São Paulo School of Advanced Science on Algorithms, Combinatorics and Optimization

The Perfect Matching Polytope, Solid Bricks and the Perfect Matching Lattice

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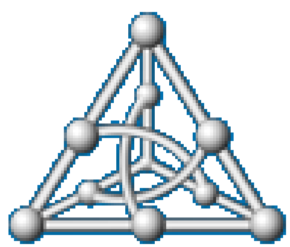


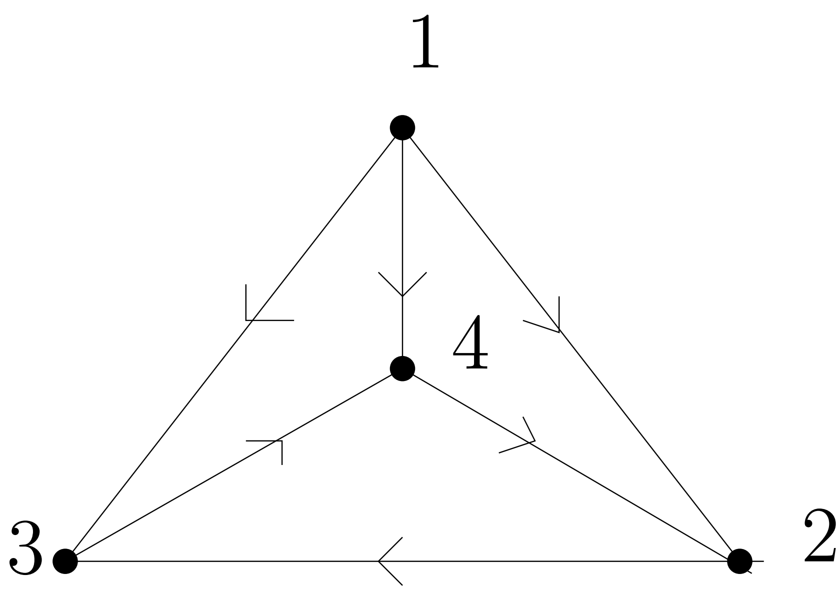
Open Problems

- Is solidity of a brick in P? Is it in NP?
  - solidity of bricks is in co-NP
  - Kawarabayashi and Ozeki's $\Rightarrow$ polynomial algorithm for the recognition of odd intercyclic graphs
- Conjecture [Murty]
  $\exists c$ s.t. $\forall$ simple solid graph G of order $2n \geq c$, $|E| \leq n^2$ with equality only if $G = K_{n,n}$
- Conjecture [Murty]
  $\exists k$ s.t. $\forall$ simple solid brick of order n, $|E| \leq kn$
- Conjecture [Murty]
  Every cubic solid brick is odd intercyclic

Pfaffian Orientations

- Theorem [Kasteleyn (1963)]
  Every planar graph G has an orientation D s.t.

$$\det(\mathrm{Adj}(D)) = |\mathcal{M}|^2$$



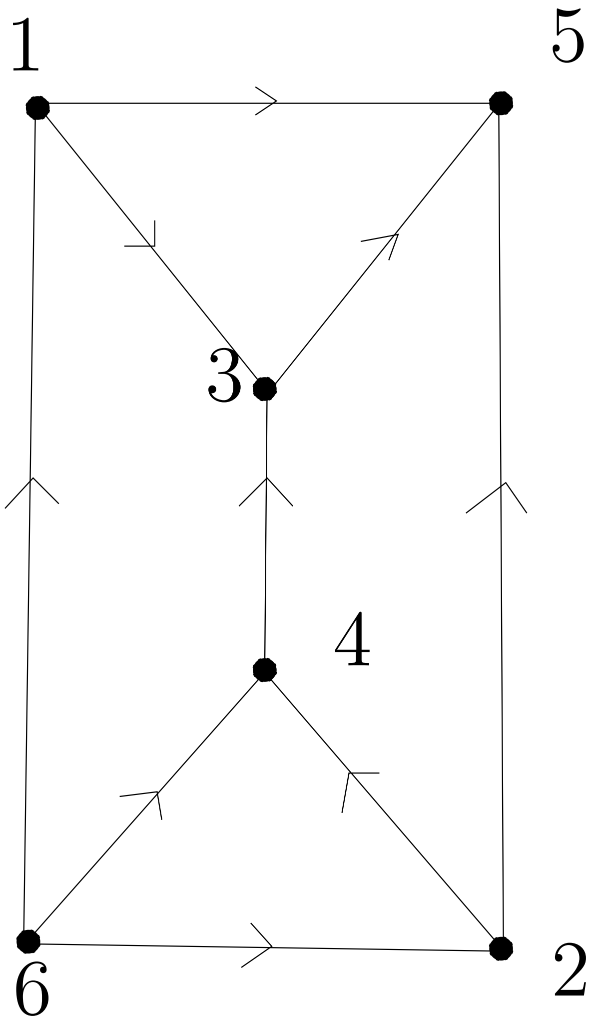
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{bmatrix}$$

- Theorem [Valiant (1979)]
  Determinantion of $|\mathcal{M}|$ is NP-hard

Pfaffian Orientations

- Not all graphs have such orientations
- Even if a graph has such an orientation, not all orientations may work
- The problem of deciding which bipartite graphs have such orientations is related to many seemingly unrelated and fundamental problems

The Sign of a Perfect Matching in a Digraph



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 3 & 6 & 2 \end{pmatrix} \qquad -$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 4 & 3 & 2 & 5 \end{pmatrix} \qquad +$$

The Pfaffian

- Let $\mathbf{A} = (a_{ij})$ denote the adjacency matrix of D.
- $\mathbf{A}$ is a *skew symmetric matrix*

$$\forall i, j \; a_{ij} = -a_{ji}$$

- The Pfaffian of $\mathbf{A}$ is:

$$\mathrm{Pf}(\mathbf{A}) = \sum_{M \in \mathcal{M}} \mathrm{sign}(M) \, a_{u_1 v_1} a_{u_2 v_2} \ldots a_{u_k v_k}$$

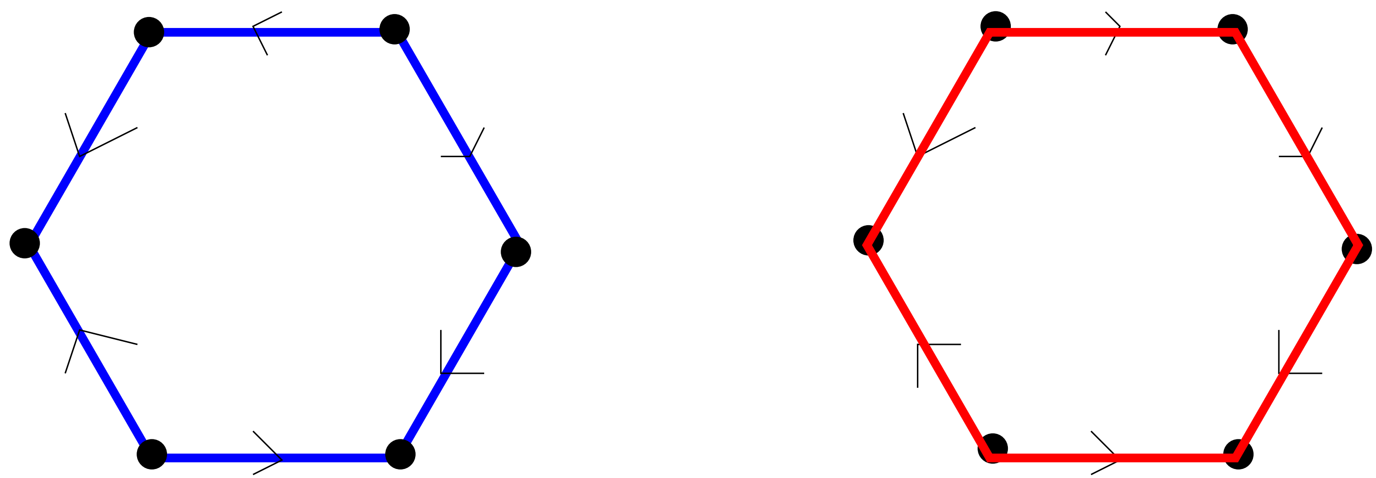
- if all pms have the same sign: $|\mathcal{M}| = |\mathrm{Pf}(\mathbf{A})|$
- Theorem [Muir (1882)]
  If $\mathbf{A}$ *is skew symmetric then* $\det(\mathbf{A}) = (\mathrm{Pf}(\mathbf{A}))^2$

Two Pfaffian Problems

- *The Pfaffian recognition problem*
  - Given: A digraph D
  - Decide: if D is Pfaffian
- *The Pfaffian orientation problem*
  - Given: A graph G
  - Decide: if G is Pfaffian (that is, it admits a Pfaffian orientation).
- Theorem [Vazirani and Yanakakis (1989)]
  A polynomial-time solution to one of the two problems implies a polynomial-time solution to the other problem

Oddly oriented cycles

- An orientation of an even cycle is *odd* if an odd number of edges are oriented in one of the senses of traversal, and the rest (also odd in number) are oriented in the other sense. Otherwise, the orientation is *even*



Sign-Product Lemma

- Lemma *Let M_1 and M_2 be two perfect matchings of a digraph, and let k be the number of evenly directed $M_1 M_2$-alternating cycles. Then*

$$\text{sign } M_1 \times \text{sign } M_2 = (-1)^k$$

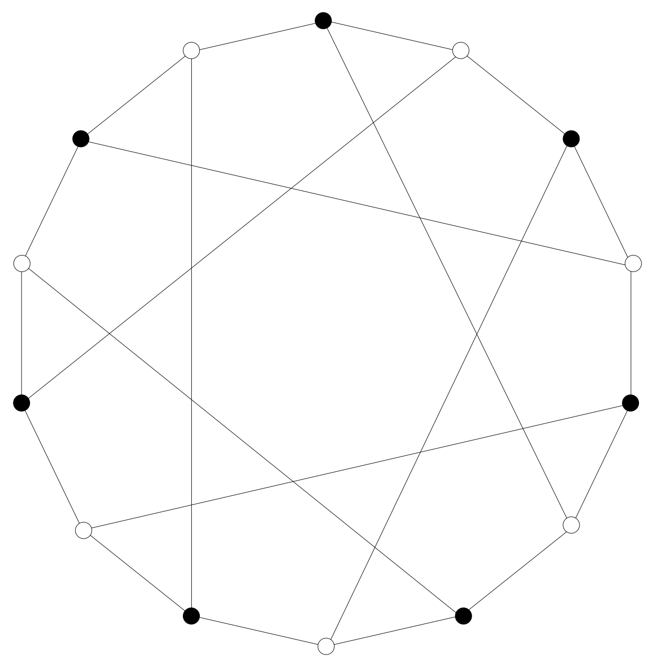
- Corollary *Let M be any perfect matching of a digraph D. Then D is Pfaffian iff all its M-alternating cycles are oddly oriented*

Conformal Subgraphs

- A subgraph H of a graph G is *conformal* if $G - V(H)$ has a perfect matching.
- Corollary *A digraph is Pfaffian iff all conformal even cycles are oddly oriented*
- J is a conformal subgraph H
- H is a conformal subgraph G
- $\Rightarrow$ J is a conformal subgraph of G
- Corollary *A digraph is Pfaffian iff all conformal subgraphs are Pfaffian*

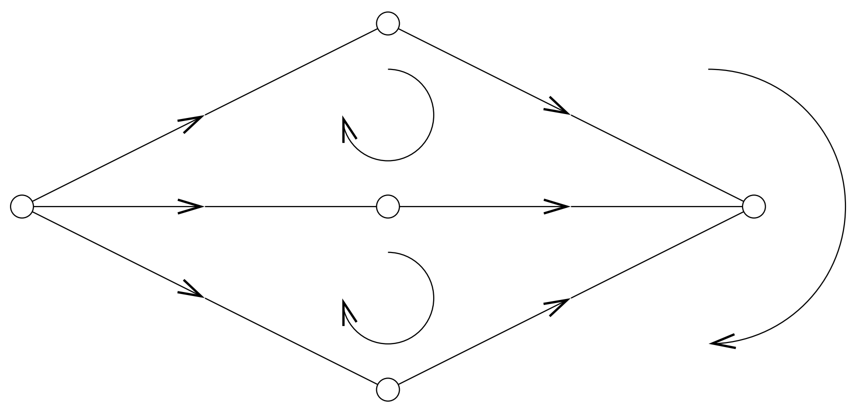
The Heawood graph is Pfaffian

- The Heawood graph is the smallest cubic graph of girth six. It is bipartite and has no conformal cycles of lengths 8 or 12



- Direct each edge from its black end to its white end

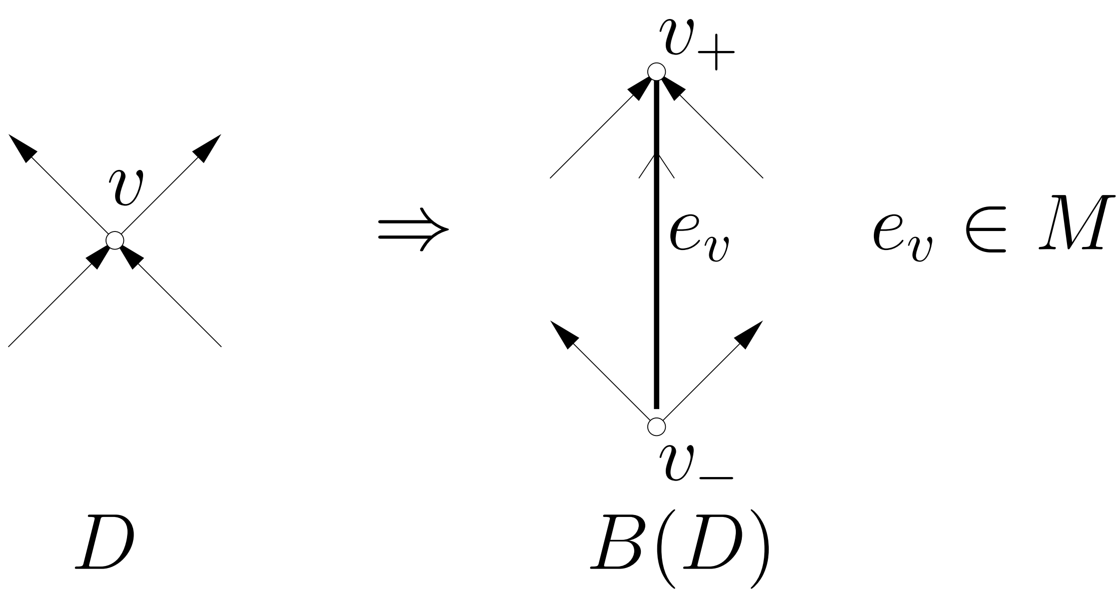
$K_{3,3}$ is not Pfaffian



- the three cycles are conformal in $K_{3,3}$
- every edge lies in precisely two of the three cycles
- every change of orientation of an edge changes the parity of two of the three cycles
- the three cycles are even
- every orientation renders an odd number of such cycles as even

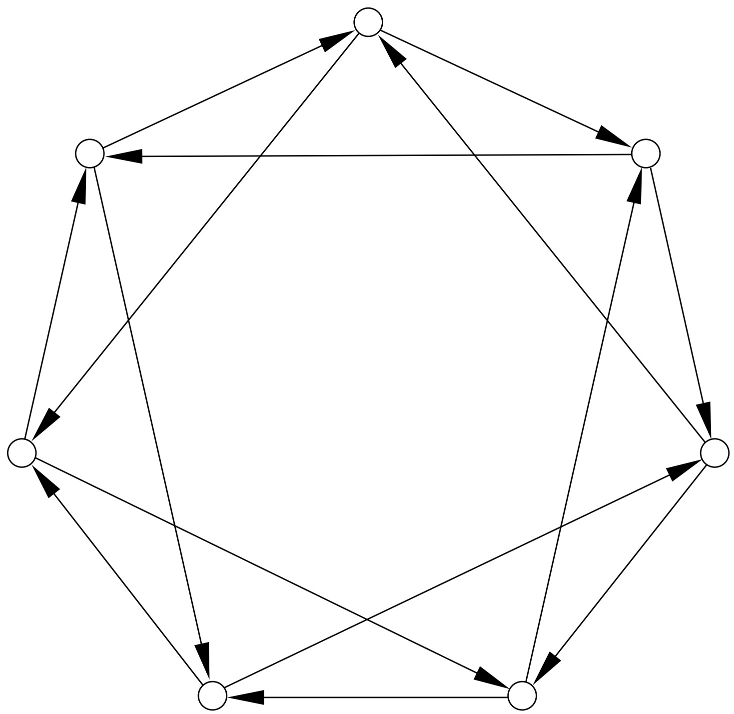
The Even Directed Cycle Problem

- Given: A digraph D
- Decide: if D has a directed cycle of even length



- D has a directed cycle of even length iff the associated bipartite digraph $B(D)$ is not Pfaffian

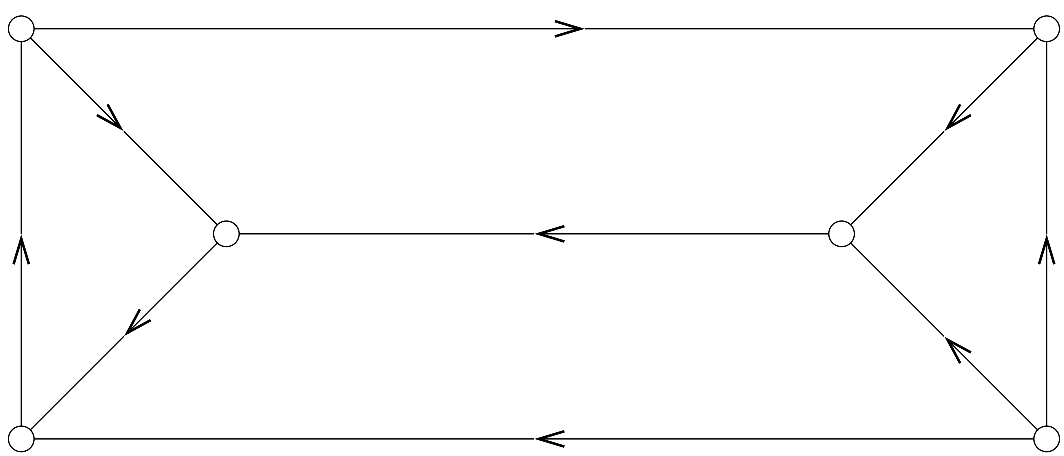
Koh-Tindell $\to$ Heawood



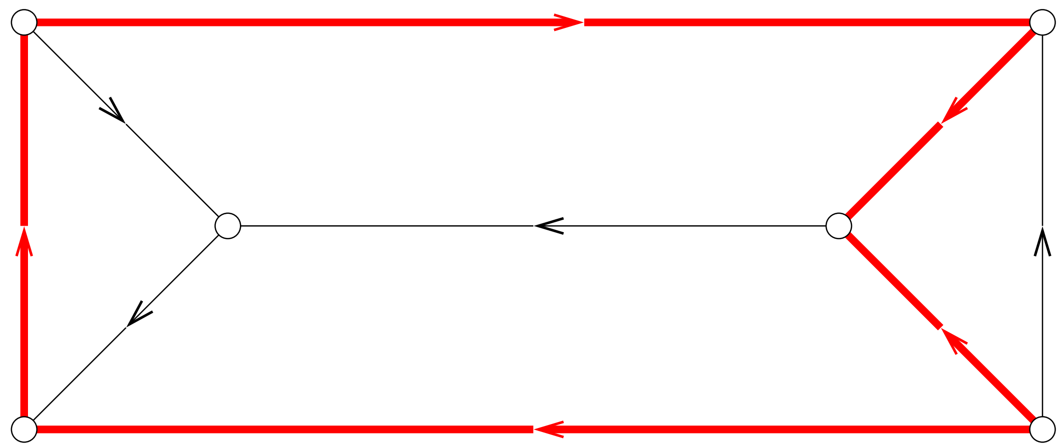
- The associated bipartite digraph of this graph is the Heawood graph with the orientation mentioned earlier

Kasteleyn's Algorithm

- An orientation D of a 2-connected plane graph G is *odd* if every facial cycle, except possibly the the one bounding the outer face, contains an odd number of edges directed clock-wisely
- An odd orientation of a 2-connected plane graph can be found inductively using an ear decomposition
- An odd orientation of a 2-connected plane graph is a Pfaffian orientation



Kasteleyn's Algorithm



- $\forall$ cycle Q,

$$|\mathrm{clock}(Q)| \equiv 1 + |I| \pmod 2,$$

where I is the number of vertices in the interior of Q

- $\therefore$ If Q is alternating then $|I|$ is even
- $\therefore$ the parity of Q is odd

Minors—Little's Theorem

- A graph H is a *minor* of a graph G if H may be obtained from G by edge deletions and tight cut contractions
- (Every graph is a minor of itself)
- Theorem [Little (1975)]
  A bipartite mc graph is Pfaffian iff it has no $K_{3,3}$-minor

A Polynomial Algorithm

- Let $G_1, G_2, \ldots, G_n$ be $n \geq 2$ graphs such that $|V(G_i)| \geq 6$ and let Q be a quadrilateral such that $G_i \cap G_j = Q$ for $1 \leq i < j \leq n$
- The *4-sum* of the n graphs is

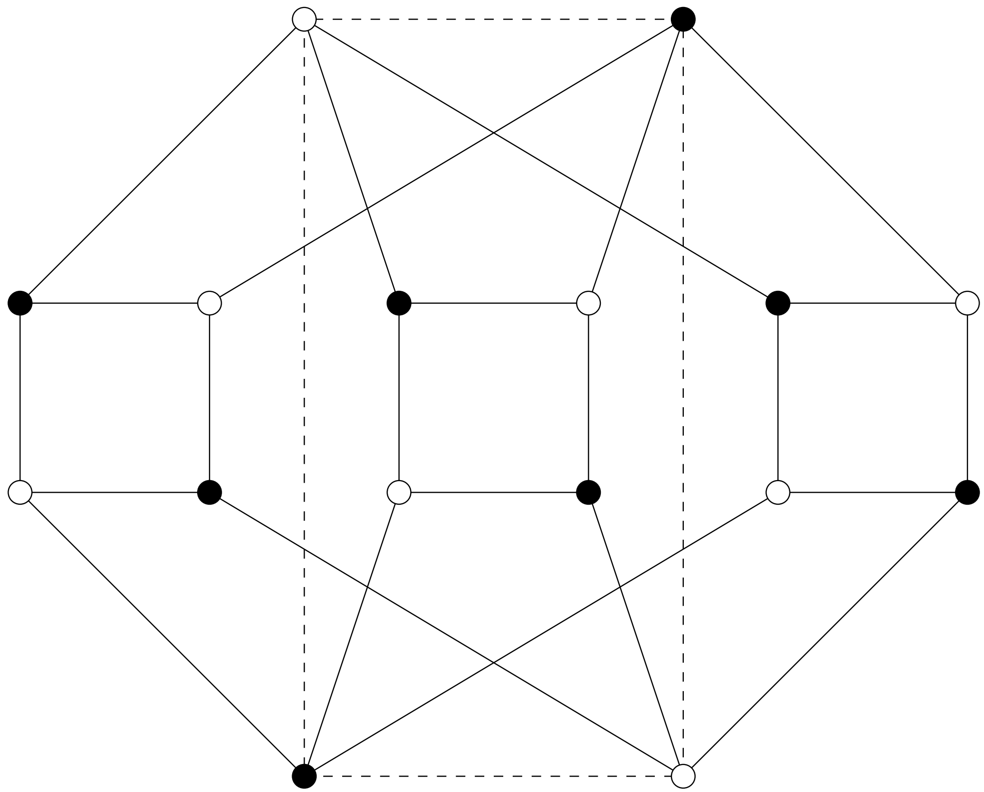
$$[\cup_{i=1}^{n} G_i] - R, \text{ where } R \subseteq E(Q)$$

- If a brace is a 4-sum of two or more graphs, then each summand is a brace
- If a brace G is the 4-sum of three or more braces, then G is Pfaffian iff each summand is a Pfaffian brace
- A brace is *reducible* if it is a 4-sum of three or more graphs

A Polynomial Algorithm

- We are thus left with (simple) irreducible braces
- Every planar brace is clearly Pfaffian
- We are thus left with (simple) irreducible nonplanar braces
- <u>Theorem</u> [McCuaig (2004) and Robertson, Seymour and Thomas (1999)]
  The only simple, Pfaffian, nonplanar, irreducible brace is the Heawood graph
- $\Rightarrow$ polynomial time algorithm for the Pfaffian recognition problem for bipartite graphs

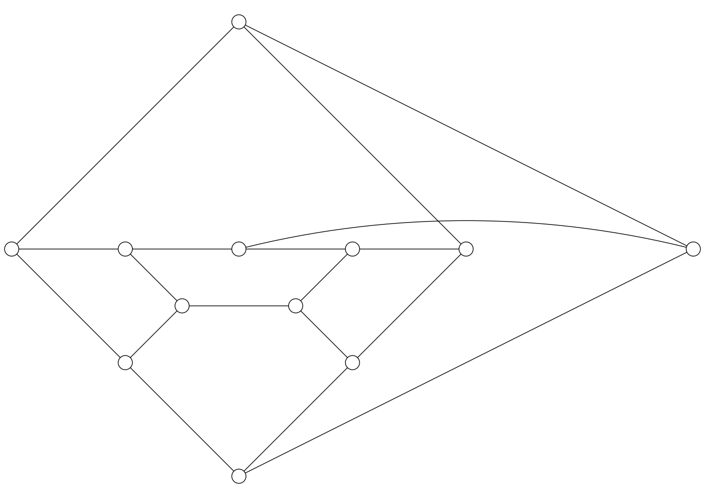
The Rotunda is Pfaffian



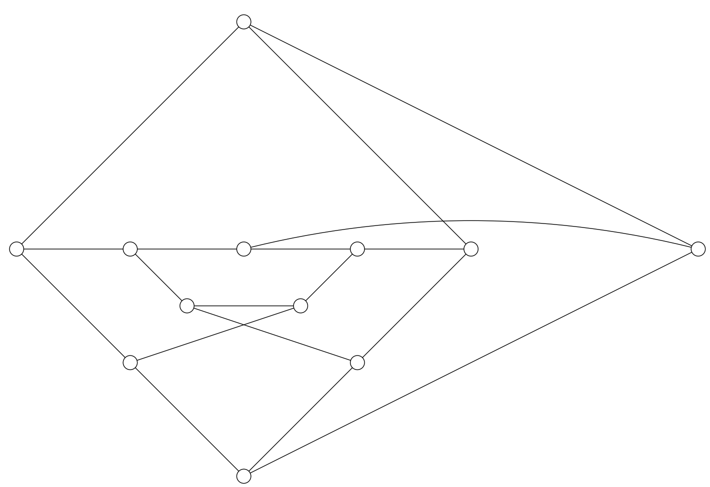
- 4-sum of three cubes, with $R = E(Q)$
- $\therefore$ the rotunda is Pfaffian

Other Characterizations of Pfaffian Graphs

- a mc graph G is *near-bipartite* if G is not bipartite but it contains two edges, e_1 and e_2, s.t. $G - e_1 - e_2$ is bipartite mc
- Theorem [Fischer and Little (2001)]
  A near-bipartite brick is non-Pfaffian iff it has a minor $\in \{K_{3,3}, \Gamma_1, \Gamma_2\}$



Γ_1



Γ_2

A Polynomial Time Algorithm

- A. Miranda and L (2010):
- Fischer and Little's characterization
- McCuaig–RST algorithm
- $\Rightarrow$ polynomial time algorithm for recognizing Pfaffian near-bipartite graphs

Other Pfaffian Graphs

- not very much is known
- Norine and Thomas (2008) have conjectured the (infinite) list of minimal non-Pfaffian bricks

thank you!