A stability version for a theorem of Erdős on nonhamiltonian graphs

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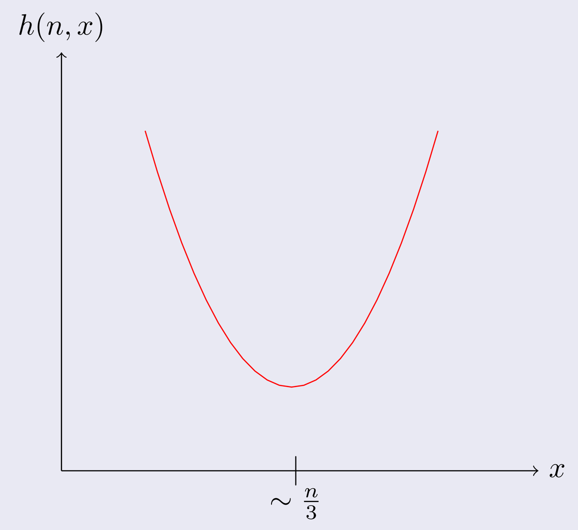
Classical results

A simple graph $G = (V, E)$ is called *hamiltonian* if there exists a cycle that covers every vertex of the graph. Conditions for hamiltonicity is a widely studied field, although testing if a graph is hamiltonian is *NP*-complete. Classical theorems for hamiltonian graphs include those of Dirac, Pósa, Bondy, Chvátal, Ore, and Erdős, among others.

Consider the function

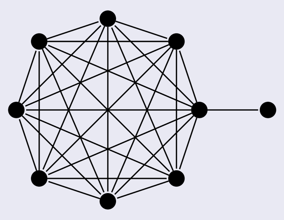
$$h(n, x) := \binom{n-x}{2} + x^2.$$



The first Turán-type result for hamiltonian graphs was due to Ore:

Ore (1959): If G is an n-vertex graph and $e(G) > h(n, 1)$, then G is hamiltonian.

The extremal example is a complete graph on $n-1$ vertices plus a single edge. Erdős later refined the bound in terms of the minimum degree of the graph.



Erdős (1962): Let G be an n-vertex graph with minimum degree $\delta(G) \geq d$. If

$$e(G) > \max\{h(n, d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\},$$

then G is hamiltonian.

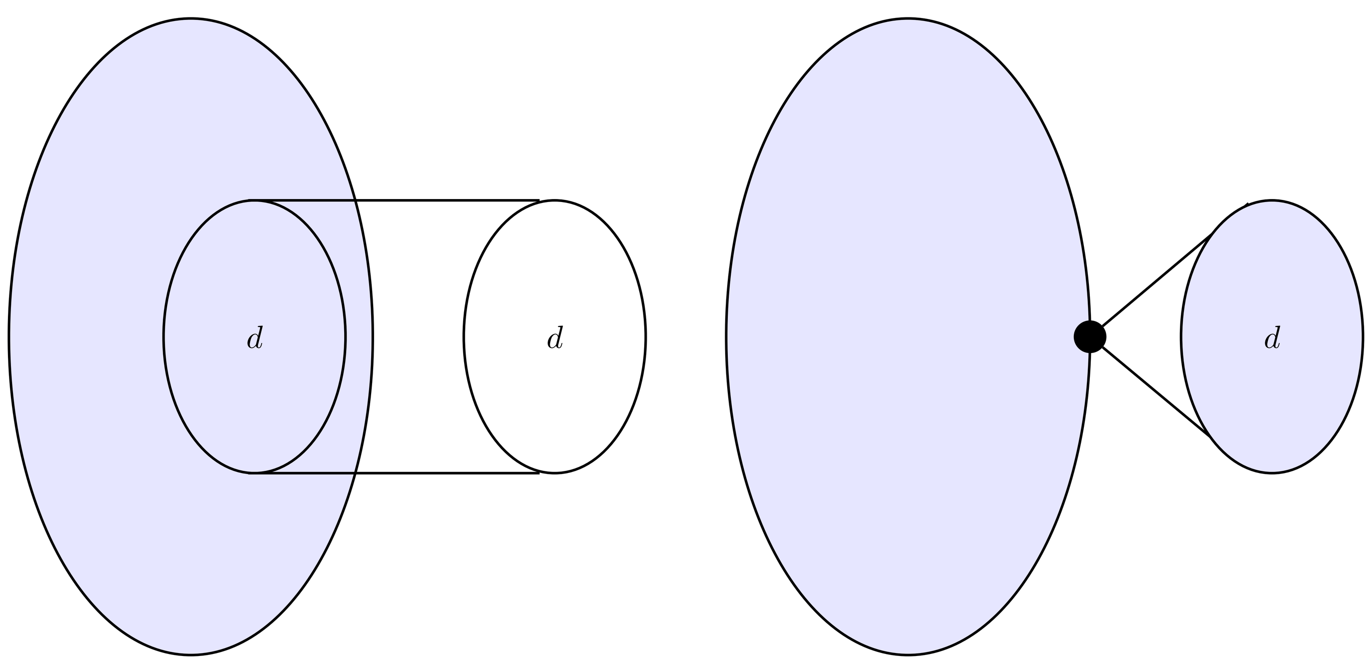


Figure: $H_{n,d}$ and $H'_{n,d}$ (blue background denotes complete graph).

Sharpness example

For $d \leq \lfloor \frac{n-1}{2} \rfloor$, define the graph

$$H_{n,d} := A \cup B$$

where A is a clique of order $n - d$, B is an independent set of order d, and there exists a set of d vertices, $\{a_1, \ldots, a_d\}$ such that for each $b \in B$, $N(b) = \{a_1, \ldots, a_d\}$.

Note that $e(H_{n,k}) = h(n, k)$, and $H_{n,k}$ is nonhamiltonian for all k. $H_{n,d}$ and $H_{n,\lfloor (n-1)/2 \rfloor}$ are sharpness examples for the bound given by Erdős.

Also, define $H'_{n,d}$ to be the edge disjoint union of K_{n-d} and K_{d+1} sharing exactly one vertex.

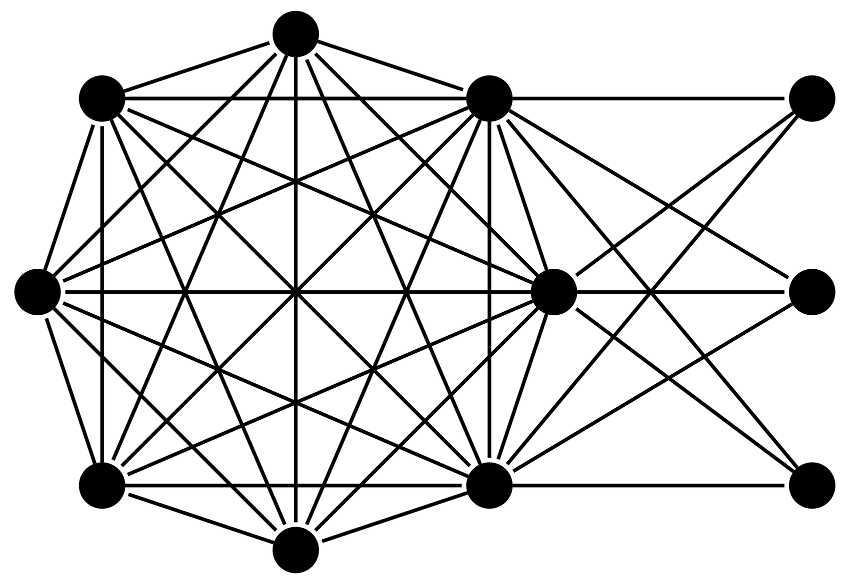


Figure: $H_{11,3}$.

Main theorem (2016)

Let $n \geq 3$ and $d \leq \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \geq d$ such that

$$e(G) > \max\{h(n, d+1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\},$$

Then G is a subgraph of either $H_{n,d}$ or $H'_{n,d}$.

Elementary calculation shows that $h(n, d) > h(n, \lfloor \frac{n-1}{2} \rfloor)$ in the range $1 \leq d \leq \lfloor \frac{n-1}{2} \rfloor$ if and only if $d < (n+1)/6$ (for n is odd) or if $d < (n+4)/6$ for n is even. In this range of d,

$$h(n, d) - h(n, d+1) = n - 3d - 2 \geq n/2.$$

The theorem is a stability result in the sense that for $d < n/6$, each 2-connected, nonhamiltonian n-vertex graph with minimum degree at least d and within $n/2$ edges of $h(n, d)$ is a subgraph of the extremal graph $H_{n,d}$.

Since a hamiltonian graph is necessarily 2-connected, this theorem implies that for $d < n/6$, the only 2-connected extremal example of a nonhamiltonian graph with $h(n, d)$ edges is $H_{n,d}$.

Fewer edges

What kind of graphs appear when we relax the bound on the edges? Most nonhamiltonian graphs have few edges, and so if we allow a lower bound on number of edges that is too small, any stability theorem would be weak.

We recently proved another step of the stability theorem.
Define

$$H''_{n,d} := A \cup B$$

where A is a complete graph of order $n - d - 1$, B is a set of $d + 1$ vertices such that $e(B) = 1$, and there exists a set of vertices $\{a_1, \ldots, a_d\} \subseteq A$ such that for all $b \in B$, $N(b) - B = \{a_1, \ldots, a_d\}$. Note that contracting the edge in B yields $H_{n,d}$.

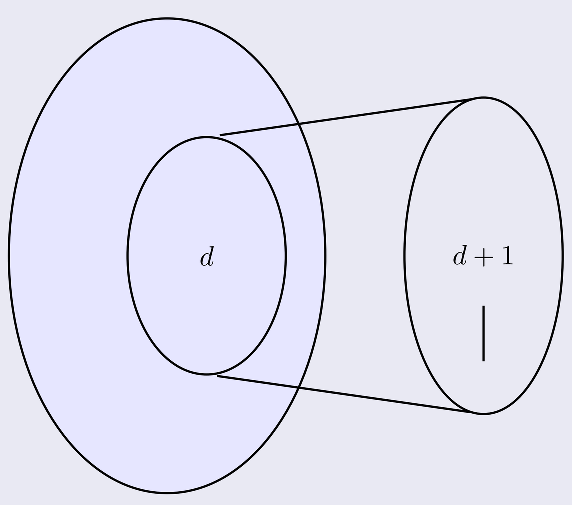


Figure: $H''_{n,d}$.

Theorem (2016+): If G is a nonhamiltonian graph with $\delta(G) \geq d \geq 4$ and $e(G) > \max\{h(n, d+2), h(n, \lfloor \frac{n-1}{2} \rfloor)\}$, then G is a subgraph of

$$H_{n,d}, H'_{n,d}, H_{n,d+1}, H'_{n,d+1}, \text{ or } H''_{n,d}.$$

Furthermore, for $d \leq 3$, there are only two additional cases.

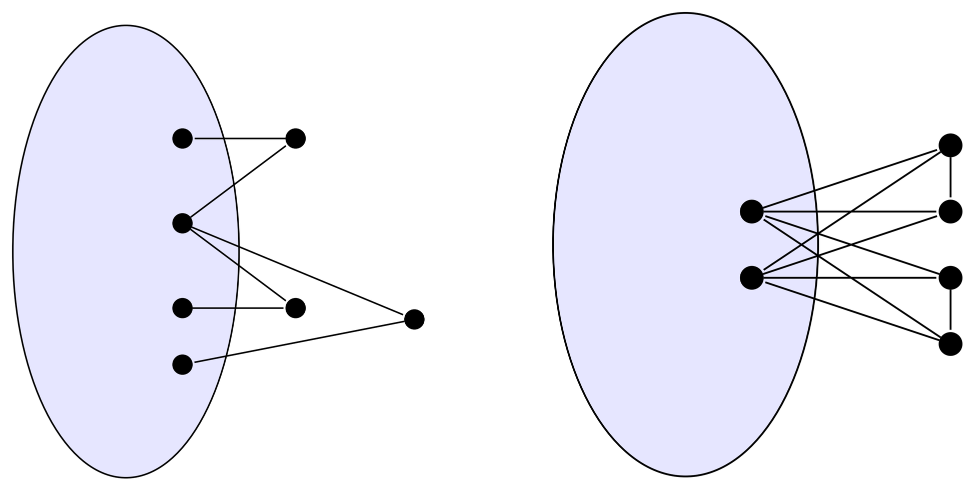


Figure: Additional cases for $d = 2$ and $d = 3$.

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