São Paulo School of Advanced Science on Algorithms, Combinatorics and Optimization
The Perfect Matching Polytope, Solid Bricks and the Perfect Matching Lattice

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1 Notation

Throughout, G denotes a graph with vertex set V and edge set E and $\mathbb{R}^E$ denotes the set of all real-valued vectors whose coordinates are indexed by the edges of G.

For a vector $\mathbf{x}$ in $\mathbb{R}^E$ and a set F of edges, $\mathbf{x}(F) = \sum_{e \in F} x(e)$.

$\mathcal{M}$ denotes the set of all perfect matchings of G.

χ^M denotes the incidence vector of M.

Cuts: A *cut* of G is a subset of E that is the coboundary $\partial(S)$ of some subset S of V. For a cut $C := \partial(S)$, S and $\overline{S}$ are the *shores* of C. A cut is *trivial* if one of its shores is a singleton. A cut is *odd* if both its shores have odd cardinality.

Note: This is the notation used in Bondy and Murty's book "Graph Theory (2008)" [1]. Most optimizers use $\delta(S)$, and we ourselves used $\nabla(S)$ instead of $\partial(S)$ in some of our papers [3]-[9].

- For any graph G, $\mathcal{O}(G)$ denote the set of odd components of G.
- $M_{2n}, n \geq 2$, denotes the Móbius ladder of order $2n$
- $B_{2n}, n \geq 3$, denotes the biwheel of order $2n$
- $P_{2n}, n \geq 3$, denote the prism of order $2n$
- $W_n, n \geq 3$, denotes the n-wheel

*Based on joint work with Marcelo H. de Carvalho and U. S. R. Murty

2 Matching Covered Graphs

2.1 Classical Results

Theorem 2.1 (Tait (1880) [26])
A 2-connected cubic planar graph is 4-face-colourable iff it has a 3-edge-colouring.

Theorem 2.2 (Petersen (1891) [22])
Every 2-connected cubic graph has a perfect matching.

Theorem 2.3 (Tutte (1947) [27])
A graph G has a perfect matching iff $|\mathcal{O}(G-S)| \leq |S| \quad \forall S \subseteq V$.

Corollary 2.4 *Every edge of a 2-connected cubic graph lies in a perfect matching.*

Exercise 2.5 *Deduce Corollary 2.4 from Theorem 2.3.*

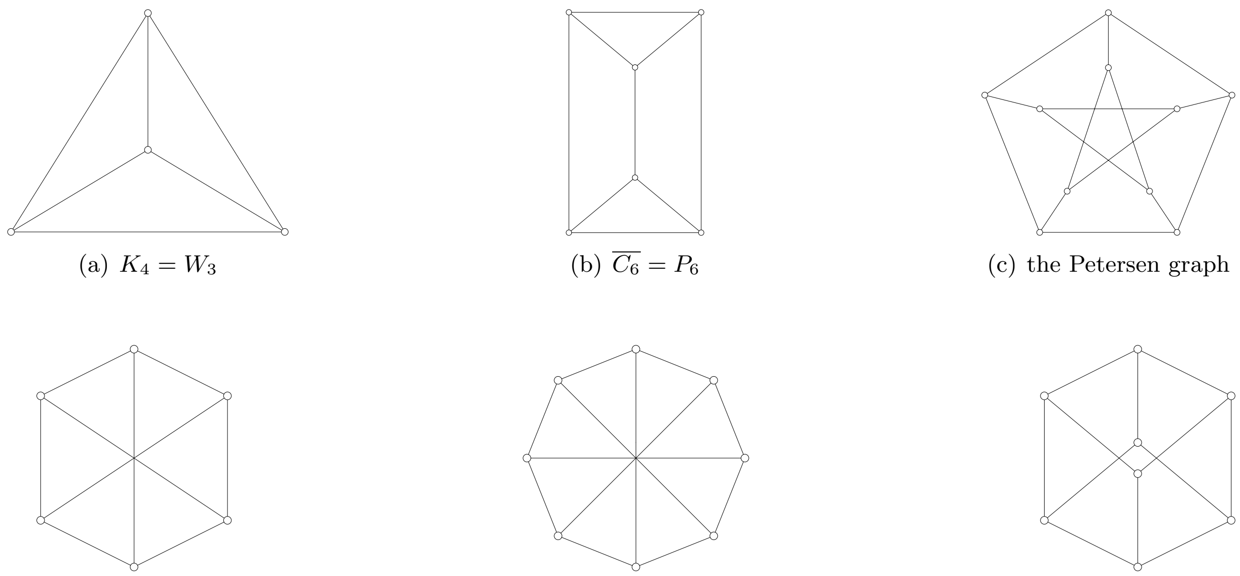
Barriers: In a graph with a perfect matching, a *barrier* is a subset S of V s.t. $|\mathcal{O}(G-S)| = |S|$.

Admissible edges: An edge e of G is *admissible* if $e \in M$ for some $M \in \mathcal{M}$.

Exercise 2.6 *In a graph with a perfect matching, show that an edge e is admissible iff there is no barrier that contains both ends of e.*

A Matching Covered Graph is a connected graph on two or more vertices in which every edge is admissible. We restrict our attention to matching covered graphs.

Thus, every 2-connected cubic graph is matching covered. Prisms, Möbius ladders and Biwheels are examples of matching covered graphs. Figure 1 depicts several cubic matching covered graphs.

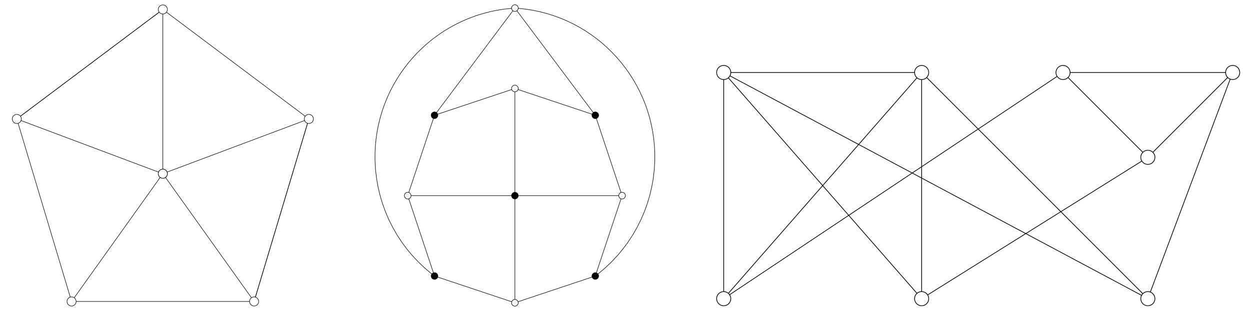


(a) $K_4 = W_3$ (b) $\overline{C_6} = P_6$ (c) the Petersen graph

(d) $K_{3,3} = M_6$ (e) M_8 (f) cube $B_8 = P_8$

Figure 1: Illustrious Cubic Graphs

Figure 2 depicts noncubic matching covered graphs.



(a) W_5 (b) B_{10} (c) Murty's graph

Figure 2: Examples of noncubic matching covered graphs

3 Building Blocks

Let G_1 and G_2 denote two disjoint matching covered graphs, let v_1 denote a vertex of G_1, let v_2 denote a vertex of G_2, such that the degree of v_1 in G_1 is equal to the degree of v_2 in G_2. Denote their common degree by d.

Enumerate the edges incident with v_1 as $\theta(v_1) := u_1v_1, u_2v_1, \ldots, u_dv_1$. Enumerate the edges incident with v_2 as $\theta(v_2) := v_2w_1, v_2w_2, \ldots, v_2w_d$. The *splicing* of G_1 and G_2 induced by θ_1 and θ_2 is the graph obtained from $(G_1 - v_1) \cup (G_2 - v_2)$ by the addition of the d edges u_iw_i, for $i = 1, 2, \ldots, d$. Figure 3 depicts a splicing of two 5-wheels that generates the pentagonal prism P_{10}. Figure 4 depicts a splicing of two 5-wheels that generates the Petersen graph.

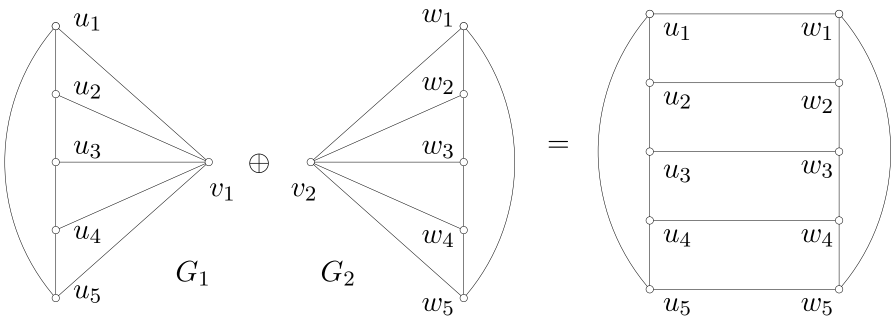


Figure 3: A splicing that generates P_{10}

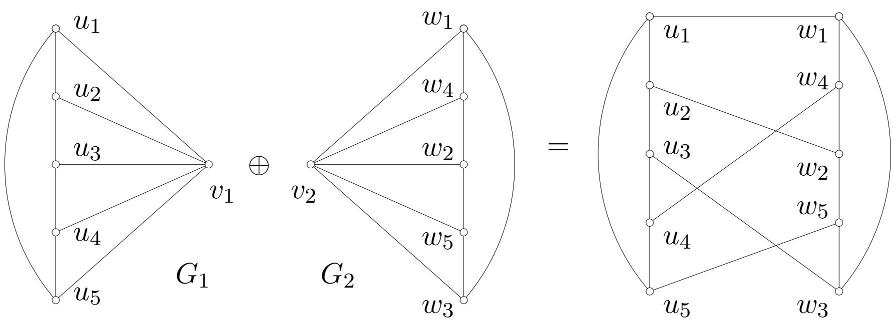


Figure 4: A splicing that generates $\mathbb{P}$, the Petersen graph

4 Separating Cuts and Tight Cuts

C-contractions: Let $C := \partial(X)$ be a cut of a matching covered graph G, where $|X|$ is odd. We denote the graph obtained by shrinking X to a single vertex x by $G/(X \to x)$ and, similarly, the graph obtained from G by shrinking $\overline{X}$ to a single vertex $\overline{x}$ by $G/(\overline{X} \to \overline{x})$. The two graphs $G/(X \to x)$ and $G/(\overline{X} \to \overline{x})$ are the *C-contractions* of G (Figure 5).

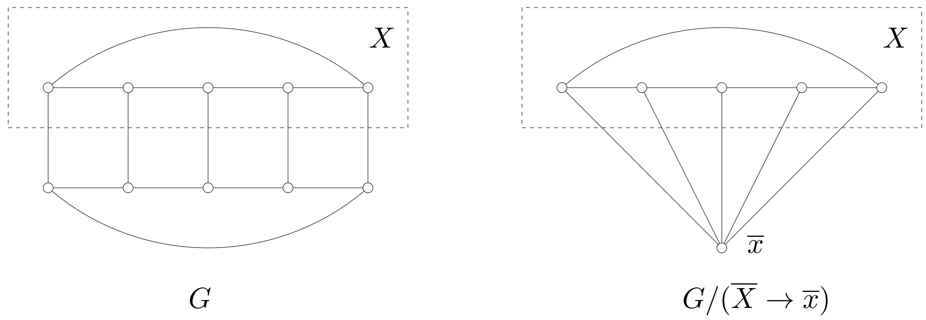


Figure 5: A C-contraction

Separating Cuts: A cut C of G is *separating* if both C-contractions are also matching covered.

Exercise 4.1 *Show that cut C of a matching covered graph G is separating iff, for any $e \in E$, there is a perfect matching M_e such that $e \in M_e$ and $|C \cap M_e| = 1$.*

Tight Cuts: A cut C of G is *tight* if $|C \cap M| = 1$ for all $M \in \mathcal{M}$.

Figure 6 shows several examples of tight cuts.

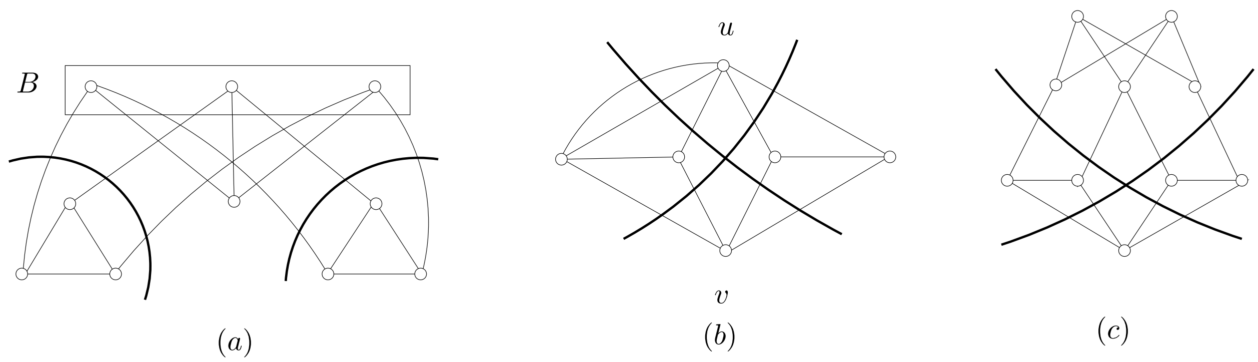


Figure 6: Tight Cuts

Not every separating cut is tight. For example, the cut C in Figure 7 is a separating cut, but it is not tight! However

Proposition 4.2 *Every tight cut is a separating cut.*

Exercise 4.3 *Let $C := \partial(X)$ be a separating cut in a matching covered graph G that is not tight. In this case, show that both shores of C induce graphs that are not bipartite.*

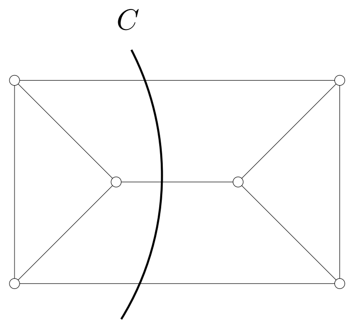


Figure 7: Cut C is separating but not tight

Exercise 4.4 *Deduce from the above exercise that in a bipartite matching covered graph, every separating cut is a tight cut.*

solid matching covered graphs A matching covered graph is *solid* if every separating cut is tight.

odd intercyclic graphs A graph is *odd intercyclic* if every pair of distinct odd cycles shares at least one vertex.

Exercise 4.5 *Prove that every odd intercyclic matching covered graph is solid.*

Problem 4.6 (Unsolved) *Is there a polynomial time algorithm to determine whether a given matching covered graph is solid?*

We do have a polynomial time algorithm to recognize solid planar graphs.

The following exercise provides a simple characterization of tight cuts in bipartite graphs. If X is an odd subset of the vertex set of a bipartite matching covered graph G with bipartition (A, B), clearly, one of $|X \cap A|$ and $|X \cap B|$ is larger than the other; the larger of the two sets is called the *majority part* and is denoted by X_+, and the smaller is called the *minority part* and is denoted by X_-. (Similarly, the majority and minority parts of $\overline{X} = V \setminus X$ are $\overline{X}_+$ and $\overline{X}_-$, respectively.)

Exercise 4.7 *Let $\partial(X)$ be a tight cut in a bipartite matching covered graph $G[A, B]$. Show that*

(i) $|X_+| = |X_-| + 1$, and $|\overline{X}_+| = |\overline{X}_-| + 1$, and

(ii) all edges in the cut $\partial(X)$ have one end in X_+ and one end in $\overline{X}_+$.

Exercise 4.8 *Using Exercise 4.3, deduce the following:*

(i) Each odd prism (of order $4k + 2$, $k \geq 1$) has precisely one separating cut that is not tight

(ii) The Petersen graph has precisely six separating cuts that are not tight.

5 Bricks and Braces

Barrier cuts: For any barrier B and any odd component K of $G - B$, $\partial(V(K))$ is a tight cut. Such cuts are called *barrier cuts.* (See Figure 6(a))

2-separation cuts: For any 2-separation $\{u, v\}$ of G and any even component L of $G - \{u, v\}$, $\partial(V(L) \cup \{u\})$ and $\partial(V(L) \cup \{v\})$ are tight cuts. Such cuts are called 2-*separation cuts.* (See Figure 6(b).)

A graph may have tight cuts that are neither barrier cuts nor 2-separation cuts. (See Figure 6(c).) However:

THEOREM 5.1 (EDMONDS, LOVÁSZ, PULLEYBLANK, 1982 [13]) *Every graph that has a nontrivial tight cut either has a nontrivial barrier or a 2-separation.*

New proofs of this result appear in Szigeti, 2002 [25] and in CLM, 2014 [10].

Braces: A *brace* is a bipartite matching covered graph that has no nontrivial tight cuts. (A bipartite graph G with bipartition (A, B), $|V| \geq 4$, is a brace iff, for any $a_1, a_2 \in A$ and $b_1, b_2 \in B$, the graph $G - \{a_1, a_2, b_1, b_2\}$ has a perfect matching.) The cube and $K_{3,3}$ and all prisms of order $4k$, $k \geq 2$ and all Möbius ladders of order $4k + 2$, $k \geq 1$, are braces.

Bricks: A *brick* is a nonbipartite matching covered graph that has no nontrivial tight cuts. (A graph G is a brick iff it is bicritical and 3-connected. A proof of this requires Theorem 5.1. There is a polynomial-time algorithm for deciding whether or not a given G is a brick.) The graphs K_4, $\overline{C_6}$, all Möbius ladders of order $4k$, $k \geq 1$, and all prisms of order $4k + 2$, $k \geq 1$, are bricks.

Tight Cut Decomposition: By repeatedly taking contractions with respect to nontrivial tight cuts, any graph may be decomposed into bricks and braces. For example, up to multiple edges, the tight cut decompositions of the graphs in Figure 6 produce, respectively, (a) two K_4's and $K_{3,3}$, (b) two K_4's, and (c) two K_4's and $K_{3,3}$.

THEOREM 5.2 (LOVÁSZ, 1987 [18]) *Any two tight cut decompositions of a matching covered graph G yield the same list of bricks and braces (except possibly for multiplicities of edges).*

Figure 8 depicts an example of the uniqueness of tight cut decompositions, up to multiple edges.

EXERCISE 5.3 *Find all the tight cut decompositions of the graph in Figure 6(c).*

Crossing Cuts: Two cuts $\partial(X)$ and $\partial(Y)$ *cross* if each of the four *quadrants* is nonempty: $X \cap Y$, $X \cap \overline{Y}$, $\overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$.

Laminar Collection of Cuts: A collection of cuts is *laminar* if no two of its cuts cross. There is a one-to-one correspondence between the set of tight cut decompositions of a matching covered graph G and the set of maximal laminar collections of nontrivial tight cuts of G.

THEOREM 5.4 (UNCROSSING TIGHT CUTS) *Let $C := \partial(X)$ and $D := \partial(Y)$ be two tight cuts of a matching covered graph G, where $|X \cap Y|$ is odd. Then:*

1. *No edge of G joins a vertex in $X \cap \overline{Y}$ to a vertex in $\overline{X} \cap Y$, and*
2. *the cuts $\partial(X \cap Y)$ and $\partial(\overline{X} \cap \overline{Y})$ are both tight in G.*

The following corollary will prove to be very useful in the proof of Theorem 5.2.

COROLLARY 5.5 *Let G be a matching covered graph, let $C := \partial(X)$ and $D := \partial(Y)$ be two tight cuts of G that cross, where $|X \cap Y|$ is odd. Then the underlying simple graphs of the two graphs $G_1 := G/X/\overline{X} \cap \overline{Y}$ and $G_2 := G/\overline{Y}/X \cap Y$ are isomorphic.*

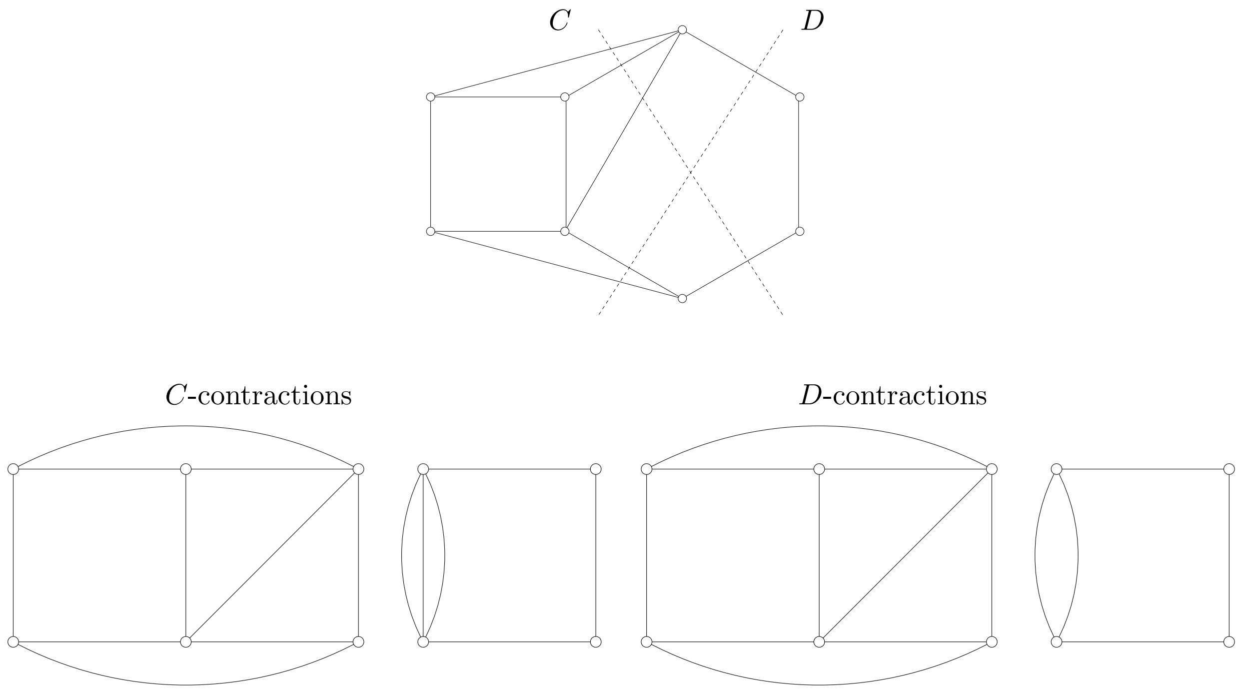


Figure 8: An example of the uniqueness of tight cut decompositions

Proof: Each of the two graphs G_1 and G_2 has two contraction vertices. For the convenience of referring to them, let us write:

$$G_1 = (G/X \to x)/(\overline{X} \cap \overline{Y}) \to s, \text{ and } G_2 = (G/\overline{Y} \to \overline{y})/(X \cap Y) \to t$$

See Figure 9.

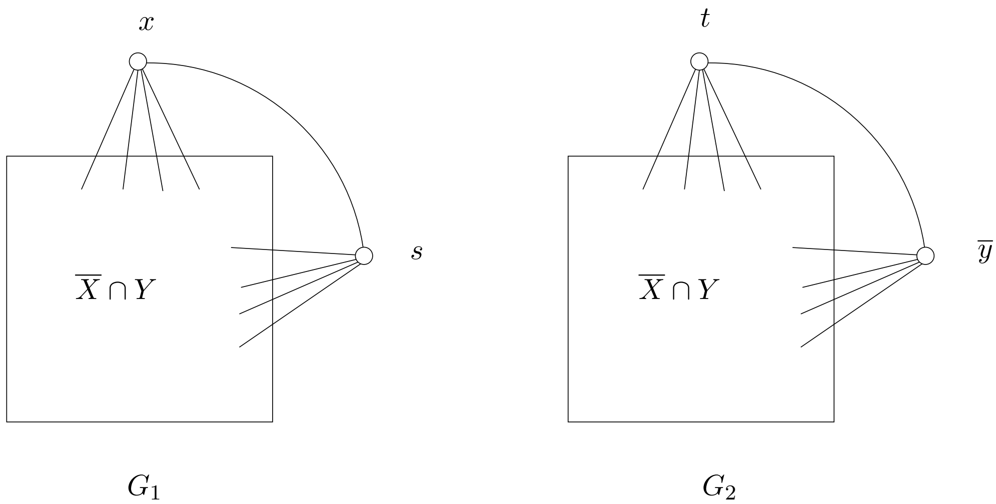


Figure 9: Isomorphic contractions

Observe that $V(G_1) = (\overline{X} \cap Y) \cup \{x, s\}$, and $V(G_2) = (\overline{X} \cap Y) \cup \{\overline{y}, t\}$. As D is a tight cut, $G[\overline{Y}]$ is connected. This implies that x and s are adjacent in G_1. Similarly, since C is a tight cut,

$G[X]$ is connected, implying that t and $\overline{y}$ are adjacent in G_2. Furthermore, as there are no edges between $\overline{X} \cap Y$ and $X \cap \overline{Y}$ by Theorem 5.4, it follows that the mapping θ, where $\theta(v) = v$, for each $v \in \overline{X} \cap Y$, $\theta(x) = t$, and $\theta(s) = \overline{y}$ is an isomorphism between the underlying simple graphs of G_1 and G_2. We leave the details as Exercise 5.6. □

EXERCISE 5.6 *Supply the missing details in the proof of Corollary 5.5.*

Proof of Theorem 5.2: We shall refer to two maximal laminar families $\mathcal{C}$ and $\mathcal{D}$ of nontrivial tight cuts of G as *equivalent*, and write $\mathcal{C} \equiv \mathcal{D}$, if they produce the same list of bricks and braces, up to multiple edges. We prove that any two maximal laminar collections of nontrivial tight cuts of G are equivalent, by induction on the number of vertices.

For $i = 1, 2$, let $\mathcal{C}_i$ be two maximal laminar collections of nontrivial tight cuts of G. We consider various cases.

CASE 1 *Collections $\mathcal{C}_1$ and $\mathcal{C}_2$ contain a common cut C.*

Let G_1 and G_2 denote the two C-contractions of G. For $i, j \in \{1, 2\}$, let $\mathcal{C}_{ij}$ denote the restrictions of $\mathcal{C}_i \setminus \{C\}$ to G_j. As any tight cut of G_i, $i = 1, 2$, is also a tight cut of G, it follows that $\mathcal{C}_{ij}$ is a maximal laminar collection of nontrivial tight cuts of G_j. By induction, $\mathcal{C}_{1j}$ and $\mathcal{C}_{2j}$ are equivalent. Thus, $\mathcal{C}_1$ and $\mathcal{C}_2$ are also equivalent. The assertion holds in this case.

CASE 2 *There are cuts $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ that do not cross.*

By hypothesis, $\{C_1, C_2\}$ is laminar. Let $\mathcal{C}_3$ denote a maximal laminar collection of nontrivial tight cuts of G that includes $\{C_1, C_2\}$. For $i = 1, 2$, $\mathcal{C}_i$ and $\mathcal{C}_3$ have cut C_i in common. By Case 1,

$$\mathcal{C}_1 \equiv \mathcal{C}_3 \equiv \mathcal{C}_2.$$

Thus, $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent. The assertion holds in this case.

CASE 3 *There are cuts $C_1 := \partial(X_1) \in \mathcal{C}_1$ and $C_2 := \partial(X_2) \in \mathcal{C}_2$ such that $|X_1 \cap X_2|$ is odd and nontrivial.*

If C_1 and C_2 do not cross then Case 2 is applicable. We may thus assume that C_1 and C_2 cross. Let $C_3 := \partial(X_1 \cap X_2)$. Then, C_3 is nontrivial. By Theorem 5.4, cut C_3 is tight in G.

Let $\mathcal{C}_4$ denote a maximal laminar collection of nontrivial tight cuts of G that contains C_3. Cuts C_1 and C_3 do not cross. By Case 2, collections $\mathcal{C}_1$ and $\mathcal{C}_4$ are equivalent. Likewise, cuts C_2 and C_3 do not cross. By Case 2, collections $\mathcal{C}_2$ and $\mathcal{C}_4$ are equivalent. In sum,

$$\mathcal{C}_1 \equiv \mathcal{C}_4 \equiv \mathcal{C}_2.$$

Thus, $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent. The assertion holds in this case.

CASE 4 *None of the previous cases are applicable.*

If G is a brick or a brace then the assertion holds trivially. We may thus assume that G has nontrivial tight cuts. Then, $\mathcal{C}_1$ and $\mathcal{C}_2$ are both nonempty. For $i = 1, 2$, let $C_i := \partial(X_i)$ denote a cut in $\mathcal{C}_i$. If C_1 and C_2 do not cross then Case 2 applies. We may thus assume that C_1 and C_2 cross.

Adjust notation so that $|X_1 \cap X_2|$ is odd, whereupon $|\overline{X_1} \cap \overline{X_2}|$ is also odd. If $|X_1 \cap X_2| > 1$ or if $|\overline{X_1} \cap \overline{X_2}| > 1$ then Case 3 applies. We may thus assume that $X_1 \cap X_2$ and $\overline{X_1} \cap \overline{X_2}$ are both singletons. Let u denote the only vertex of $X_1 \cap X_2$, let v denote the only vertex of $\overline{X_1} \cap \overline{X_2}$.

Assume that $\mathcal{C}_i = \{C_i\}$, for $i = 1, 2$. By Corollary 5.5, G/X_1 and $G/\overline{X_2}$ are isomorphic, up to multiple edges. Likewise, $G/\overline{X_1}$ and G/X_2 are also isomorphic, up to multiple edges. Thus, $\mathcal{C}_1 \equiv \mathcal{C}_2$. The assertion holds in this case.

We now prove that $\mathcal{C}_i = \{C_i\}$, thereby completing the proof. Assume, to the contrary, that one of $\mathcal{C}_1$ and $\mathcal{C}_2$ contains two or more cuts. Adjust notation so that $\mathcal{C}_1 \setminus \{C_1\}$ contains a cut $D := \partial(Y)$. By hypothesis, $\mathcal{C}_1$ is laminar, therefore C_1 and D do not cross. Adjust notation so that one of Y and X_1 is a subset of the other. Adjust notation, by complementing the three sets X_1, X_2 and Y if necessary, so that $Y \subset X_1$. Then,

$$Y \cap X_2 \subseteq X_1 \cap X_2 = \{u\} \quad \text{and} \quad \{v\} = \overline{X_1} \cap \overline{X_2} \subseteq \overline{Y} \cap \overline{X_2}. \tag{1}$$

Cuts C_2 and D cross, otherwise Case 2 applies. Thus, $Y \cap X_2$ is nonempty. From (1) above we deduce that $Y \cap X_2 = X_1 \cap X_2 = \{u\}$. Then, $|\overline{Y} \cap \overline{X_2}|$ is odd. If $|\overline{Y} \cap \overline{X_2}| > 1$ then Case 3 applies. We may thus assume that $|\overline{Y} \cap \overline{X_2}| = 1$. From (1), we deduce that $\overline{Y} \cap \overline{X_2} = \overline{X_1} \cap \overline{X_2}$, whence $Y \cup X_2 = X_1 \cup X_2$. In sum,

$$Y \cap X_2 = X_1 \cap X_2 \quad \text{and} \quad Y \cup X_2 = X_1 \cup X_2.$$

We conclude that Y and X_1 coincide, a contradiction. As assumed, $\mathcal{C}_i = \{C_i\}$, for $i = 1, 2$. □

The number of bricks: The number of bricks resulting from a tight cut decomposition of G, denoted by $b(G)$, is an invariant of G. A graph G is a *near-brick* if $b(G) = 1$.

6 The Perfect Matching Polytope

The Perfect Matching Polytope ($\mathcal{P}oly(G)$) is the convex hull of $\{\chi^M : M \in \mathcal{M}\}$.

THEOREM 6.1 (EDMONDS, 1965 [11]) *A vector* $\mathbf{x}$ *in* $\mathbb{R}^E$ *belongs to the perfect matching polytope* $\mathcal{P}oly(G)$ *of a graph* G *if and only if it satisfies the following system of linear inequalities:*

$$\begin{array}{rcll} \mathbf{x} & \geq & 0 & & \textit{(nonnegativity)} \\ \mathbf{x}(\partial(v)) & = & 1 & \textit{for all } v \in V & \textit{(degree constraints)} \\ \mathbf{x}(\partial(S)) & \geq & 1 & \textit{for all odd } S \subset V & \textit{(odd set constraints)} \end{array}$$

When G is bipartite, the first two conditions imply the third. This is in general not true, see Figure 10.

EXERCISE 6.2 *Prove that for bipartite graphs the nonnegativity and the degree constraints imply the odd set constraints.*

EXERCISE 6.3 *Prove that for near-bricks the nonnegativity and the degree constraints imply the odd set constraints.*

In the case of the Petersen graph, instead of 2^8 odd set constraints, only six constraints are necessary, the six cuts whose shores are pentagons (Figure 11).

EXERCISE 6.4 *Prove that in the case of the Petersen graph, the six odd constraints are necessary.*

We now try to identify the graphs that need odd set constraints. We study this question in the context of matching covered graphs.

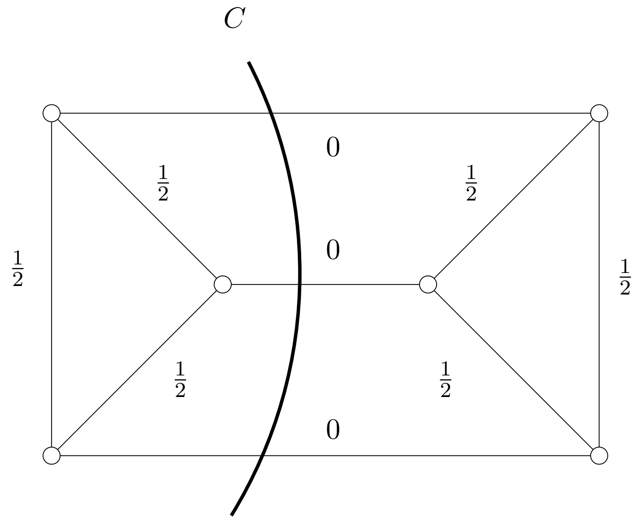


Figure 10: A nonnegative 1-regular vector that is not in the polytope

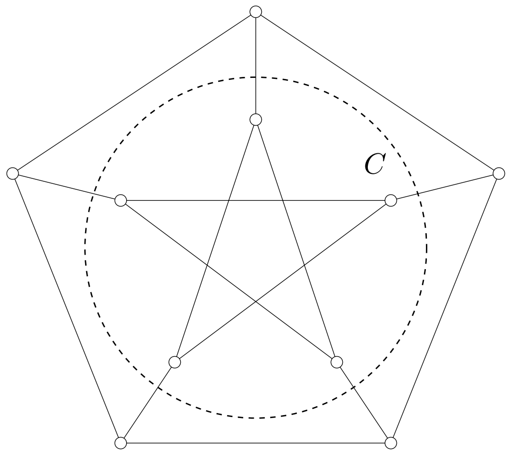


Figure 11: The Petersen graph needs only six odd set constraints

THEOREM 6.5 *Let C be a tight cut of G and let G_1 and G_2 be the two C-contractions of G. A vector $\mathbf{x}$ in $\mathbb{R}^E$ belongs to $\mathcal{P}oly(G)$ iff the restrictions of $\mathbf{x}$ to $E(G_1)$ and $E(G_2)$ belong, respectively, to $\mathcal{P}oly(G_1)$ and $\mathcal{P}oly(G_2)$.*

Thus, to check if a vector $\mathbf{x}$ is in $\mathcal{P}oly(G)$, it suffices to check whether or not the restrictions of $\mathbf{x}$ to the edge sets of the bricks and braces are in the perfect matching polytopes of those graphs. For this reason, in seeking an answer to Problem 4.6, we may restrict our attention to bricks.

7 Solid Bricks

A matching covered graph G is *solid* if it has no separating cuts other than tight cuts. Since a brick has no tight cuts other than the trivial cuts, it follows that a brick is solid if and only if it has no nontrivial separating cuts.

We introduced and made use of solid bricks in proving a conjecture of Lovász ([4] and [5]).

(This will be described later on.) One of the notions that played a useful role in that work was a relation defined on the set of cuts of a graph.

A precedence relation on cuts: Let C and D be two cuts of a graph G. Cut D *precedes* cut C (written as $D \preceq C$) if $|M \cap D| \leq |M \cap C|$ for each perfect matching M of G.

Example 7.1 *Let G be a brick and let $C := \partial(X)$ be a nontrivial odd cut of G. If C is not a separating cut, then one of the two C-contractions is not matching covered. Suppose that $G_1 := G/(X \to x)$ is not matching covered. Then, either (i) G_1 has no perfect matching, or (ii) G_1 has a perfect matching, but it has an edge that is inadmissible. In the first case, there exists a subset S of $V(G_1)$ such that $|\mathcal{O}(G_1 - S)| > |S|$, and in the second case, there is a barrier S of G_1 that contains both ends of some edge e of G. Since G is a brick there is no subset S of $V(G)$ with either of these properties. In both alternatives, the contraction vertex x lies in S (Figure 12).*

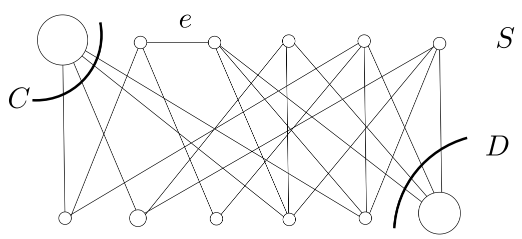


Figure 12: The case $|\mathcal{O}(G - S)| = |S|$

Suppose that K is an odd component of $G_1 - S$ and let $D := \partial(V(K))$. In case (i), $|D \cap M| < |C \cap M|$ for every perfect matching M of G. In case (ii), $|D \cap M| \leq |C \cap M|$ for every perfect matching M of G, with equality only if e does not lie in M. It follows that, in either case, D strictly precedes C.

If a brick G is nonsolid then, by definition, it has a nontrivial separating cut, say C, and the two C-contractions G_1 and G_2 of G are matching covered. But, in general, G_1 and G_2 need not be bricks or even near-bricks. For the purpose of applying induction to prove Lovász's conjecture, it was necessary for us to find a separating cut C such that both G_1 and G_2 are near-bricks. We called such a separating cut a *robust* cut and proved the following theorem.

Theorem 7.2 *Every nonsolid brick has a robust cut.*

Given any separating cut C of a brick G, we showed that either C is a robust cut or there is a separating cut D that precedes C strictly. Thus, any separating cut that is minimal with respect to the precedence relation is a robust cut. We also proved the following generalization of the above theorem.

Theorem 7.3 *In any nonsolid brick G there are two separating cuts $\partial(X)$ and $\partial(Y)$, $X \cap Y = \emptyset$, such that $G/\overline{X}$ and $G/\overline{Y}$ are bricks and the graph obtained from G by shrinking X to x and Y to y is bipartite.*

8 A Solution to the Problem for Bricks

Theorem 8.1 *For a brick G, $\mathcal{P}oly(G)$ consists of all nonnegative 1-regular vectors if and only if G is solid.*

Proof: Firstly suppose that G is not solid. We wish to show that there is some nonnegative 1-regular vector in $\mathbb{R}^E$ that does not belong to $\mathcal{P}oly(G)$. Since G is nonsolid, it has a nontrivial separating cut C. Let M_0 be a perfect matching of G such that $|M_0 \cap C| > 1$. (Such a perfect matching must exist; otherwise C would be tight.) Also, since C is separating, for every edge e of G, there is a perfect matching M_e of G such that $M_e \cap C = \{e\}$. Now let

$$\mathbf{x} := \frac{1}{|M_0| - 1}\left(\left(\sum_{e \in M_0} \chi^{M_e}\right) - \chi^{M_0}\right)$$

Clearly the vector $\mathbf{x}$ is nonnegative, 1-regular with $x(C) < 1$.

Conversely, suppose that G is solid. We wish to prove that every nonnegative 1-regular vector in $\mathbb{R}^E$ belongs to $\mathcal{P}oly(G)$. Assume to the contrary that there is a nonnegative 1-regular vector $\mathbf{x}$ that does not belong to $\mathcal{P}oly(G)$. Then, by Theorem 6.1 there must exist odd cuts C with $\mathbf{x}(C) < 1$. Let $\mathcal{C}$ denote the set of all cuts C for which $\mathbf{x}(C) < 1$ and let $D := \partial(Y)$ be a cut in $\mathcal{C}$ that is minimal with respect to the precedence relation $\preceq$ defined in the previous section. We shall obtain a contradiction by showing that D is a separating cut.

Consider the D-contraction $G_1 := G/Y$. We wish to show that G_1 is matching covered. If it is not, then either there is a subset S of $V(G_1)$ such that either (i) $|\mathcal{O}(G_1 - S)| > |S|$ or (ii) $|\mathcal{O}(G_1 - S)| = |S|$, but there is an edge e of G_1 with both its ends in S. In either case, there must be an odd component K of $G_1 - S$ for which $\mathbf{x}(D') < 1$, where $D' := \partial(V(K))$. Such a component K is clearly nontrivial. One may verify that D' strictly precedes D (see Example 7.1), contradicting the choice of D. Therefore G_1 is matching covered. Similarly, $G_2 := G/\overline{S}$ is also matching covered and D is a separating cut. A contradiction. □

Exercise 8.2 *Prove that the cut D above is in fact robust.*

Using Theorem 7.3 it is possible to establish the following characterization of nonsolid bricks.

Theorem 8.3 *A brick G has a nontrivial separating cut iff it there exists two disjoint subsets X and Y of V such that (i) $G[X]$ and $G[Y]$ are nontrivial critical graphs, and (ii) $G - (X \cup Y)$ has a perfect matching.*

The above theorem is a variant of the following attractive theorem.

Theorem 8.4 (Reed and Wakabayashi) *A brick G has a nontrivial separating cut iff it has two odd circuits C_1 and C_2 such that the graph $G - (V(C_1) \cup V(C_2))$ has a perfect matching.*

9 Examples of Solid Bricks

A graph is *odd-intercyclic* if any two odd circuits of G have at least one vertex in common. By Theorem 8.4, all odd-intercyclic bricks are solid. (This can be proved by elementary arguments quite easily.) Odd wheels and Möbius ladders described below are examples of odd-intercyclic solid bricks.

Odd Wheels: The *wheel of order* $n \geq 3$, denoted by W_n, is obtained by adjoining a vertex h to a circuit R of length n and joining it to each vertex of R; h and R are referred to as the *hub* and the *rim* of W_n, respectively. (The wheel W_3 of order three is isomorphic to K_4; any one of its four vertices may be regarded as its hub.) A wheel is *odd* or *even* according to the parity of its order. It is easy to show that every odd wheel is an odd-intercyclic brick.

Möbius ladder: The ladder L_{2n}, $n \geq 2$, is obtained from two disjoint paths $(x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ by adding the edges $x_i y_i$, $1 \leq i \leq n$. The Möbius ladder M_{2n}, $n \geq 2$, is obtained from L_{2n} by joining x_1 to y_n and y_1 to x_n. This graph is Hamiltonian and is isomorphic to the cubic graph obtained from the circuit $(0, 1, ..., 2n-1)$ by joining each vertex i to the vertex $i + n$ $(mod\ 2n)$. Figure 13 depicts the Möbius ladderM_8.

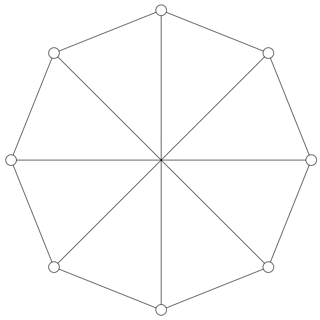


Figure 13: The Möbius ladder M_8

It can be shown that, for any odd integer $n \geq 3$, M_{2n} is a brace, and for any even integer $n \geq 2$, M_{2n} is an odd-intercyclic brick.

9.1 An infinite family of solid bricks that are not odd-intercyclic

Let $n \geq 3$ be an odd integer. Consider the brace $\mathbb{M}_{2n}$, which is the Möbius ladder on $2n$ vertices. Obtain the cubic graph H from $\mathbb{M}_{2n}$ by deleting the vertex n, adding three new vertices u, v and w, and joining u to $n-1$, v to 0, w to $n+1$ and u, v and w to each other. (Thus H is obtained by splicing $\mathbb{M}_{2n}$ and K_4.) Now obtain the graph S_{2n+2} from H by joining 1 and $2n-1$. See Figure 14. It can be shown that S_{2n+2} is a solid brick for every odd integer $n \geq 3$. In fact, Murty devised this family as a generalization of the first graph in the family, S_8, which he discovered, and is the smallest non-odd-intercyclic solid matching covered graph.

Exercise 9.1 *Show that the graph S_8 (Figure 14, $n = 3$) is a non-odd-intercyclic solid brick.*

There is a characterization of odd-intercyclic graphs:

Theorem 9.2 (Kawarabayashi and Ozeki [15]) *Let G be an internally 4-connected graph. Then G is odd-intercyclic if and only if G satisfies one of the following:*

1. *either $G - x$ is bipartite for some vertex $x \in V$; or*
2. *G has a triangle T such that $G - T$ is bipartite; or*
3. *$|V| \leq 5$; or*
4. *G can be embedded into the projective plane so that every face boundary has even length.*

One general result that we have been able to prove is that odd wheels are the only simple planar solid bricks [9].

Solid bricks behave much as bipartite graphs. We have proved that every cubic solid brick has a 3-edge-colouring [20].

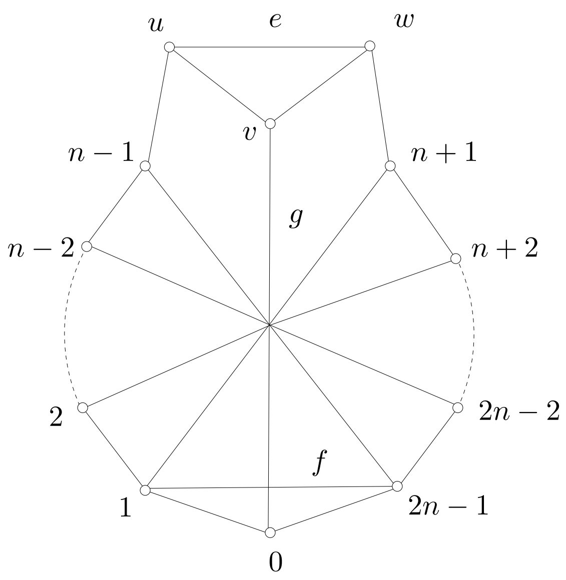


Figure 14: The solid brick $G := S_{2n+2}$

10 The Matching Lattice and its Bases

10.1 Regular Vectors

For a matching covered graph G and $r \in \mathbb{R}$, a vector $\mathbf{x} \in \mathbb{R}^E$ is *r-regular* if $\mathbf{x}(C) = r$ for each tight cut C of G. A vector $\mathbf{x} \in \mathbb{R}^E$ is *regular* if it is r-regular, for some $r \in \mathbb{R}$.

10.2 The Linear Space

$\mathcal{L}in(G)$ denotes the set of linear combinations of characteristic vectors of perfect matchings of a matching covered graph G. That is,

$$\mathcal{L}in(G) := \sum_{M \in \mathcal{M}} \alpha(M)\chi^M,$$

where each coefficient $\alpha(M)$ is a real number. Note that if $\mathbf{x} = \sum \alpha(M)\chi^M$ is any vector in $\mathcal{L}in(G)$, and C is any tight cut of G, then

$$\mathbf{x}(C) = \sum_{M \in \mathcal{M}} \alpha_M \chi^M(C) = \sum_{M \in \mathcal{M}} \alpha_M$$

As this is true for any tight cut C, it follows that every vector in $\mathcal{L}in(G)$ is regular. Conversely, it can be shown that all regular vectors are in $\mathcal{L}in(G)$.

Theorem 10.1 (Edmonds, Lovász, Pulleyblank [13]) *Let G be a matching covered graph. A vector $\mathbf{x}$ in $\mathbb{R}^E$ lies in $\mathcal{L}in(G)$ if and only if it is regular. Moreover, the dimension of $\mathcal{L}in(G)$ satisfies the formula*

$$\dim(\mathcal{L}in(G)) = m - n + 2 - b,$$

where m, n and b denote respectively the number of edges, vertices and bricks of G.

We shall abuse the language and refer to a set B of perfect matchings as a basis, meaning actually the set $\{\chi^M : M \in B\}$ of the corresponding incidence vectors. We shall denote the basis by using boldface, thus, $\mathbf{B}$ denotes the basis consisting of the incidence vectors of matchings in B.

Exercise 10.2 *Figure 15 depicts an example of a matching covered graph G and a regular vector $\mathbf{x} \in \mathbb{R}^E$. The parameters for graph G are $m = 15$, $n = 10$ and $b = 2$. Thus, $\dim(\mathcal{L}in(G)) = 5$. Find a basis of $\mathcal{L}in(G)$ consisting of 5 perfect matchings and express $\mathbf{x}$ as a linear combination of these five matchings.*

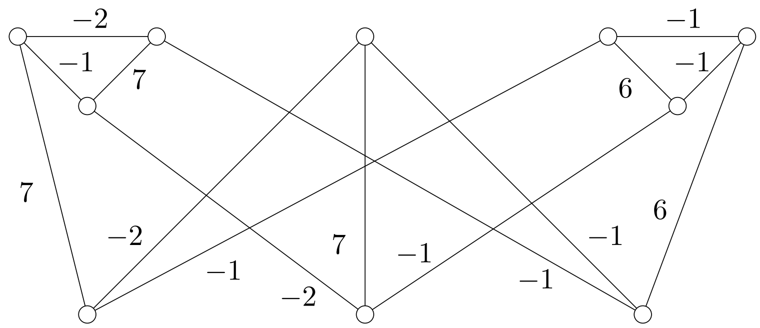


Figure 15: An illustration of Theorem 10.1

Exercise 10.3 *For the graph G of Figure 15 give an example of a nonregular vector $\mathbf{x}$ in $\mathbb{R}^E$ such that $\mathbf{x}(\partial(v)) = \mathbf{x}(\partial(w))$ for any each pair $\{v, w\}$ of vertices of G.*

Exercise 10.4

(i) *Let r be any real number. If $\mathbf{x}$ and $\mathbf{y}$ are any two r-regular vectors in $\mathbb{R}^E$, then show that the vector $\alpha\mathbf{x} + \beta\mathbf{y}$ is also an r-regular vector, for any α and β such that $\alpha + \beta = 1$.*

(ii) *By Edmonds' theorem all non-negative 1-regular vectors are in $\mathcal{L}in(G)$. Suppose that $\mathbf{y}$ is a 1-regular vector that is not in $\mathcal{P}oly(G)$. Express $\mathbf{y}$ as a linear combination of two vectors in $\mathcal{P}oly(G)$ and thereby show that $\mathbf{y}$ is also $\mathcal{L}in(G)$. (Hint: Take $\mathbf{x}$ to be any strictly positive vector in $\mathcal{P}oly(G)$ (such a vector must exist because every edge of G is in a perfect matching). Then, by the first part, the vector $\mathbf{z} := (1 - \epsilon)\mathbf{x} + \epsilon\mathbf{y}$ is 1-regular for any real number ϵ. Clearly, for small enough values of ϵ, the vector $\mathbf{z}$ is also non-negative, and hence is in $\mathcal{P}oly(G)$.)*

(iii) *Now suppose $r \neq 1$, and let $\mathbf{y}$ be an r-regular vector. Show that $\mathbf{y}$ is in $\mathcal{L}in(G)$ by showing that, for any 1-regular vector $\mathbf{x}$, the vector $\frac{1}{1-r}(\mathbf{x} - \mathbf{y})$ is 1-regular.*

Exercise 10.5

(i) *Let $\mathbf{T}$ denote the matrix whose rows are the incidence vectors of all tight cuts of a matching covered graph G. For any fixed real number r, show that the set of all r-regular vectors of G is the set of solutions to the system $\mathbf{Tx} = \mathbf{r}$ of linear equations, where $\mathbf{r}$ is a column vector each of whose entries is r.*

(Thus, the set of all regular vectors is the set of solutions to the system $\mathbf{Tx} = \mathbf{r}$, where r is treated as a variable).

When G is either a brace or a brick, all tight cuts are trivial. Thus, in this case, $\mathbf{T}$ is the same as the incidence matrix $\mathbf{A}$ of G. Consequently, $\mathcal{L}in$ is the set of solutions to the system $\mathbf{Ax} - \mathbf{r} = \mathbf{0}$ of homogeneous linear equations in $m+1$ variables, and its dimension is $(m+1) - \mathrm{rank}(\mathbf{A})$.)

(ii) *Show that the rank of $\mathbf{A}$ is $n-1$ when G is bipartite, and is n, when G non-bipartite, and deduce the validity of the dimension formula for braces and bricks.*

10.3 Robust Cuts and Regularity

The regularity of a vector $\mathbf{x} \in \mathbb{R}^E$ is obviously necessary for $\mathbf{x}$ to be in $\mathcal{L}in(G)$. Let us now consider matching covered graphs G that are either bipartite or near-bricks. That is, $b(G) \leq 1$. Let $\mathbf{x}$ be a vector in $\mathbb{R}^E$ and $r \in \mathbb{R}$ such that $\mathbf{x}(\partial(v)) = r$ for any each vertex v of G. For any tight cut C of G, the inequality $b(G) \leq 1$ implies that one C-contraction of G is bipartite, therefore $\mathbf{x}(C) = r$. In other words, whenever $b(G) \leq 1$ the "regularity on the vertices" implies regularity over all tight cuts. We then have the following very important observation.

LEMMA 10.6 *Let G be a brick, let C be a robust cut of G, let $\mathbf{x} \in \mathbb{R}^E$ be an r-regular vector, $r \in \mathbb{R}$. If $\mathbf{x}(C) = r$ then the restriction of $\mathbf{x}$ to each C-contraction of G is r-regular.*

10.4 The Matching Lattice

$\mathcal{L}at(G)$ denotes the set of integral linear combinations of characteristic vectors of perfect matchings of a matching covered graph G. That is,

$$\mathcal{L}at(G) := \sum_{M \in \mathcal{M}} \alpha(M)\chi^M,$$

where each coefficient $\alpha(M)$ is an integer. Certainly,

$$\mathcal{L}at(G) \subseteq \mathbb{Z}^E \cap \mathcal{L}in(G).$$

If equality held then the characterizations and results for $\mathcal{L}in(G)$ could be easily adapted to $\mathcal{L}at(G)$. However, equality does not hold: let P be the Petersen graph. It has precisely six perfect matchings, and each edge lies in precisely two perfect matchings. Thus the vector $\mathbf{x} \in \mathbb{R}^E$ of all 1's is equal to $\sum_{M \in \mathcal{M}} \frac{1}{2}\chi^M$. So, $\mathbf{x}$ lies in $\mathcal{L}in(P)$. By Theorem 10.1, $\dim(\mathcal{L}in(P)) = 6$, therefore the linear combination is unique. We deduce that $\mathbf{x}$ does not lie in $\mathcal{L}at(P)$.

We now state a fundamental result. A *Petersen brick* is a brick whose underlying simple graph is the Petersen graph.

THEOREM 10.7 (LOVÁSZ [17, 18]) *Let G be a matching covered graph. The dimension of $\mathcal{L}at(G)$ satisfies the equality*

$$\dim(\mathcal{L}at(G)) = \dim(\mathcal{L}in(G)) = m - n + 2 - b.$$

Moreover, if no brick of G is a Petersen brick then a vector $\mathbf{x} \in \mathbb{Z}^E$ lies in $\mathcal{L}at(G)$ if and only if $\mathbf{x}$ is regular.

10.5 Two Important Conjectures

Let G be a matching covered graph. An edge e of G is *removable* if $G - e$ is also matching covered.

THEOREM 10.8 (LOVÁSZ) *Every brick distinct from K_4 and $\overline{C_6}$ has a removable edge.*

An edge e of G is *b-invariant* if e is removable and $b(G-e) = b(G)$. Figure 16 shows an edge e that is removable but not b-invariant and also a b-invariant edge f. It is important to notice that the Petersen graph has no b-invariant edge.

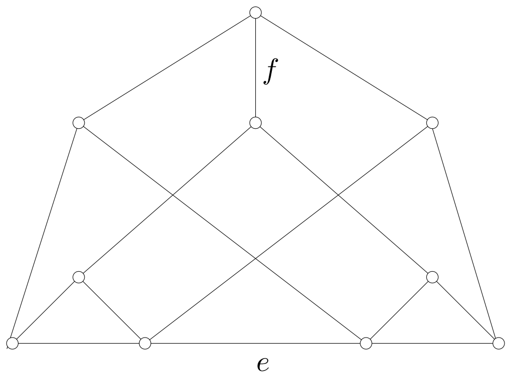


Figure 16: Edge e is removable in the brick G, but $b(G - e) = 2$; edge f is b-invariant

Lovász considered the possibility of a simpler proof of Theorem 10.7 if the following result were true:

CONJECTURE 10.9 (LOVÁSZ) *Every brick distinct from K_4, $\overline{C_6}$ and the Petersen graph has a b-invariant edge.*

Also, Murty posed the following conjecture, which extends a trivial result for $\mathcal{L}in(G)$ to $\mathcal{L}at(G)$.

CONJECTURE 10.10 (MURTY) *Every matching covered graph G has a basis for $\mathcal{L}at(G)$ that consists solely of perfect matchings.*

In his Ph. D. Thesis, Marcelo Carvalho proved Lovász' Conjecture. He in fact proved a result that is slightly stronger:

THEOREM 10.11 (CARVALHO 1996 [2], CLM 2002 [4, 5]) *Let G be a brick that is distinct from K_4 and $\overline{C_6}$. If G is not a Petersen brick then G has a b-invariant edge e such that the brick of $G - e$ is not the Petersen brick.*

10.6 Proof of Theorem 10.7 and Conjecture 10.10

Let us prove Lovász' Theorem 10.7 and Murty's Conjecture 10.10 by induction, following seminal ideas used by Seymour [24]. We shall prove the sufficiency of regularity and also the existence of bases for $\mathcal{L}at(G)$ consisting of $m - n + 2 - b$ perfect matchings. In order to reduce to braces and bricks, we use the operation of *merging*.

10.6.1 The Merger Operation

Let C be a separating cut of a matching covered graph G. We denote by $\mathcal{L}at(G, C)$ the subspace of $\mathcal{L}in(G)$ spanned by the collection of perfect matchings of G that contain just one edge in C. In particular, if C is a tight cut then $\mathcal{L}at(G, C) = \mathcal{L}at(G)$.

Let G_1 and G_2 denote the two C-contractions of G. For $i = 1, 2$, let B_i be a collection of perfect matchings of G_i such that $\mathbf{B}_i$ is a basis for $\mathcal{L}at(G_i)$. For each edge e in C, let B_i^e denote the subcollection of B_i consisting of those perfect matchings that contain edge e, let F_i^e be a fixed matching in B_i^e. See Figure 17. Let

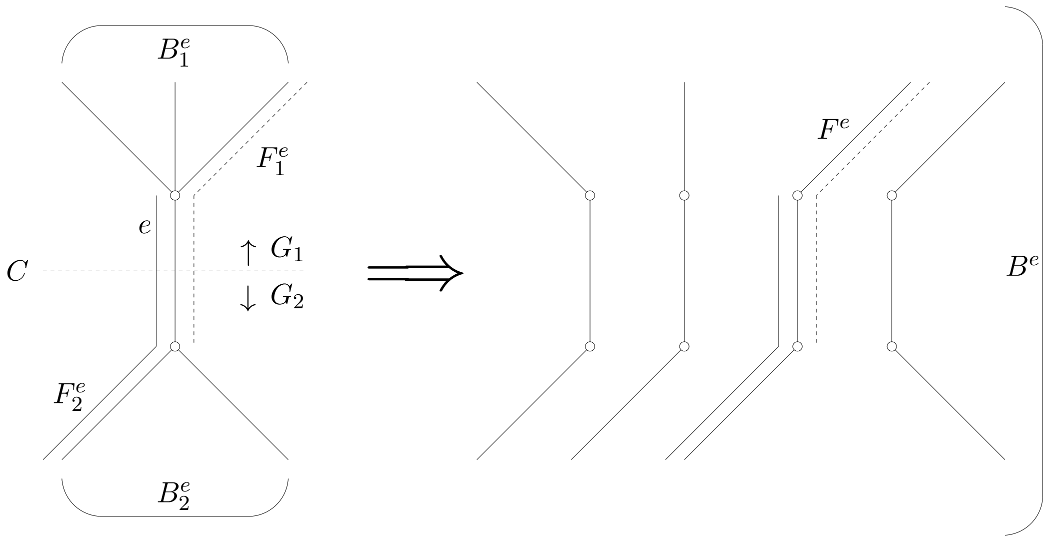


Figure 17: Illustration of the merging operation, with $|B_1^e| = 3$ and $|B_2^e| = 2$

$$B^e := \{M_1 \cup F_2^e : M_1 \in B_1^e\} \cup \{F_1^e \cup M_2 : M_2 \in B_2^e\} \quad \text{and let} \quad F^e := F_1^e \cup F_2^e.$$

Clearly, the set $B := \cup_{e \in C} B^e$ is a set of perfect matchings of G. We shall denote it by $B_1 \vee B_2$ and refer to it as the *merger* of B_1 and B_2.

It follows that $|B^e| = |B_1^e| + |B_2^e| - 1$, since F^e is counted twice in the sum $|B_1^e| + |B_2^e|$. Consequently,

$$|B_1 \vee B_2| = |B_1| + |B_2| - |C|. \tag{2}$$

EXERCISE 10.12 *Prove that B^e is linearly independent. Conclude that $B_1 \vee B_2$ is linearly independent.*

LEMMA 10.13 *Assume that, for $i = 1, 2$, the restriction $\mathbf{x}_i$ of a vector $\mathbf{x} \in \mathbb{Z}^E$ to G_i lies in $\mathcal{L}at(G_i)$. Then, $\mathbf{x}$ lies in $\mathcal{L}at(G, C)$ and is an integral linear combination of matchings in $B_1 \vee B_2$.*

Proof: By hypothesis, $\mathbf{B}_i$ is a basis of $\mathcal{L}at(G_i)$. Thus, there exist integral coefficients $\alpha(M)$, $M \in B_i$, such that

$$\mathbf{x}_i = \sum_{M \in B_i} \alpha(M) M \quad i = 1, 2.$$

It follows that

$$\mathbf{x} = \sum_{e \in C} \Big[\sum_{M \in B_1^e} \alpha(M)(M \cup F_2^e) + \sum_{M \in B_2^e} \alpha(M)(F_1^e \cup M) - x(e) F^e \Big].$$

Consequently, $\mathbf{x}$ lies in $\mathcal{L}at(G, C)$ and is an integral linear combination of matchings in $B_1 \vee B_2$. $\square$

COROLLARY 10.14 *The merger $B_1 \vee B_2$ is a basis for $\mathcal{L}at(G, C)$.*

Proof: Let $\mathbf{x} \in \mathcal{L}at(G, C)$. Then, $\mathbf{x}$ is an integral linear combination of perfect matchings of G that contain precisely one edge in C. Thus, the restrictions $\mathbf{x}_i$ of $\mathbf{x}$ to G_i lie in $\mathcal{L}at(G_i)$. It follows that $B_1 \vee B_2$ spans $\mathbf{x}$. This conclusion holds for each $\mathbf{x} \in \mathcal{L}at(G, C)$. $\square$

For $i = 1, 2$, let m_i, n_i and b_i denote, respectively, the number of edges, vertices and bricks of G_i. By induction, $|B_i| = m_i - n_i + 2 - b_i$. Then,

$$\begin{aligned} \dim(\mathcal{L}at(G, C)) &= |B_1 \vee B_2| = |B_1| + |B_2| - |C| \\ &= (m_1 - n_1 + 2 - b_1) + (m_2 - n_2 + 2 - b_2) - |C| \\ &= (m_1 + m_2 - |C|) - (n_1 + n_2 - 2) + 2 - (b_1 + b_2) \\ &= m - n + 2 - (b_1 + b_2). \end{aligned} \tag{3}$$

10.6.2 Reduction to Bricks and Braces

Let C be a nontrivial tight cut of G, let G_1 and G_2 denote the two C-contractions of G. From (3), we deduce that $\mathcal{L}at(G)$ has a basis consisting of $m - n + 2 - b$ perfect matchings.

Assume that a vector $\mathbf{x} \in \mathbb{Z}^E$ is r-regular, $r \in \mathbb{Z}$. Every tight cut of G_i is also a tight cut of G. Thus, the restriction $\mathbf{x}_i$ of $\mathbf{x}$ to G_i is r-regular. Assume also that no brick of G_i is a Petersen brick. By induction, $\mathbf{x}_i$ lies in $\mathcal{L}at(G_i)$. By Lemma 10.13, $\mathbf{x}$ lies in $\mathcal{L}at(G)$.

Using induction and the merger operation, we have reduced the proof of Theorem 10.7 to the case where G is a brace or a brick.

10.6.3 Reduction to Braces and Solid Bricks

We now use the merger operation and the fact that a nonsolid brick has a robust cut to reduce the problem to braces and solid bricks.

Let G be a nonsolid brick. If G is the Petersen graph then its set of six perfect matchings are linearly independent. Moreover, $m - n + 1 = 6$, the assertion holds.

If G is a Petersen brick, we may "coalesce" parallel edges and use the previous case to prove that it also has a basis consisting of $m - 9$ perfect matchings.

We may thus assume that G is not a Petersen brick. The following result is a fundamental property of bricks which are not Petersen bricks.

THEOREM 10.15 (CARVALHO 1996 [2], CLM 2002 [4, 5]) *Let G be a brick that is not a Petersen brick. Then, G has a robust cut C such that (i) the bricks of the C-contractions of G are not Petersen bricks and (ii) G has a perfect matching M_0 such that $|M_0 \cap C| = 3$.*

Figure 18 illustrates a brick G and a robust cut C as in the statement of Theorem 10.15.

Let C be a robust cut as indicated in Theorem 10.15. Let G_1 and G_2 denote the two C-contractions of G. Let M_0 be a perfect matching of G that contains precisely three edges in C. By induction and (3), $B_1 \vee B_2$ is a basis for $\mathcal{L}at(G, C)$ consisting of $m - n$ perfect matchings. Add to that collection M_0, thereby obtaining a collection B of $m - n + 1$ perfect matchings. The matching M_0 is not spanned by $B_1 \vee B_2$. Thus, B is linearly independent.

Let $\mathbf{x} \in \mathbb{Z}^E$ be an r-regular vector. Let

$$\beta := \frac{\mathbf{x}(C) - r}{|M_0 \cap C| - 1} \quad \text{and let} \quad \mathbf{y} := \mathbf{x} - \beta \cdot \mathbf{M_0}. \tag{4}$$

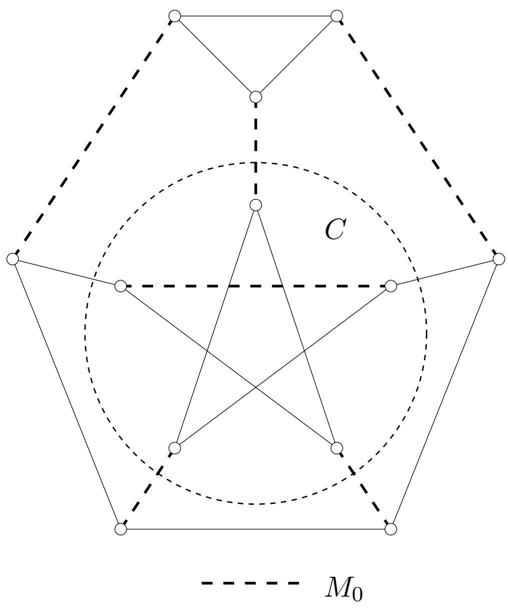


Figure 18: An illustration of Theorem 10.15

Clearly, $\mathbf{x}(C)$ and r have the same parity. It follows that β is integral. Moreover, $\mathbf{y}(C) + \beta = r$, whence $\mathbf{y}$ is s-regular, where $s = r - \beta$. Also, $\mathbf{y}(C) = s$. As G_i is a near-brick, the restriction $\mathbf{y}_i$ of $\mathbf{y}$ to G_i is s-regular. By induction, $\mathbf{y}_i$ lies in $\mathcal{L}at(G_i)$. By Lemma 10.13, $\mathbf{y}$ is spanned by $B_1 \vee B_2$. Thus, $\mathbf{x}$ is spanned by B and lies in $\mathcal{L}at(G)$. We conclude that B is a basis of $\mathcal{L}at(G)$ and spans every regular vector in $\mathbb{Z}^E$.

Again, by the use of induction and the merger operation we have advanced more, now we are left with the case in which G is either a brace or a solid brick.

10.6.4 Braces and Solid Bricks

THEOREM 10.16 (LOVÁSZ [17, 19]) *Every brace distinct from K_2 and C_4 has a removable edge. Every brick distinct from K_4 and $\overline{C_6}$ has a removable edge.*

THEOREM 10.17 (CARVALHO 1996 [2], CLM 2002 [4]) *Let e be a removable edge of a solid brick G. Then, e is b-invariant and $G - e$ is solid.*

The assertion of Theorem 10.7 holds trivially for K_2, C_4 and K_4. The brick $\overline{C_6}$ is not solid. For any other brace or solid brick G, let e be a b-invariant edge of G. Let M_e be any perfect matching of G that contains edge e. By induction, $\mathcal{L}at(G - e)$ has a basis consisting of $m - n + 1 - b$ perfect matchings. Add M_e to that basis, thereby obtaining a set B. Clearly, B is linearly independent, because $B \setminus \{M_e\}$ is linearly independent and none of its perfect matchings contains edge e. For any regular vector $\mathbf{x}$ in $\mathbb{Z}^E$, the restriction $\mathbf{y}$ of vector $\mathbf{x} - \mathbf{x}(e) \cdot \mathbf{M_e}$ to $E(G - e)$ is regular. Moreover, $G - e$ is a solid matching covered graph, therefore its brick is not a Petersen brick. By induction hypothesis, $\mathbf{y}$ lies in $\mathcal{L}at(G - e)$. We conclude that $\mathbf{x}$ lies in $\mathcal{L}at(G)$. Indeed, B spans every regular vector $\mathbf{x}$ in $\mathbb{Z}^E$. Thus, B is a basis of G consisting of $m - n + 1$ perfect matchings and every regular vector in $\mathbb{Z}^E$ lies in $\mathcal{L}at(G)$. The proof of Theorem 10.7 is complete.

EXERCISE 10.18 *Find a basis for the matching lattice of the graphs depicted in Figures 15, 16 and 18.*

11 Algorithmic Proof

The proof we gave so far may not yield a polynomial algorithm for determining a basis for $\mathcal{L}at(G)$. The reason is that we do not know how to find separating cuts efficiently (Problem 4.6).

In his Ph. D. thesis. [2], (see also [4, 5]) Marcelo Carvalho proved the following fundamental result, fully solving Conjecture 10.9. We denote by $(b+p)(G)$ the invariant consisting of the number of bricks of G, where the Petersen bricks are counted twice.

THEOREM 11.1 (CARVALHO 1996 [2], CLM 2002 [4, 5]) *Every brick G distinct from K_4 and $\overline{C_6}$ has a removable edge e such that $(b+p)(G-e) = (b+p)(G)$.*

With this result, we get in polynomial time a basis for $\mathcal{L}at(G)$ for every matching covered graph G, as (i) Theorem 5.1 indicates how to find tight cuts in polynomial time, (ii) there are polynomial time algorithms for determining whether a graph has a perfect matching [12].

12 Open Problems

PROBLEM 12.1 *Is solidity of a brick in P? Is it in NP?*

We know that solidity of bricks is in co-NP. Kawarabayashi and Ozeki [15] have a characterization of odd intercyclic graphs that gives a polynomial algorithm for the recognition of odd intercyclic graphs.

CONJECTURE 12.2 (MURTY) *There exists a constant c such that every simple solid graph of order $2n \geq c$ has at most n^2 edges and this limit is only attained by $K_{n,n}$*

CONJECTURE 12.3 (MURTY) *There exists a constant k such that every simple solid brick of order n has at most kn edges.*

CONJECTURE 12.4 (MURTY) *Every cubic solid brick is odd intercyclic.*

13 Pfaffian Orientations

THEOREM 13.1 (KASTELEYN (1963) [14]) *Every planar graph G has an orientation D such that determinant of the adjacency matrix of D is equal to the square of the number of perfect matchings of G.*

Sign of a perfect matching: Suppose that $(1, 2, \ldots, 2k)$ is an enumeration of the vertices of a digraph D. With each perfect matching $M = \{e_1, e_2, \ldots, e_k\}$, where, for $1 \leq i \leq k$, u_i and v_i denote, respectively, the tail and the head of e_i, we associate the permutation $\pi(M)$, where:

$$\pi(M) := \begin{pmatrix} 1 & 2 & 3 & 4 & . & . & . & 2k-1 & 2k \\ u_1 & v_1 & u_2 & v_2 & . & . & . & u_k & v_k \end{pmatrix}$$

The *sign* of M, denoted by $\mathrm{sign}(M)$, is the sign of the permutation $\pi(M)$. (The sign of M does not depend on the order in which the edges of M are listed. But it does depend on the enumeration of the vertices. However, the effect of permuting the labels, say by α, on $\mathrm{sign}(M)$, merely consists of multiplying $\mathrm{sign}(M)$ by the sign of α.)

Pfaffian of the adjacency matrix $\mathbf{A} = (a_{ij})$ of D:

$$\mathrm{Pf}(\mathbf{A}) := \sum \mathrm{sign}(M)\ a_{u_1v_1}a_{u_2v_2}\ldots a_{u_kv_k} \tag{5}$$

where the sum is taken over the set of all perfect matching of D.
Pfaffian orientation: An orientation D of G is called a *Pfaffian orientation* if all perfect matchings in D have the same sign. In this case, $|\mathrm{Pf}(\mathbf{A})|$ is the number of perfect matchings of D.
A determinantal identity: For a skew-symmetric matrix $\det \mathbf{A} = (\mathrm{Pf}(\mathbf{A}))^2$. Thus, if a graph G has a Pfaffian orientation D, the number of perfect matchings of G can be computed by computing the determinant of $\mathbf{A}(D)$ and taking its square root.
The Pfaffian Recognition Problem:
Given: A digraph D
Decide: whether or not D is Pfaffian.
The Pfaffian Orientation Problem:
Given: A graph G
Decide: whether or not G has a Pfaffian orientation.

Theorem 13.2 (Vazirani and Yanakakis (1989) [28], CLM (2005) [8])
A polynomial-time algorithm for one of the problems implies a polynomial-time algorithm for the other problem.

Pfaffian orientations of an even cycle: An orientation of an even cycle is *odd* if an odd number of edges of are oriented 'clockwise', and the rest of the edges (also odd in number) are oriented 'anti-clockwise'.

An orientation of an even cycle is Pfaffian iff it is odd.

Lemma 13.3 (Sign-Product Lemma) *Let D be a directed graph, and let M_1 and M_2 be two perfect matchings of D. Let k denote the number evenly directed M_1M_2-alternating cycles in D. Then $\mathrm{sign}(M_1)\mathrm{sign}(M_2) = (-1)^k$. (See Lemma 8.3.1, Lovász and Plummer's book [19].)*

Corollary 13.4 *Let D be a digraph and let M be a perfect matching of D. Then, D is Pfaffian iff each M-alternating cycle is oddly oriented.*

Conformal subgraphs: A subgraph H of a graph G is *conformal* if $G-V(H)$ has a perfect matching. An 8-cycle in the Petersen graph is conformal, but a 6-cycle is not.

Corollary 13.5 *A digraph D is Pfaffian iff each of its conformal even cycles is oddly oriented.*

Example 13.6 *The* Heawood graph *is the smallest cubic graph of girth six. It is bipartite, and has no conformal cycles of lengths* 8 *or* 12*. Let (A, B) be a bipartition of this graph. Then, the orientation which directs all edges from A to B is a Pfaffian orientation. To see this, let M denote the perfect matching consisting of the seven vertical edges. Then, in an M-alternating cycle of length $2k$, the k edges of M are oriented the same way, but the remaining k edges are oriented the other way. When k is even, such a cycle would be evenly oriented, but the Heawood graph does not have conformal cycles of length* 8*, or* 12*!*

Pfaffian Orientations of Planar Graphs: An orientation of a 2-connected plane graph G is *odd* if each facial cycle of G, except possibly the cycle bounding the outer face, is oddly oriented. Such an orientation can be found using an appropriate ear decomposition of G.

LEMMA 13.7 *If D is an odd orientation of a 2-connected plane graph G and Q is a cycle of G then the parity of Q in D is the opposite of the parity of the number of vertices of G that lie in the interior of Q.*

COROLLARY 13.8 *Every odd orientation of a 2-connected plane graph G is a Pfaffian orientation of G.*

EXERCISE 13.9 *Prove Lemma 13.7*

Reversing the orientations on the edges of a cut: Let D be an orientation of a graph D, and let $C := \partial(X)$ be a cut of G. Then D rev C denotes the digraph obtained from D by reversing the orientations on all the edges in the cut C.
Similarity of Orientations: Two orientations D and D' of G are *similar* if $D' = D$ rev C, for some cut C.

EXERCISE 13.10 *Prove that for any sets X and Y of vertices of a graph G, $\partial(X \oplus Y) = \partial(X) \oplus \partial(Y)$, where $\oplus$ is the operation of symmetric difference. Deduce that similarity is an equivalence relation on the set of all possible orientations of G.*

If an orientation is Pfaffian, then any orientation similar to it is also Pfaffian. As an application of the above statement one can show the following: Suppose that G is a Pfaffian graph and that T is a spanning tree of G. Let $\overrightarrow{T}$ be any orientation of T. Then $\overrightarrow{T}$ can be extended to a Pfaffian orientation $\overrightarrow{G}$ of G.
Pfaffian Orientations of Bipartite Graphs A variety of algorithmic problems in graph theory and matrix theory are reducible to the problem of recognizing whether or not a given directed bipartite graph is Pfaffian. One such problem is the following:
The Even Directed Cycle Problem:
Given: A (strict) digraph D;
Decide: whether or not D has a directed cycle of even length.

The above problem may be reduced to the Pfaffian Recognition Problem as follows. Let D be any digraph. Obtain the bipartite digraph $B(D)$ from D. (Recall how the vertex-disjoint path problem is converted to an arc-disjoint path problem.)

EXERCISE 13.11 *Prove that D has an even cycle iff $B(D)$ is not Pfaffian.*

A good example to try is the Koh-Tindell digraph (see Bondy and Murty's book). This graph has no even cycles. The bipartite digraph associated with the Koh-Tindell digraph is the Heawood graph with the orientation described earlier.

Similarity of Pfaffian Orientations for Bipartite Graphs

EXERCISE 13.12 *Let G be a Pfaffian bipartite matching covered graph, let e be an edge of G such that $G-e$ is (Pfaffian, bipartite) matching covered graph. Assume that any two Pfaffian orientations of $G-e$ are similar. Prove that any two Pfaffian orientations of G are similar. Conclude that any Pfaffian orientation of $G-e$ may be extended to a Pfaffian orientation of G.*

The Ear Decomposition Theorem: Given any bipartite matching covered graph G, there exists a sequence
$(G_1, G_2, \ldots, G_r)$ of bipartite matching covered graphs such that: (i) $G_1 = K_2$, $G_r = G$, and (ii) for $1 \le i < r$, G_{i+1} is obtained from G_i by adding an odd ear.

With the aid of the above theorem, one can now show that any two Pfaffian orientations of a Pfaffian bipartite matching covered graph are similar.

THEOREM 13.13 *Let $G(A, B)$ be a Pfaffian bipartite matching covered graph. Then, any two Pfaffian orientations of G are similar (that is, one can be obtained from the other by reversing the orientations on the edges of some cut). Furthermore, if $G \neq K_2$, and $(G_1, G_2, \ldots, G_r)$ is an ear decomposition of G, then a Pfaffian orientation of $H := G_{r-1}$ can be extended to a Pfaffian orientation of $G = G_r$.*

Little (1975) proved that the Pfaffian orientation problem for bipartite graphs is in co-$\mathcal{NP}$ by showing that, in a certain sense, $K_{3,3}$ is the only minimal non-Pfaffian bipartite graph. To make this precise, we need to define the operations of contractions and deletions appropriate for matching covered graphs.

THEOREM 13.14 *[Little and Rendl (1991) [16]]Let G be a matching covered graph and C a tight cut in G. Then G is Pfaffian iff both C-contractions of G are Pfaffian*

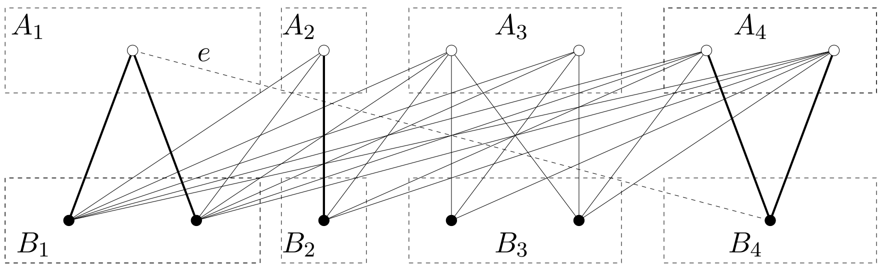
COROLLARY 13.15 *A bipartite matching covered graph is Pfaffian if and only if each of its braces is Pfaffian.*

Removable edges: An edge e of a matching covered graph G is *removable* if $G - e$ is also matching covered.

We shall make use of the following lemmas concerning removable edges in the proof of Little's theorem.

LEMMA 13.16 *Every edge of a brace on six or more vertices is removable.*

The graph obtained by removing an edge e from a brace G need not be a brace. If it is not, it has a 'nested family' of tight cuts. This is illustrated in the figure below. The only non-removable edges of $G - e$ are the edges indicated by solid lines.



LEMMA 13.17 *Let G be a brace on six or more vertices, and let e be an edge of G. Suppose that v is a vertex of G. If the degree of v is greater than two in $G - e$, then at most one edge of G incident with v is not removable in G.*

We shall say that a matching covered graph H is a *minor* of another matching covered graph G if there exists a sequence $(G_1, G_2, \ldots, G_r)$ of graphs such that (i) $G_1 = G$, $G_r = H$, and (ii) for $0 \leq i < r$, G_{i+1} is obtained from G_i by either deleting a removable edge, or by contracting a shore of a nontrivial tight cut to a single vertex.

EXERCISE 13.18 *Prove that a bipartite matching covered graph is Pfaffian iff all its minors are Pfaffian.*

THEOREM 13.19 (LITTLE'S THEOREM) *The only minor-minimal non-Pfaffian bipartite matching covered graph is $K_{3,3}$.*

Proof: (Sketch [8]) Let G be a minor-minimal non-Pfaffian bipartite graph on $\{1, 2, \ldots, 2n\}$. By Theorem 13.14 , G is a brace. Since all braces on fewer than six vertices are Pfaffian, it follows that G has at least six vertices. Also, since all multiple edges are removable, G is simple.

Adjust the enumeration of $V(G)$ so that edge $e = \{2n-1, 2n\}$ is an edge of G. By the minimality of G, it follows that $G - e$ is Pfaffian. Let D' be a Pfaffian orientation of $G - e$. Note that all perfect matchings of G which do not contain e have the same sign, say positive, in D'.

e-pairs: A pair $\{F_1, F_2\}$ of perfect matchings of $G - \{2n-1, 2n\}$ is an *e-pair* if they have different signs in $D' - \{2n-1, 2n\}$. (Such a pair must exist because if all of them are positive, then D' can be extended to a Pfaffian orientation of G by orienting e from $2n-1$ to $2n$; and if all of them are negative, then D' can be extended to a Pfaffian orientation of G by orienting e from $2n$ to $2n-1$.)

LEMMA 13.20 *Let $\{F_1, F_2\}$ be an e-pair. Then, every removable edge f of $G-e$ belongs to $F_1 \cup F_2$,*

Proof: Assume that there there is an edge $f \notin F_1 \cup F_2$ which is removable in $G - e$. We first note that this assumption implies that $G - f$ is matching covered. (Any edge of $G - f$, different from e, is admissible because it also belongs to the matching covered graph $G - e - f$. Also, e is admissible in $G - f$ because $F_1 \cup \{e\}$ is a perfect matching of $G - f$ because it does not contain f.)

Now observe that $D' - f$ is a Pfaffian orientation of $G - e - f$. This orientation of $G - e - f$ can be extended, by Theorem 19, to a Pfaffian orientation D'' of $G - f$. But then F_1 and F_2 would have the same sign in $D'' - \{2n-1, 2n\} = D' - f - \{2n-1, 2n\}$. This contradicts the hypothesis that $\{F_1, F_2\}$ is an e-pair. □

COROLLARY 13.21 *The ends of e have degree two in $G - e$.*

Proof: If the degree of an end v of e has degree three or more in $G - e$, then, by Lemma 26, some edge incident with v, say f, is removable in $G - e$. Clearly such an edge f is not in $F_1 \cup F_2$. □

COROLLARY 13.22 *The set $F_1 \cap F_2$ is empty.*

Proof: Assume to the contrary that $f \in F_1 \cap F_2$. Let v denote an end of f. Clearly, v is not an end of e. Thus, v has degree at least three in $G - e$ and, by Lemma 26, $\partial(v) - f$ contains a removable edge of $G - e$. That edge does not lie in $F_1 \cup F_2$. □

COROLLARY 13.23 *$F_1 \cup F_2$ is a single cycle (which is evenly oriented).*

Proof: Suppose that $F_1 \cup F_2$ consists of more than one cycle. As F_1 and F_2 have different signs, by Lemma 8, at least one cycle, say C, is evenly oriented. Then F_1 and $F_2' = F_1 \oplus C$ have different signs. We thus have a new e-pair $\{F_1, F_2'\}$, But this pair have common edges, which is impossible by Corollary 13.22. □

COROLLARY 13.24 *The cycle $C := F_1 \cup F_2$ is chordless.*

Proof: Suppose that C has a chord f. Then, $C \cup \{f\}$ consists of two cycles of even length. One of them, say C_1, is F_1-alternating and the other, say C_2, is F_2-alternating. As C is evenly oriented, one of C_1 and C_2 is evenly oriented. Suppose that C_1 is evenly oriented. Then, $\{F_1, F_2'\}$, where $F_2' := F_1 \oplus C_1$ is an e-pair. We again have a contradiction because $F_1 \cap F_2' \neq \emptyset$. □

To summarize, G consists of a chordless cycle on $2n-2$ vertices, the edge $e = \{2n-1, 2n\}$, and the edges linking the ends of e with the cycle. But, the ends of e have degree two in $G - e$ by Corollary 13.21. Clearly G is $K_{3,3}$. □

A polynomial algorithm

4-sums Let $G_1, G_2, \ldots, G_n$ be $n \geq 2$ graphs and let Q be a square such that $|V(G_i)| \geq 6$ and $G_i \cap G_j = Q$ for $1 \leq i < j \leq n$. The *4-sum* of the n graphs is

$$\left[\cup_{i=1}^{n} G_i\right] - R, \text{ where } R \subseteq E(Q).$$

Exercise 13.25 *Prove that if a brace is a 4-sum of two or more graphs, then each summand is a brace.*

Exercise 13.26 *Prove that if a brace G is the 4-sum of three or more braces, then G is Pfaffian iff each summand is a Pfaffian brace*

Reducible Braces A brace is *reducible* if it is a 4-sum of three or more graphs

In view of the above results, and taking into account that every planar graph is Pfaffian, in order to solve the Pfaffian recognition problem for bipartite matching covered graphs it suffices to solve it for irreducible nonplanar (simple) braces.

This fundamental result is due independently to McCuaig (2004) and to Robertson, Seymour and Thomas (1999)

Theorem 13.27 ([21, 23]) *The only simple, Pfaffian, nonplanar, irreducible brace is the Heawood graph*

This result shows how to solve the Pfaffian recognition problem for bipartite graphs in polynomial time. An example of a brace that is Pfaffian and nonplanar is the rotunda, which is a 4-sum of three cubes (Figure 19).

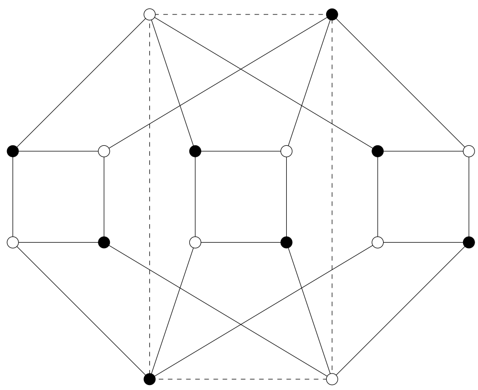


Figure 19: The rotunda is a nonplanar Pfaffian brace

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