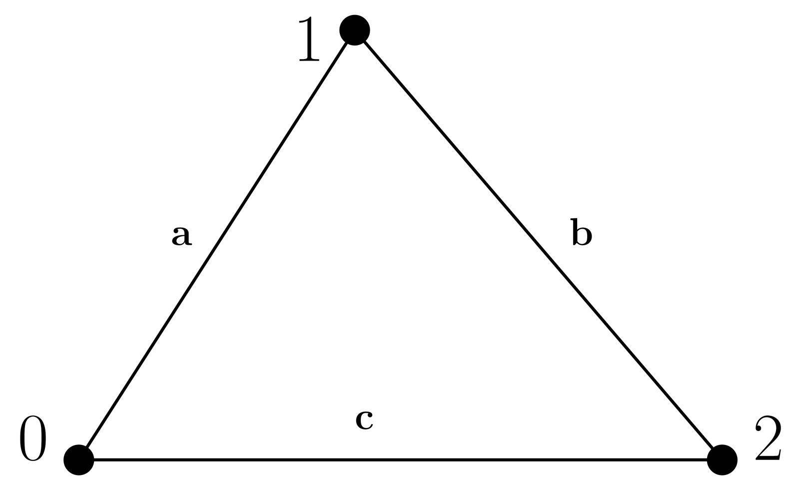
Theorem(N.) *Given two graphs G_1 and G_2, algebras C_{G_1} and C_{G_2} are isomorphic if and only if their graphical matroids are isomorphic. (The algebraic isomorphism can be thought of either as graded or as non-graded.)*

Comments

Here all graphs without loops, but might have multiple edges.

We fix a field $\mathbb{K}$ of zero characteristic.

Example



Φ_Δ generated by ϕ_a, ϕ_b and ϕ_c with relations

$$\phi_a^2 = \phi_b^2 = \phi_c^2 = 0.$$

Generators of C_G are

$$X_0 = \phi_a + \phi_c,$$
$$X_1 = -\phi_a + \phi_b,$$
$$X_2 = -\phi_b - \phi_c.$$

Algebra C_G has a natural graded structure and its graded components are

- $\mathbb{K} = \text{span}\{1\}$, $dim = 1$;
- $\text{span}\{\phi_a - \phi_b,\ \phi_a + \phi_c\}$, $dim = 2$;
- $\text{span}\{\phi_a\phi_b,\ \phi_b\phi_c,\ \phi_a\phi_c\}$, $dim = 3$;
- $\text{span}\{\phi_a\phi_b\phi_c\}$, $dim = 1$.

Generators of $\mathcal{K}_\Delta$ are

$$Y_0 - 1 = e^{X_0} - 1 = \phi_a + \phi_c + \phi_a\phi_c;$$
$$Y_1 - 1 = e^{X_1} - 1 = -\phi_a + \phi_b - \phi_a\phi_b;$$
$$Y_2 - 1 = e^{X_2} - 1 = -\phi_b - \phi_c + \phi_b\phi_c.$$

The filtered structure is

- $F_0 = \text{span}\{1\}$, $dim = 1$;
- $F_1 = \text{span}\{1,\ Y_0,\ Y_1,\ Y_2\}$, $dim(F_1) - dim(F_0) = 3$;
- $F_2 = \text{span}\{1,\ Y_0,\ Y_1,\ Y_2,\ Y_0^2,\ Y_1^2,\ Y_0Y_1, \dots\}$, $dim(F_2) - dim(F_1) = 3$.

"K-theoretic" filtration

1st Let $\mathcal{K}_G \subset \Phi_G$ be subalgebra defined by the generators:

$$Y_i = \exp(X_i) = \prod_{e \in G}(1 + c_{i,e}\phi_e),\ \ i = 0, \dots, n.$$

2nd Define J_G in $\mathcal{K}[y_0, y_1, \dots, y_n]$ as the ideal generated by the polynomials

$$q_I = \left(\prod_{i \in I} y_i - 1\right)^{D_I+1},$$

where I ranges over all nonempty subsets in $\{0, 1, \dots, n\}$.
Set $D_G := \mathcal{K}[y_0, \dots, y_n]/J_G$.

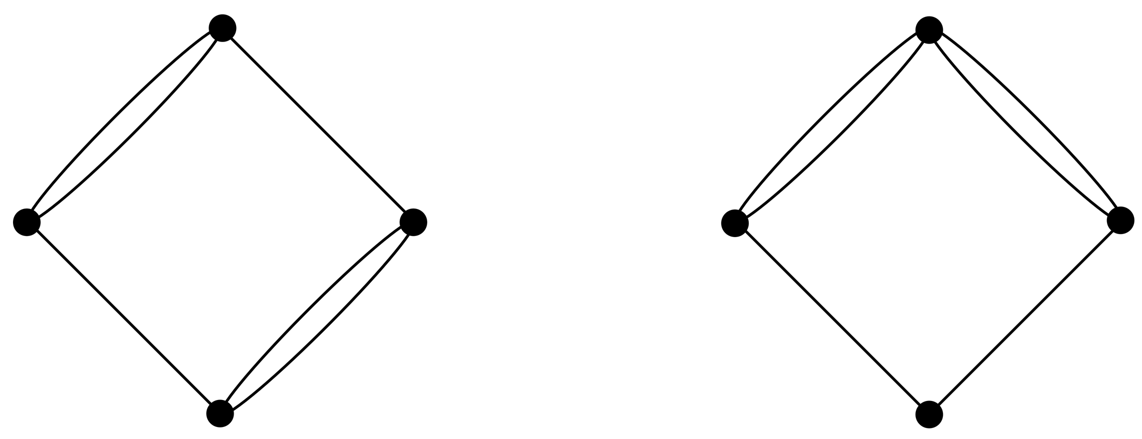
Theorem(N., Shapiro) *For any graph G, algebras B_G, C_G, D_G and $\mathcal{K}_G$ are isomorphic as non-filtered algebras. Their total dimension is equal to the number of subforests in G.*

Furthermore, for any graph G, algebras D_G and $\mathcal{K}_G$ are isomorphic as filtered algebras.

Question *What is sense of Hilbert series $HS_{\mathcal{K}_G}$?*

Theorem(N., Shapiro) *Given two graphs G_1 and G_2 without isolated vertices, $\mathcal{K}_{G_1}$ and $\mathcal{K}_{G_2}$ are isomorphic as filtered algebras if and only if G_1 and G_2 are isomorphic.*

Example



G_1 G_2

Graded algebras C_{G_1} and C_{G_2} are **isomorphic**.

Filtered algebras $\mathcal{K}_{G_1}$ and $\mathcal{K}_{G_2}$ are **not isomorphic**.

Algebras counting spanning trees

Let Φ_G^T be the algebra generated by $\{\phi_e : \ e \in G\}$ with relations $\phi_e^2 = 0$, for any $e \in G$, and $\prod_{e \in C}\phi_e = 0$, for any cut C.

Let $C_G^T \subset \Phi_G^T$ be subalgebra generated by $X_i = \sum_{e \in G} c_{i,e}\phi_e$, for $i = 0, \dots, n$.

Let $\mathcal{K}_G^T \subset \Phi_G^T$ be filtered subalgebra generated by $Y_i = e^{X_i} = \prod_{e \in G}(1 + c_{i,e}\phi_e)$, for $i = 0, \dots, n$.

Definitions of B_G^T and D_G^T are similar to definitions of B_G and D_G, we need to write power D_I instead $D_I + 1$.

Theorem *For any graph G, algebras B_G^T, C_G^T, D_G^T and $\mathcal{K}_G^T$ are isomorphic as non-filtered algebras. Their total dimension is equal to the number of spanning trees of G.*
Furthermore, algebras B_G^T, C_G^T are isomorphic as graded algebras and algebras D_G^T, $\mathcal{K}_G^T$ are isomorphic as filtered algebras.

Conjecture *Given two graphs G_1 and G_2, algebras C_{G_1} and C_{G_2} are isomorphic if and only if their bridge-matroids are isomorphic. (bridge-free matroid of graph G is graphical matroid of graph G', which is obtained from G by removal all its bridges)*

Question *When are $\mathcal{K}_{G_1}$ and $\mathcal{K}_{G_2}$ isomorphic?*