Pfaffian Graphs

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Abstract

Perfect matchings of k-Pfaffian graphs may be enumerated in polynomial time on the number of vertices, for fixed k. In general, this enumeration problem is #P-complete. A 1-Pfaffian graph is called a Pfaffian graph.

Introduction

Given directed graph D with $V(D) = \{1, 2, \ldots, n\}$, and a perfect matching $M = \{v_1v_2, v_3v_4, \ldots, v_{n-1}v_n\}$ of D, the *sign* of M on D is the sign of permutation

$$\pi_D(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & n-1 & n \\ v_1 & v_2 & v_3 & v_4 & \ldots & v_{n-1} & v_n \end{pmatrix},$$

Let $\mathcal{M}(D)$ be the set of perfect matchings of D. The *Pfaffian* of D is

$$Pf(D) = \sum_{M \in \mathcal{M}(D)} \operatorname{sgn}(\pi_D(M)).$$

It can be computed in polynomial time (square root of the determinant of an anti-symmetric matrix $A(D)$).

Pfaffian Graphs

If

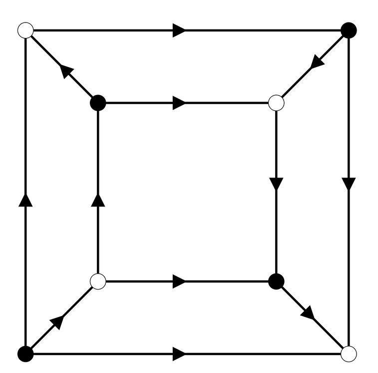
$$\operatorname{sgn}(\pi_D(M)) \text{ is constant for all } M \in \mathcal{M}(G),$$

then $|Pf(D)|$ counts the perfect matchings of D, and D is called a *Pfaffian orientation*. The underlying non-oriented graph G of D is called a *Pfaffian graph* in this case.

There is a polynomial time algorithm to determine a *canonical* orientation D of a graph G, such that D is Pfaffian if and only if G is Pfaffian (Vazirani and Yannakakis[14], Carvalho, Lucchesi and Murty[1]).

Characterizations

Theorem 1 (1963 Kasteleyn [5]). *Every planar graph is Pfaffian. ($K_{3,3}$ is not Pfaffian).*

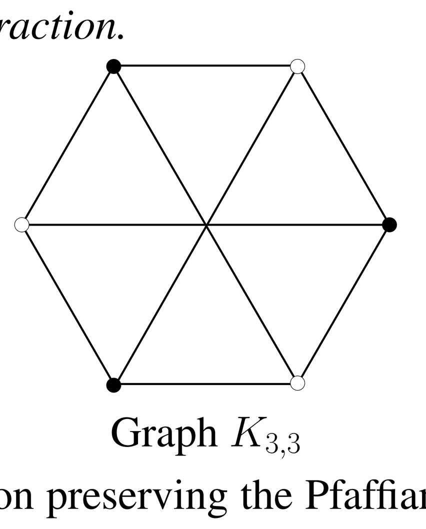


A Pfaffian orientation of the cube graph

Operations over graphs preserving the Pfaffian property may be used to characterize Pfaffian graphs. Given a set of such operations a graph is *minimally-non-Pfaffian* under those operations if it is non-Pfaffian and any application of such operations yields a Pfaffian graph.

A subgraph H is *conformal* in G if $G - V(H)$ has a perfect matching. *Bicontraction* is the operation of contracting both edges incident with a degree two vertex. The study of Pfaffian graphs restricts to graphs where each edge is in some perfect matching, the *matching covered* graphs.

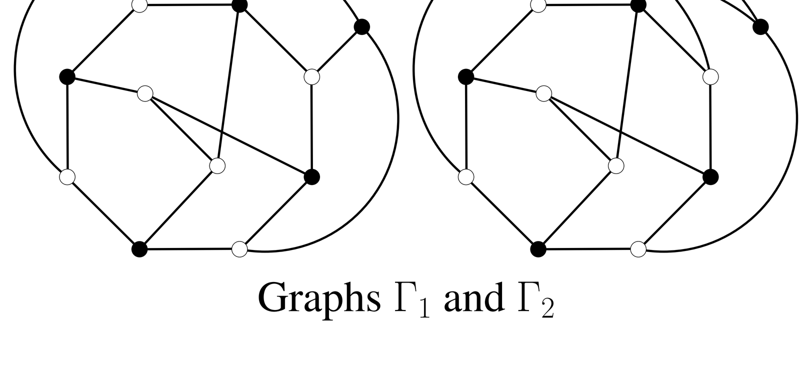
Theorem 2 (1975 Little [6]). *Graph $K_{3,3}$ is the only bipartite minimally-non-Pfaffian graph under conformal subgraph and bicontraction.*



Graph $K_{3,3}$

Another operation preserving the Pfaffian property is *odd cycle contraction*. A graph G is *near-bipartite* if it is non-bipartite, matching covered and $G - e - f$ is bipartite and matching covered.

Theorem 3 (2001 Fischer and Little [3]). *Graphs $K_{3,3}$, Γ_1 and Γ_2 are the only near-bipartite minimally non-Pfaffian graphs under conformal subgraph, bicontraction and odd cycle contraction.*



Graphs Γ_1 and Γ_2

A graph matching covered G is *solid* if there are no two odd cycles C_1 and C_2 such that $G - V(C_1) - V(C_2)$ has a perfect matching.

Theorem 4 (2012 Carvalho, Lucchesi and Murty [2]). *Graph $K_{3,3}$ is the only solid minimally-non-Pfaffian graph under conformal subgraph and bicontraction.*

A graph G is *half-bipartite* if it is matching covered and there is a partition U, W of $V(G)$ such that $|E(G[U])| = 1$ and $G - E(G[U]) - E(G[W])$ is bipartite and matching covered.

Theorem 5 (2010 Miranda and Lucchesi [9]). *Graphs $K_{3,3}$, Γ_1 and Γ_2 are the only half-bipartite minimally non-Pfaffian graphs under conformal subgraph, bicontraction and odd cycle contraction.*

Theorem 6 (2008 Norine and Thomas [11]). *There are infinitely many minimally non-Pfaffian graphs under conformal subgraph, bicontraction and odd cycle contraction.*

Open Problem 1. *Determine Pfaffian preserving operations under which there are finitely many minimally non-Pfaffian graphs and determine such graphs.*

This problem is still open to small generalizations of classes to which it is solved. For example, this problem is not solved for the generalization of half-bipartite graphs that allows two edges on part U of its defining partition instead of one edge.

Open Problem 2. *Which are the minimally non-Pfaffian graphs of class 2-half-bipartite?*

Algorithms

Open Problem 3. *Determine whether there is a polynomial time algorithm for recognizing Pfaffian graphs.*

The general problem is unsolved. However there are polynomial time algorithms to recognize whether a graph is Pfaffian for the following classes:

- Planar graphs (1963 Kasteleyn [5]) .
- Bipartite graphs (1999 McCuaig [7] and Robertson, Seymour and Thomas [12]).
- Near-bipartite graphs (2008 Miranda and Lucchesi [9]).
- Half-bipartite graphs (2008 Miranda and Lucchesi [9]).

k-Pfaffian Graphs

Let $D = (D_1, D_2, \ldots, D_k)$ be a k-tuple of orientations of a graph G, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k$. If

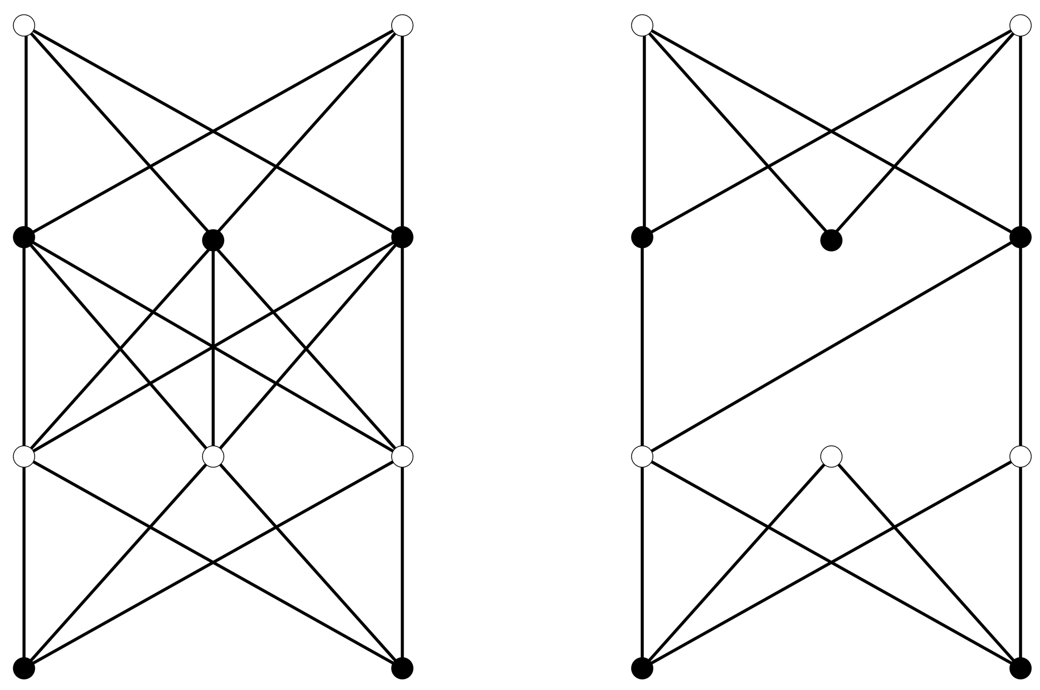
$$\sum_{i=1}^{k} \alpha_i \operatorname{sgn}(\pi_{D_i}(M)) = 1, \text{ for all } M \in \mathcal{M}(G),$$

then (D, α) is a *Pfaffian k-pair*, and G is *k-Pfaffian*. Then, it is possible to count the perfect matchings of G in polynomial time, for a fixed k. Let the pfaffian number of a graph G, named $pf(G)$, be the smallest k such that G is k-Pfaffian.

Theorem 7 (Galluccio and Loebl [4] and Tesler [13]). *If G is embeddable on an orientable surface of genus g then $pf(G) \leq 4^g$.*

Not every Pfaffian number is possible. Norine [10] proved no graph has Pfaffian number 2, 3 or 5. Moreover, he also proved that a graph only has Pfaffian number 4 if it has a special drawing on the torus. It seemed that all Pfaffian numbers were powers of 4, a conjecture stated by Norine on the same article.

Miranda and Lucchesi [8] provided a method to generate $2r$-Pfaffian graphs by combining r Pfaffian graphs. This method was used to provide a graph with Pfaffian number 6. A cut C of a graph G is *tight* if each perfect matching of G has exactly one edge in C. Let $\{C_1, C_2, \ldots, C_r\}$ be a partition of a tight cut C of G, and $G_i := G - (C - C_i)$. Then, $G_1, G_2, \ldots, G_r$ is an r-decomposition of G.



A 6-Pfaffian graph and one of the graphs of its 3-decomposition.

Theorem 8 (Miranda and Lucchesi [8]). *If G has an r-decomposition into Pfaffian graphs, then G is $2r$-Pfaffian.*

The first examples obtained for graphs with Pfaffian number 6 have 3-decompositions into Pfaffian graphs.

There are several open problems regarding k-Pfaffian graphs.

Open Problem 4. *Does every graph with Pfaffian number 6 have some variant of 3-decomposition?*

Open Problem 5. *Are minimally non-4-Pfaffian graphs always 6-Pfaffian?*

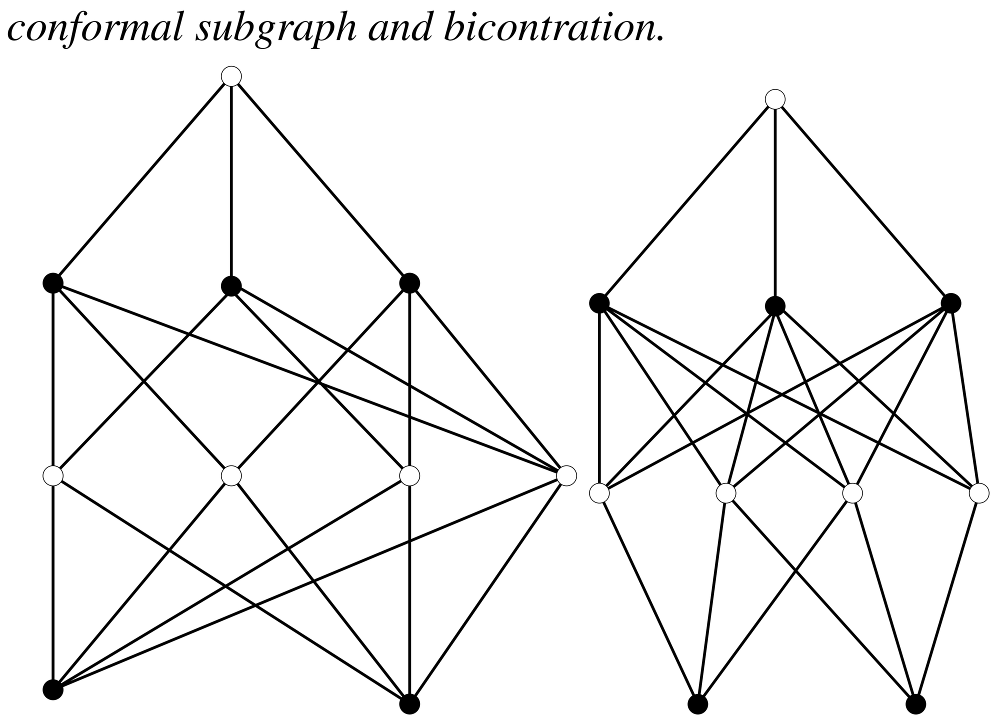
Open Problem 6. *Do minimally non-$2r$-Pfaffian graphs have an $(r+1)$-decomposition?*

Open Problem 7. *Is every even number bigger than 2 the Pfaffian number of some graph?*

Open Problem 8. *Determine the bipartite minimally non-4-Pfaffian graphs under some set of operations.*

Through the use of computer programs, we were able to obtain several bipartite minimally non-4-Pfaffian graphs under conformal subgraph and bicontraction.

Computation Result 1. *There are 3 bipartite graphs with 10 vertices that are minimally non-4-Pfaffian graphs under conformal subgraph and bicontration.*



Two minimally non-4-Pfaffian graphs.

Computation Result 2. *There are 31 bipartite graphs with 12 vertices that are minimally non-4-Pfaffian graphs under conformal subgraph and bicontration.*

Apparently, these two operations (conformal subgraph and bicontraction) are not enough to explain non-4-Pfaffian bipartite graphs.

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