C_5-Free k-Colorings of Complete Graphs

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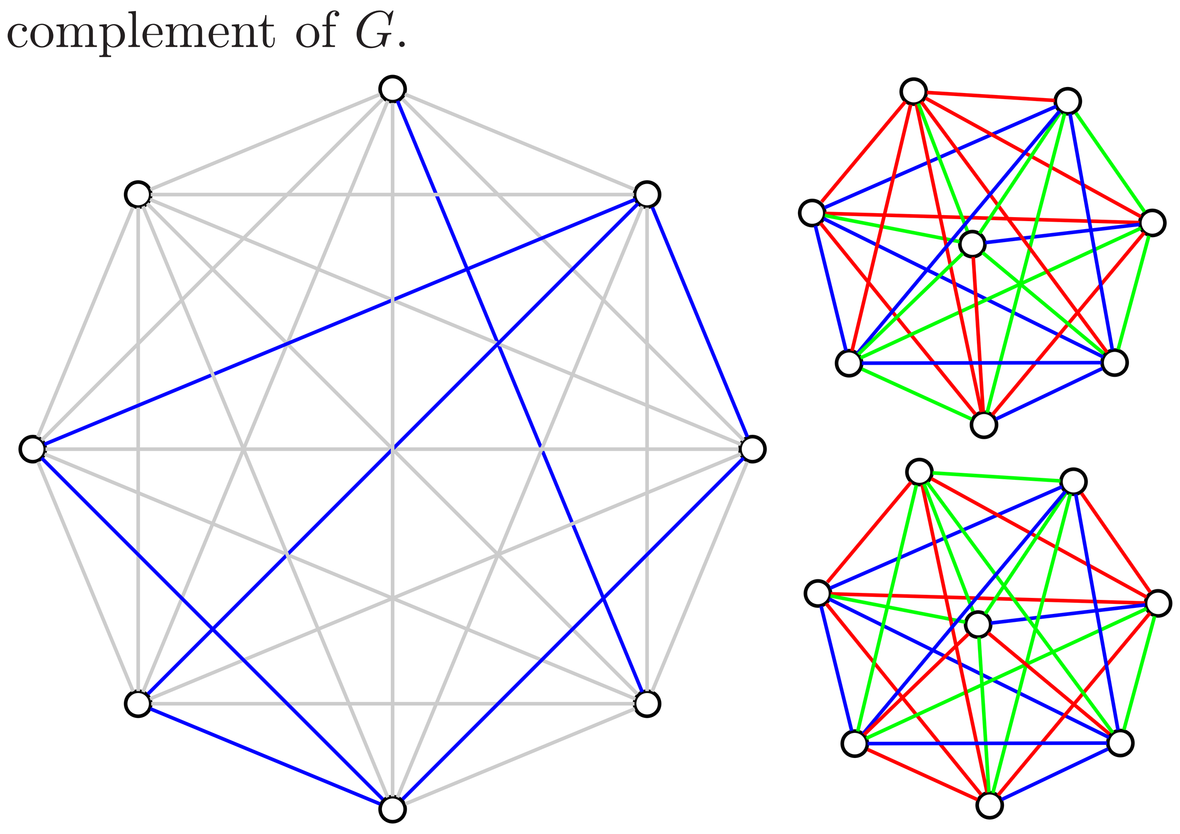
Introduction

- C_n is the cycle on n vertices.
- The notation $R_k(G) = n$ indicates n is the smallest order of a complete graph for which no k-coloring of its edges can avoid a graph G in any of its colors.
- We are interested in finding all C_5-free 3-colorings of complete graphs up to order 17, to learn about $R_4(C_5)$.
- For a thorough study of the broader field of Ramsey theory see [Gra90].

Coloring Approaches

Direct Coloring Using SAT

Use a C_5-free graph G as the fixed first color, then use SAT solvers to find C_5-free 2-colorings for the complement of G.



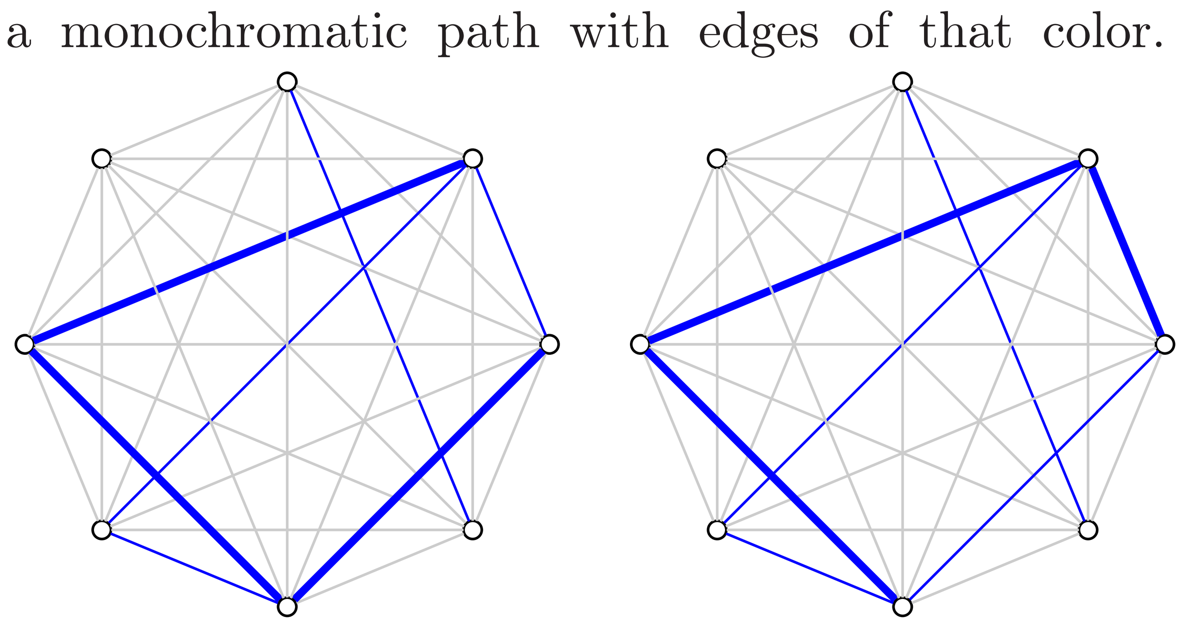
Formulas are of the form

$$\bigwedge_{e_1,e_2,e_3,e_4,e_5} \left(\bigvee x_{e_i}\right) \wedge \left(\bigvee \neg x_{e_i}\right)$$

for each set e_1, e_2, e_3, e_4, e_5 of edges that form a cycle in the complement G.

One-vertex Extension

Identify monochromatic paths of length 4, then add a new vertex and connect it to the original coloring without connecting two ends of a monochromatic path with edges of that color.



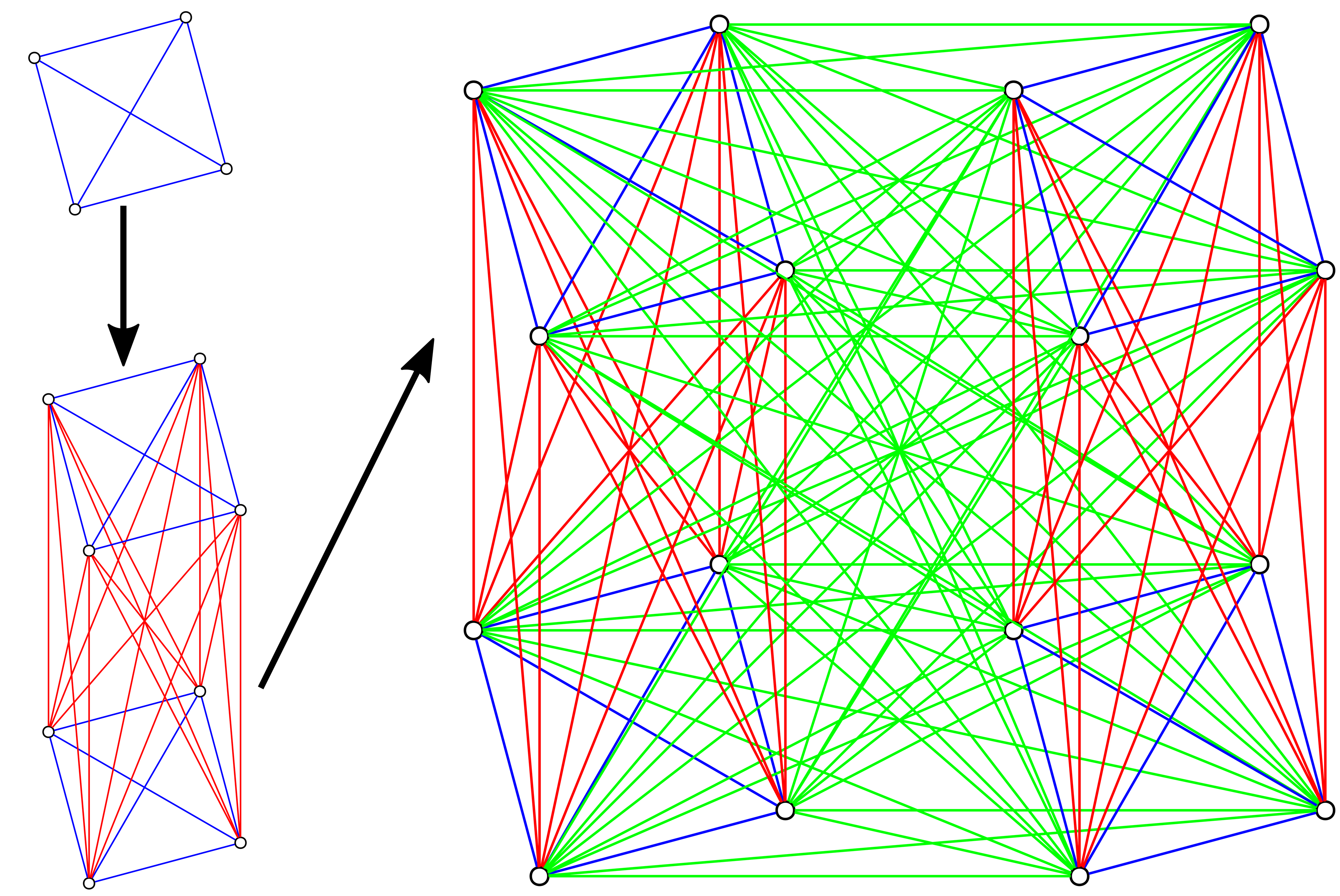
Future Work

- Use this dataset to infer densities of C_5-free 3-colorings in other families of colorings (see [KLS+13]).
- Find a succinct description of the extremal colorings, maybe through graph clustering by similarity.
- Improve bounds on $R_k(C_{2m+1})$. We believe the number of triangle-free colorings is an anomaly among cycles of odd length, so C_{2k+1} may behave more like C_5 in general.

Theorems about Odd Cycles

Here is a brief summary of what we know about Ramsey numbers for cycles. A more detailed compendium of these theorems can be found in [Rad11, Rad94].

- $2^{k+1} < R_k(C_{2n+1})$ for all n, because the following "blow-up" construction admits no C_{2n+1}:



- It is conjectured that the bound by the "blow-up" construction is tight.
- $R_3(C_n) = 4n - 3$ for all sufficiently large odd n [KSS05].

Theorems About C_5

- $R_2(C_5) = 9$.
- $R_3(C_5) = 17$ [YR92].
- $R_k(C_5) - 1 < 1 + \sqrt{18k}\,(R_{k-1}(C_5) - 1)$ [Li09] or, recursively, $R_k(C_5) < \sqrt{18^k k!}/10$.
- The above implies $33 \le R_4(C_5) \le 137$.

Theorems About Even Cycles

- Another construction by Dzido, Nowik and Szuca [DNS05] shows that $R_3(C_{2m}) \ge 4m$. In general, it shows that

$$R_k(C_{2m}) \ge \begin{cases} (k+1)\,m & \text{for } k \text{ odd} \\ (k+1)\,m - 1 & \text{for } k \text{ even} \end{cases}$$

- $R_k(C_{2m}) \le 201km$ for $k \le \frac{10^m}{201m}$ [EG73].

C_3, C_5

$R_3(C_3) = R_3(C_5) = 17$, but the behavior of the number of colorings for each case is very different:

Order	C_3-free Colorings	C_5-free Colorings
6	330	2 349
7	3 829	54 927
8	50 391	679 876
9	500 023	3 713 104
10	2 646 593	14 092 138
11	4 821 244	43 945 253
12	1 929 792	140 033 320
13	78 892	448 105 921
14	115	1 142 773 713
15	2	1 844 045 362
16	2	1 701 746 176

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