Solvable cases for the Perfect Edge Domination Problem

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Introduction

A vertex (edge) *dominates* itself and any other vertex (edge) adjacent to it. A subset $X \subset V(G)$ is a *dominating set* if every vertex of G is dominated by some vertex of X. A subset $E' \subseteq E(G)$ is a *perfect edge dominating set* (PEDS) of G, if every edge of $E(G) \setminus E'$ is dominated by exactly one edge of E'. If every edge of $E(G)$ is dominated exactly once by E' then E' is an *efficient edge dominating set* (EEDS), also called a *dominating induced matching*.

The *cardinality perfect (efficient) edge domination problem* is to determine the perfect (efficient) edge dominating set of minimum cardinality. The corresponding *weighted* problems are defined replacing minimum cardinality by minimum sum of weights of the dominating edges.

Both problems are known to be NP-complete [2] and [4], respectively. We present a dichotomy theorem for the complexity of the perfect edge domination problem for some graph classes and a linear time algorithm for P_5-free graph. This results belong to a manuscript submitted for publication.

Dichotomy Theorem

Theorem 1. *The cardinality perfect edge domination problem is NP-hard, even if restricted to graphs with bounded degree r and girth at least k, for fixed $r, k \geq 3$.*

Theorem 2. *Let H be a graph, and $\mathscr{G}$ the class of H-free graphs of degree at most d, for some fixed $d \geq 3$. Then the perfect edge domination problem is*

- *polynomial time solvable for graphs in $\mathscr{G}$ if H is a linear forest*
- *NP-complete otherwise.*

Propierties for P_k-graphs

Since P_3-free graphs are the complete graphs and P_4-free graphs are cographs. We use the following result of Bacsó and Tuza's to approach the PED problem for P_5-graphs in a robust way.

Theorem 3. *[1] Every connected graph contains an induced P_5, a dominating K_p, or a dominating P_3.*

Coloring

Let $P \subseteq E(G)$ be a PED set, then P defines a 3-coloring of the vertices as below:

- *black vertices* B have at least two incident edges of P.
- *yellow vertices* Y are incident to exactly one edge of P.
- *white vertices* W are not incident to any edge of P.

We say P is a *proper* PED set if its coloring satisfies $B, Y, W \neq \emptyset$. The *trivial* perfect edge domination is $P = E(G)$.

An efficient edge dominating set is a PED set with $B = \emptyset$.

Proposition 1. *Given $W \subseteq V(G)$, there is a linear-time algorithm to verify if there exists some PED set associated to the coloring (B, Y, W).*

Vertex dominating sets

The following results relate PED sets and the existence of vertex dominating complete subgraphs of certain sizes.

- Given a dominating set D, a PED set P and its coloring (B, Y, W), if $D \subseteq B$ then P is trivial.
- If there is a dominating K_p with $p \geq 4$ then P is trivial.
- If there is a dominating K_1 or K_3 then G has no proper PED sets.
- Given a connected P_5-free graph G, if G admits some non-trivial PED set then G has dominating induced P_3 or a dominating K_3.

Robust subroutines for P_5-free graphs

Our algorithm use a robust method to determine the situation of Theorem 3. And it represents an algorithmic proof of Bacsó and Tuza's theorem.

Definition 1. *The eccentricity of v is the maximum distance between v and any other vertex. A vertex having eccentricity at most 2 is named principal vertex.*

We need a robust method to compute a principal vertex of G, if it exists. To achieve that, we use a linear-time routine, *Test(v)*, to identify the following situations:

(i) If v has infinite eccentricity, then G is not connected.

(ii) If v has eccentricity at least 4, then G has an induced P_5.

(iii) If v has eccentricity 3, then G has an induced P_4 starting in v,

(iv) If v has eccentricity at most 2, then v is a principal vertex of G.

Theorem 4. *There is a linear-time algorithm to find a principal vertex of G or an induced P_5.*

Once a principal vertex v is obtained, we use $N(v)$ for finding a dominating K_p, a dominating P_3, or an induced P_5.

Theorem 5. *For any connected graph G, there is a linear-time algorithm to find a dominating induced P_3, a dominating K_p, or to detect that G is not a P_5-free graph.*

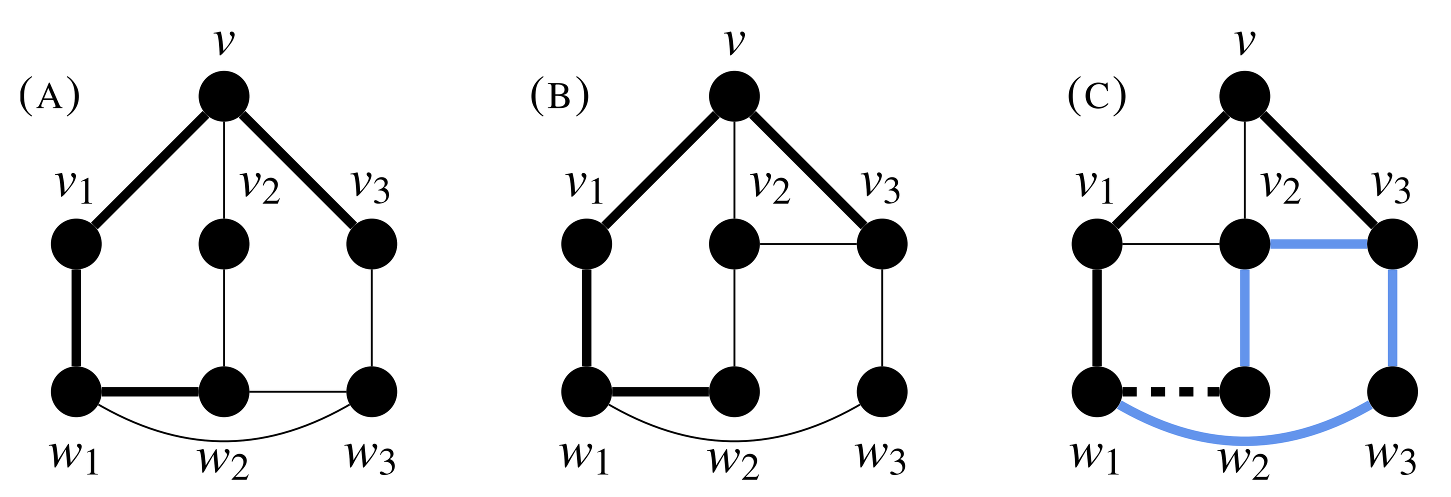

Figure 1: *Principal vertex v, its sons and proper grandsons.*

Finally, we get a vertex dominating set of G and if it is necessary we use the following result to solve efficient edge domination problem.

Theorem 6. *[3] Given a vertex dominating set of fixed size of G, there is a linear-time algorithm to solve the minimum weight efficient edge domination problem.*

Algorithm

Let G be a connected graph. The proposed algorithm, along the process, constructs a set $\mathscr{E}$ containing candidates for the minimum weight perfect edge dominating set. At the end, it selects the minimum of them or exhibits an induced P_5.

1. Define $\mathscr{E} := \{E(G)\}$
2. Find a principal vertex of G using the algorithm of Theorem 4. If no such vertex exists then return an induced P_5 and stop.
3. Using the principal vertex v and the algorithm of Theorem 5, find (i) an induced P_5, or (ii) a dominating K_p, or (iii) a dominating P_3.
4. Case (i): An induced P_5 is found. Return it and stop.
5. Case (ii): A dominating K_p is found. If $p \geq 4$ then return $E(G)$ and stop. Otherwise, $p \leq 3$ and using such dominating K_p, find a minimum weight EED set using the algorithm of Theorem 6 and, if it exists, include it in $\mathscr{E}$. If $p = 2$ then transform the dominating K_2 into a dominating P_3 by adding a third vertex to it.
6. Case (iii): A dominating P_3 is found. First, find a minimum weight EED set, and include it in $\mathscr{E}$, if it exists. Then look for a proper PED set, by considering every possible coloring of the P_3, according to Figure 2. In each case, the algorithm determines at most two proper PED sets and includes them in $\mathscr{E}$.
7. Select the minimum weight PED set of $\mathscr{E}$, return it and stop.

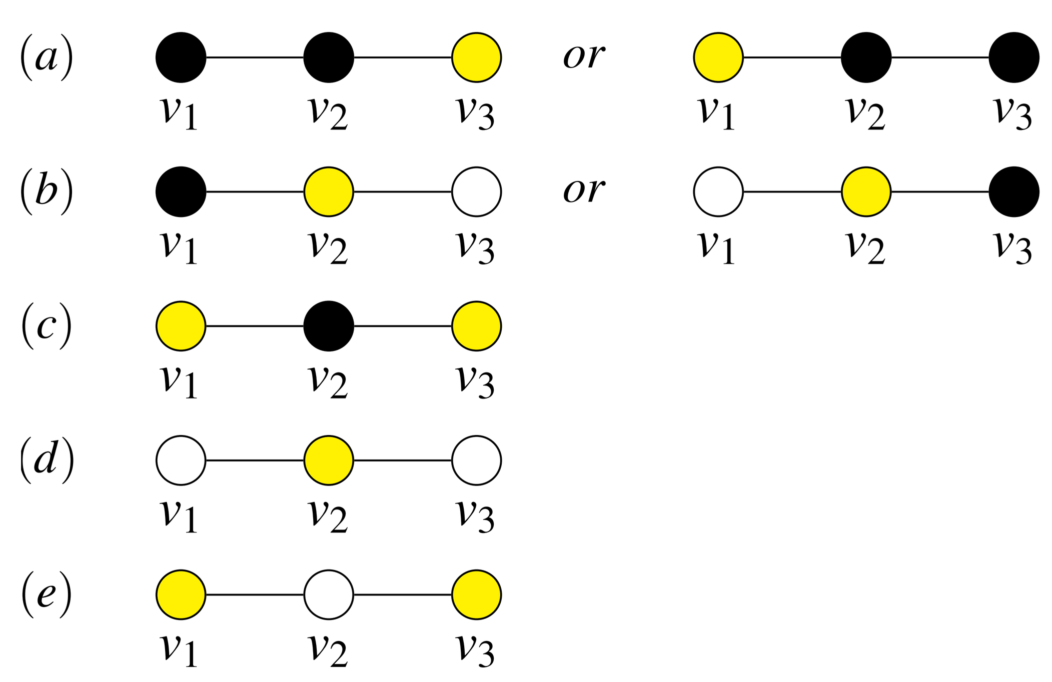

Figure 2: *Possible valid colorings of a dominating induced P_3*

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