São Paulo School of Advanced Science on Algorithms, Combinatorics and Optimization

The Perfect Matching Polytope, Solid Bricks and the Perfect Matching Lattice

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4-Flows

Theorem [Tait (1880)]
A 2-connected cubic graph is 4-face-colourable iff it is 3-edge-colourable

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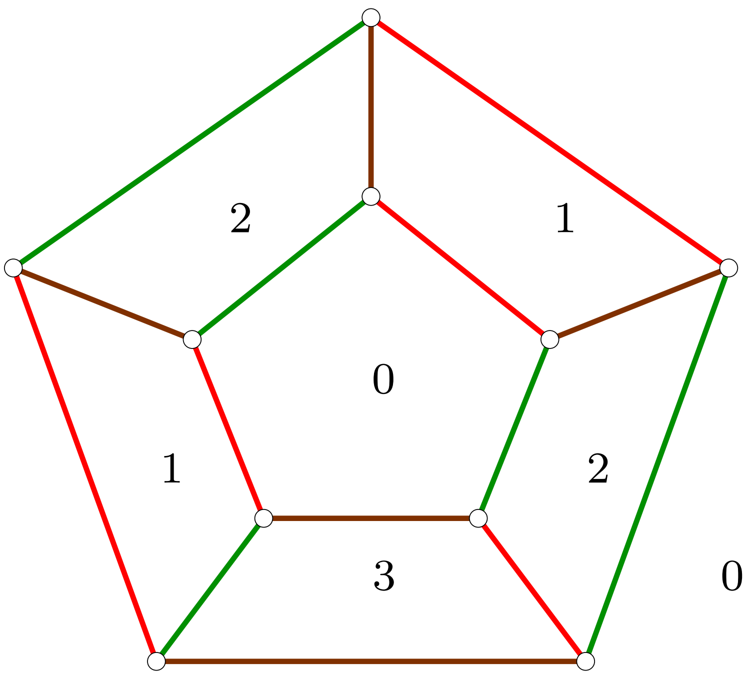

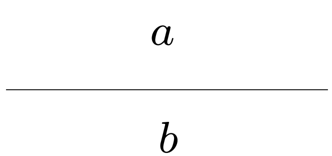

$a > b$

4-Flows

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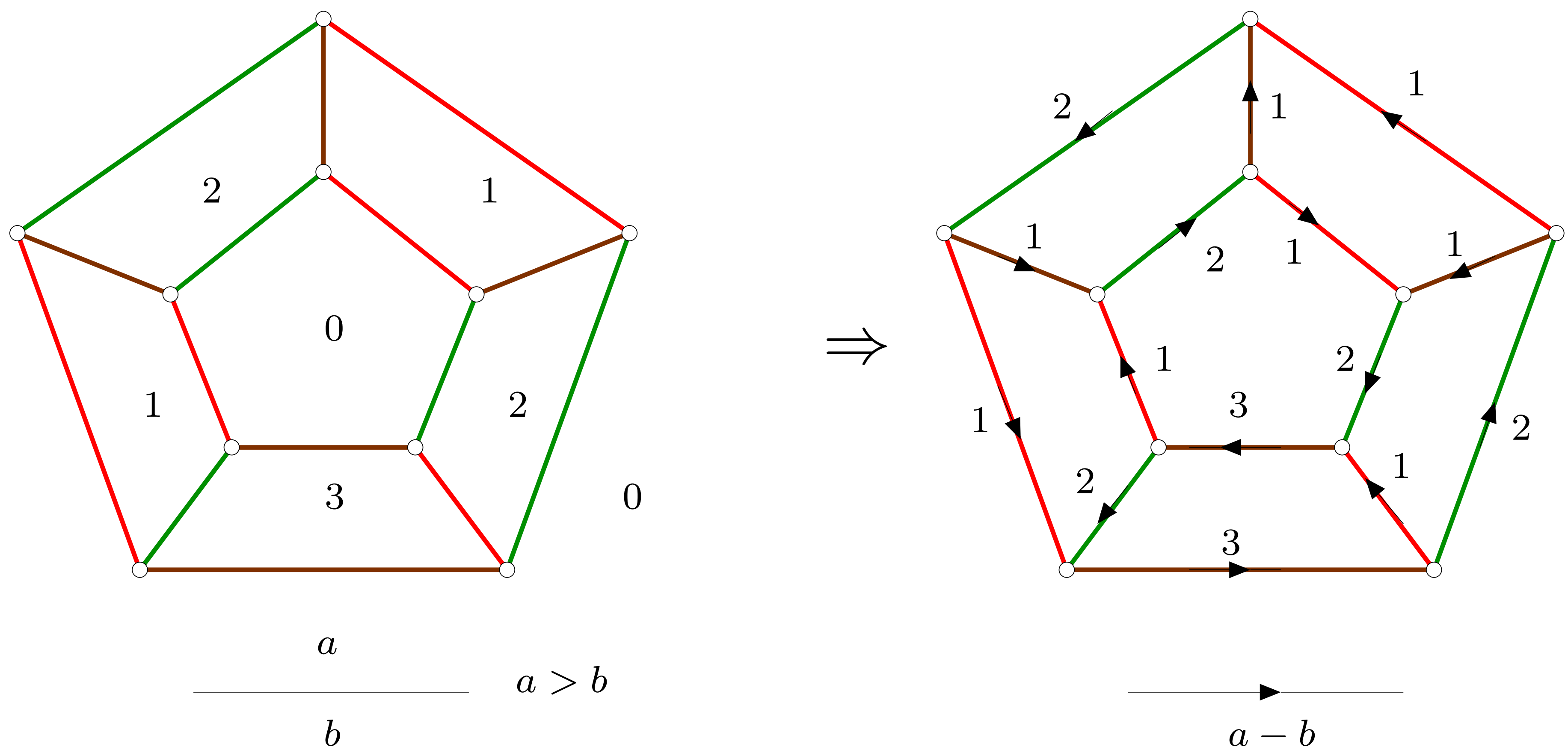
A 2-connected cubic graph is 4-face-colourable iff it is 3-edge-colourable



4-Flows

- A *4-flow* of G is an orientation D of G with weights in $\{1, 2, 3\}$ in the edges so that the in-flow equals the out-flow at each vertex
- Theorem [Tutte (1954)]
  A planar graph is 4-face-colourable iff it has a 4-flow
- H is a *minor* of G if it may be obtained from G by edge contractions and deletions

4-Flows

- Conjecture [Tutte (1966)]
  Every 2-connected graph free of Petersen minors has a 4-flow
- Theorem [Kilpatrick (1975), Jaeger (1976), Matthews (1978)]
  A 2-connected cubic graph has a 4-flow if and only if it has a 3-edge-colouring
- in her recent Ph. D. thesis, K. Edwards proved the conjecture for cubic graphs! (she coauthors a paper with Sanders, Seymour and Thomas)

The Integer Cone

- the *Integer Cone*
- $\mathcal{M}$: the set of perfect matchings of a mc graph
- $\chi^M \in 2^E$ is the incidence vector of $M \in \mathcal{M}$
- $\mathrm{Int}\mathcal{C}\mathrm{one}(G) := \sum_{\alpha_M \in \mathbb{Z}_+, M \in \mathcal{M}} \alpha_M \chi^M$
- A cubic graph G admits a 3-edge-colouring if and only if $\mathbf{1} \in \mathrm{Int}\mathcal{C}\mathrm{one}(G)$
- Theorem [L (2001)]
  In a solid brick G, every regular vector $\mathbf{x} \in \mathbb{Z}_+^E$ is in $\mathrm{Int}\mathcal{C}\mathrm{one}(G)$

Relaxations of the $\mathrm{Int}\mathcal{C}\mathrm{one}$

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$$\mathrm{Int}\mathcal{C}\mathrm{one}(G) := \sum_{\alpha_M \in \mathbb{Z}_+} \alpha_M \chi^M$$

$$\mathcal{C}\mathrm{one}(G) := \sum_{\alpha_M \in \mathbb{R}^+} \alpha_M \chi^M$$

$$\mathcal{L}\mathrm{at}(G) := \sum_{\alpha_M \in \mathbb{Z}} \alpha_M \chi^M$$

Relaxations of the $\mathrm{Int}\mathcal{C}\mathrm{one}$

$$\begin{aligned}
\mathrm{Int}\mathcal{C}\mathrm{one}(G) &:= \textstyle\sum_{\alpha_M \in \mathbb{Z}_+} \alpha_M \chi^M \\
\mathcal{C}\mathrm{one}(G) &:= \textstyle\sum_{\alpha_M \in \mathbb{R}^+} \alpha_M \chi^M \\
\mathcal{L}\mathrm{at}(G) &:= \textstyle\sum_{\alpha_M \in \mathbb{Z}} \alpha_M \chi^M \\
\mathcal{L}\mathrm{in}(G) &:= \textstyle\sum_{\alpha_M \in \mathbb{R}} \alpha_M \chi^M
\end{aligned}$$

A Relaxation of Tutte's Conjecture

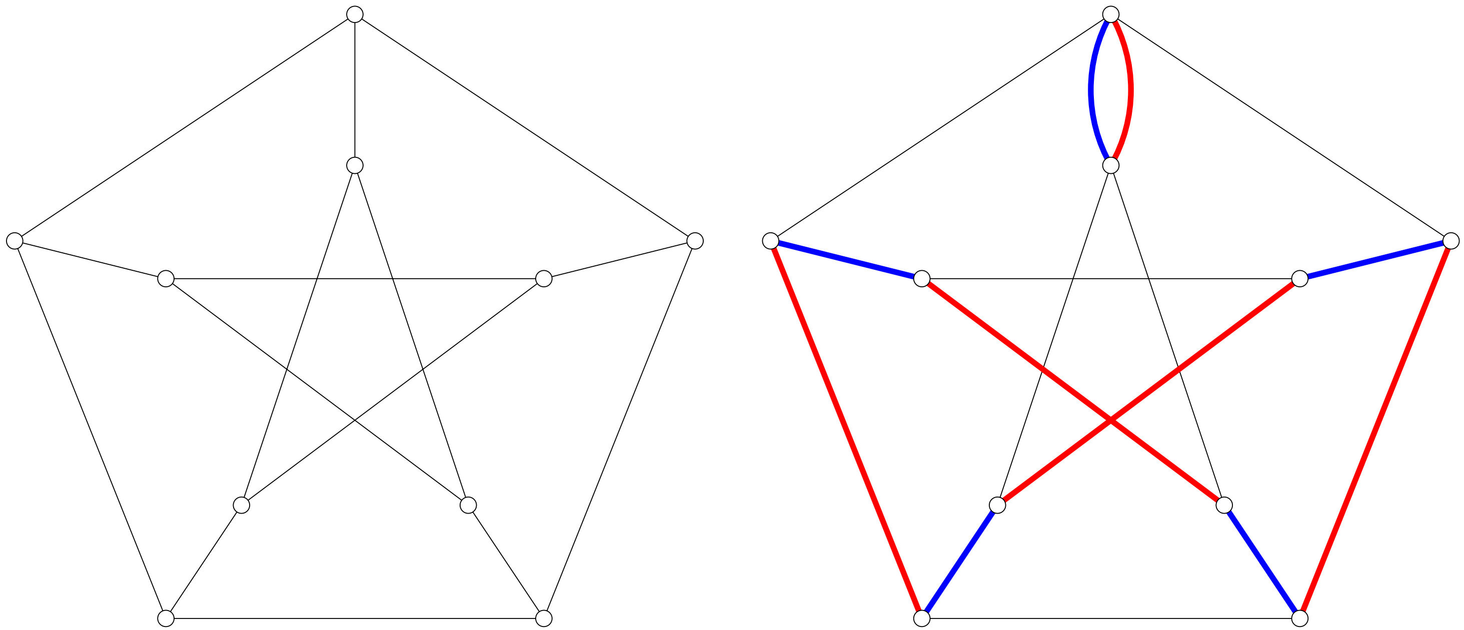
- Let G be a mc graph, let $r \in \mathbb{R}$. A vector $\mathbf{x} \in \mathbb{R}^E$ is *r-regular* if $\mathbf{x}(C) = r$ for every tight cut C of G

- Theorem [Lovász (1987)]
  For every mc graph G, every regular vector $\in \mathcal{L}\text{in}(G)$

$$\mathcal{L}\text{in}(G) := \sum_{\alpha_M \in \mathbb{R}} \alpha_M \chi^M$$

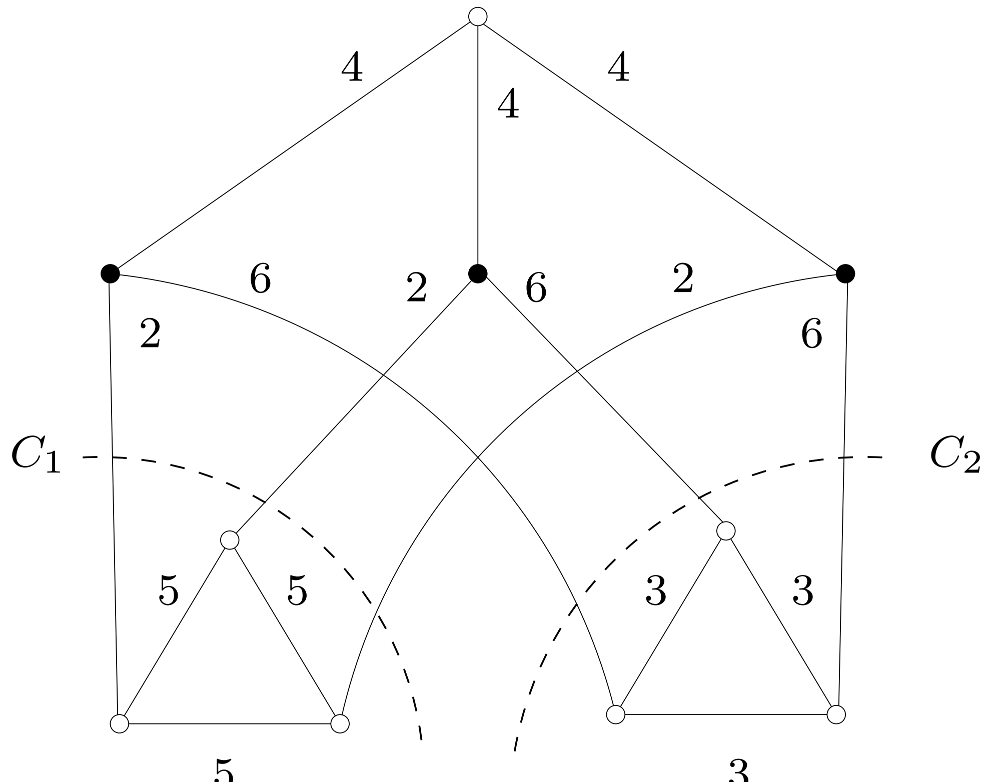
An example: $\mathbf{1} \in \mathcal{L}\mathrm{in}(P)$

- $\mathbb{P}$ has six perfect matchings



- one M_0 and five $M_1, \cdots, M_5$
- every edge in precisely two pms
- $\mathbf{1} \in \mathcal{L}\mathrm{in}(\mathbb{P}) : \mathbf{1} = \sum_{i=1}^{6} \frac{1}{2}\chi^{M_i}$

Regular in Every Tight Cut

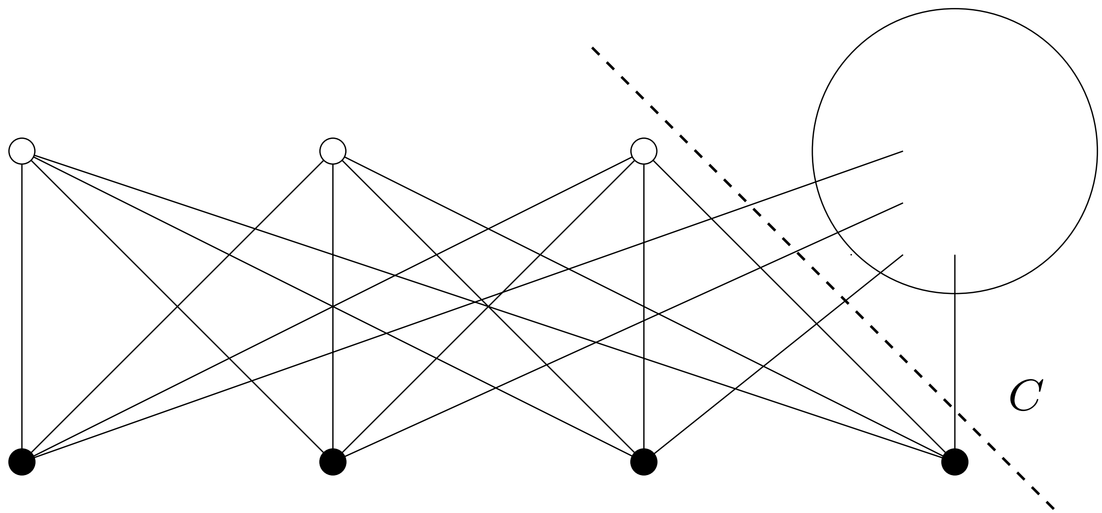


$$r = 12$$

$$\mathbf{x}(C_1) = 6 \quad \mathbf{x}(C_2) = 18$$

Regular Vectors in Bipartite Graphs

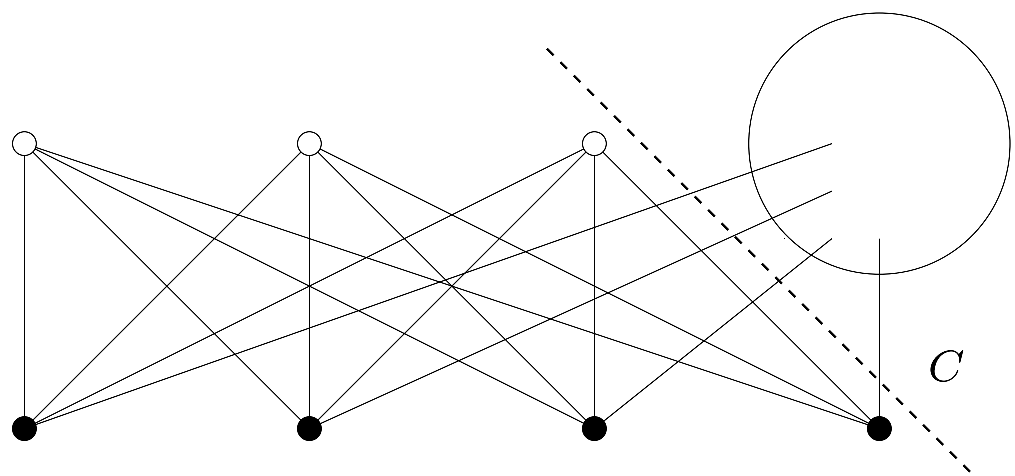
- degree constraints
  $[\forall v, w \in V \quad \mathbf{x}(\partial(v)) = \mathbf{x}(\partial(w))] \Rightarrow$ regularity



- $\mathbf{x}(C) = 4r - 3r = r$

Regular Vectors in Near-Bricks

- mc G is a *near-brick* if $b(G) = 1$
- $\forall$ tight C of near-brick G, one of the C-contractions is bipartite
- degree constraints
  $[\forall v, w \in V \quad \mathbf{x}(\partial(v)) = \mathbf{x}(\partial(w))] \Rightarrow$ regularity



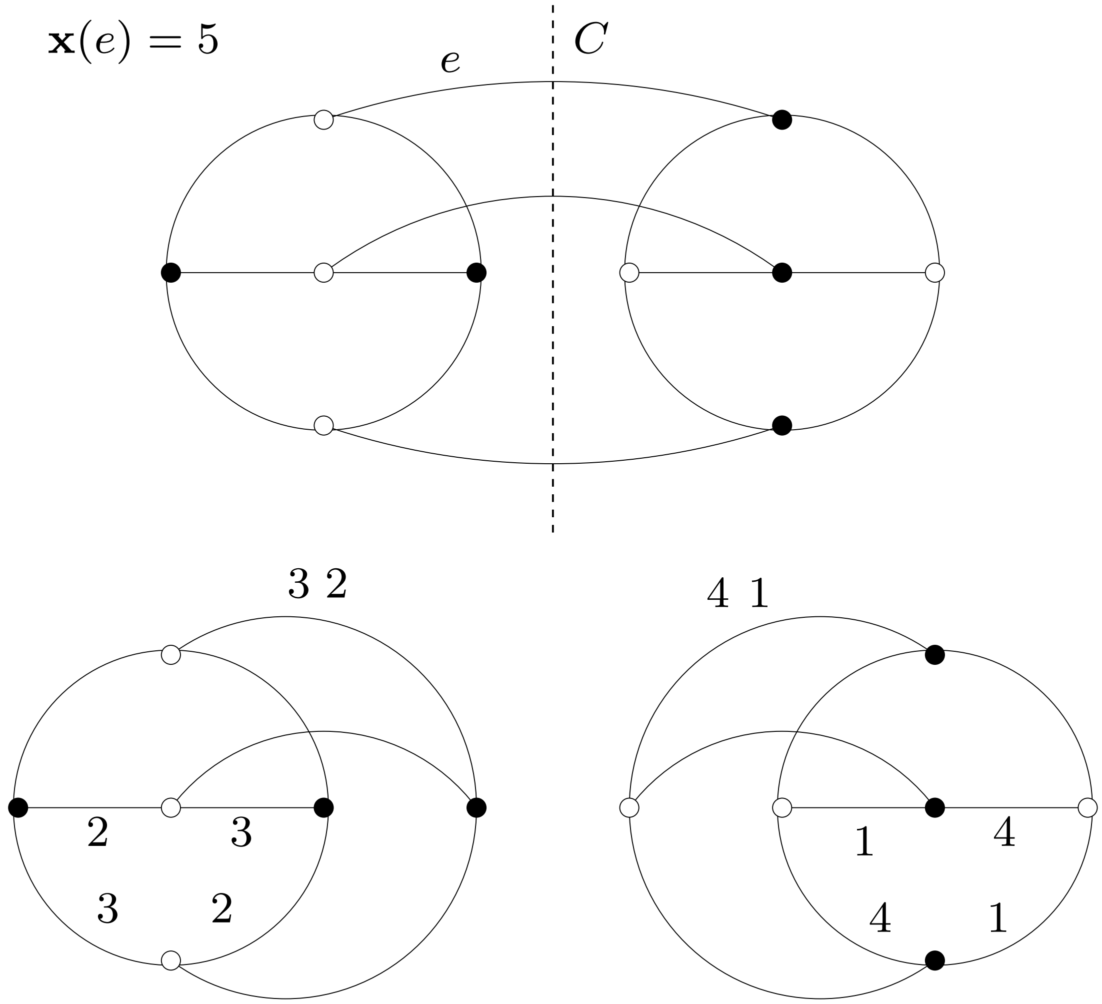
- $\mathbf{x}(C) = 4r - 3r = r$

Proof of Lovász's Theorem

- G mc, $\mathbf{x} \in \mathbb{R}^E$ r-regular, $r \in \mathbb{R}$
- $\Rightarrow \mathbf{x} \in \mathcal{L}\text{in}(G)$
- by induction on $|E|$
- Reduction to Bricks and Braces: <u>Merger</u> operation

Merger for Tight Cuts

- Reduction to Bricks and Braces



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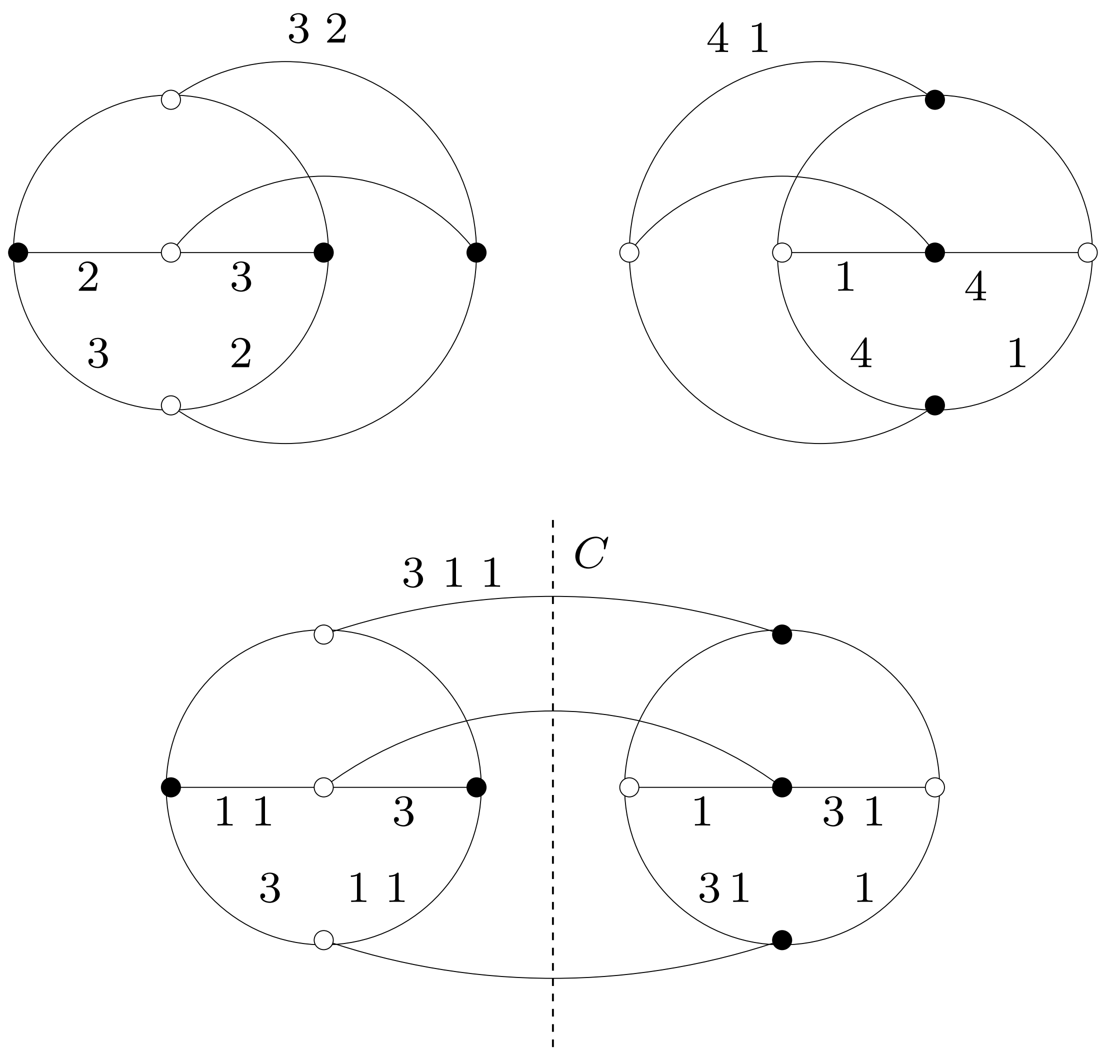


Induction Step for Braces

- G mc, e is *removable* if $G - e$ is mc
- If a brace $G \notin \{K_2, C_4\}$, it has a removable edge e
- let M_e be a pm that contains e
- let $\mathbf{y}$ be the restriction of $\mathbf{x} - \mathbf{x}(e)\chi^{M_e}$ to $G - e$
- I.H.: $\mathbf{y}$ is regular in $G - e \Rightarrow \mathbf{y} \in \mathcal{L}\text{in}(G - e)$
- $\mathbf{x} = \mathbf{x}(e)\chi^{M_e} + \mathbf{y} \Rightarrow \mathbf{x} \in \mathcal{L}\text{in}(G)$

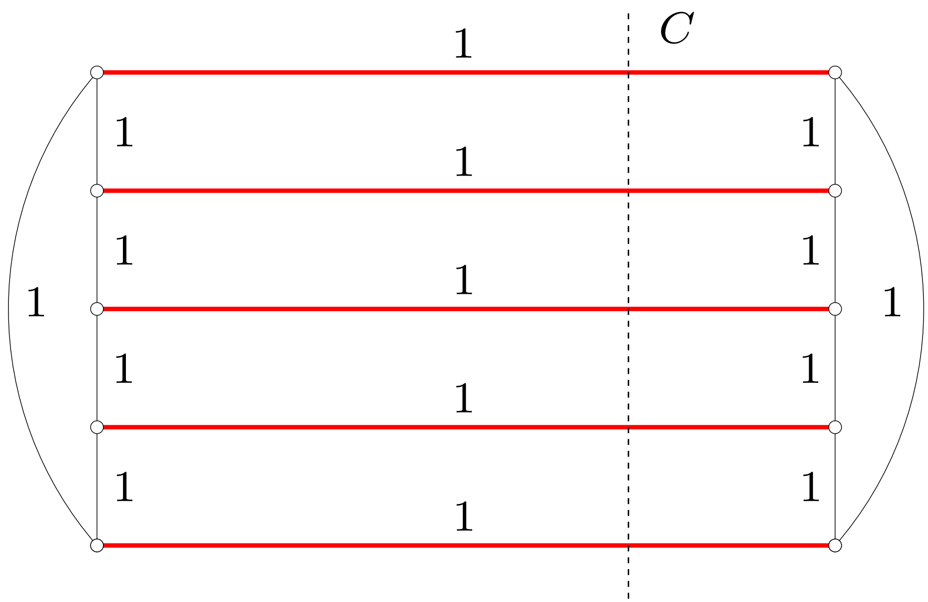
Induction Step for Solid Bricks

- Theorem [Lovász (1987)]
  $\forall$ brick $\notin \{K_4, \overline{C_6}\}$ has a removable edge
- $\overline{C_6}$ is not solid
- every solid brick $\neq K_4$ has a removable edge
- Theorem [CLM (2002)]
  e removable in solid brick $G \Rightarrow G - e$ is a near-brick
- let M_e be a pm that contains e
- let $\mathbf{y}$ be the restriction of $\mathbf{x} - \mathbf{x}(e)\chi^{M_e}$ to $G - e$
- I.H.: $\mathbf{y}$ is regular in $G - e \Rightarrow \mathbf{y} \in \mathcal{L}\mathrm{in}(G - e)$
- $\mathbf{x} = \mathbf{x}(e)\chi^{M_e} + \mathbf{y} \Rightarrow \mathbf{x} \in \mathcal{L}\mathrm{in}(G)$

The Merger - Robust Cuts

- C: robust cut of brick G
- $\mathbf{x} \in \mathbb{R}^E$: r-regular
- $\mathbf{x}(C) \neq r$
- restrictions of $\mathbf{x}$ to C-contractions of G not regular!

Merger - Robust Cuts



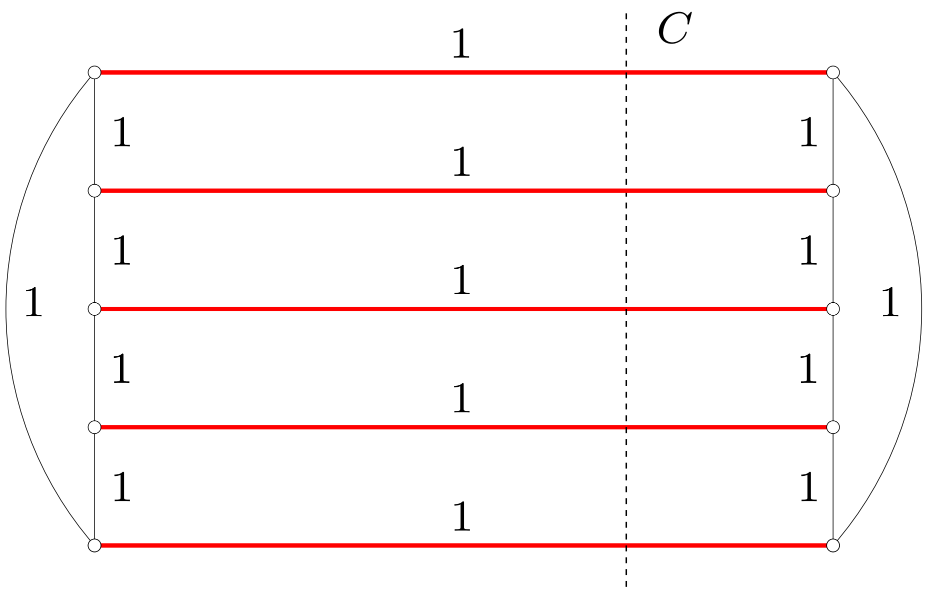
$r = 3$

$\mathbf{x}(C) = 5$

$\beta = \frac{\mathbf{x}(C) - r}{|M \cap C| - 1} = \frac{1}{2}$

M

Merger - Robust Cuts



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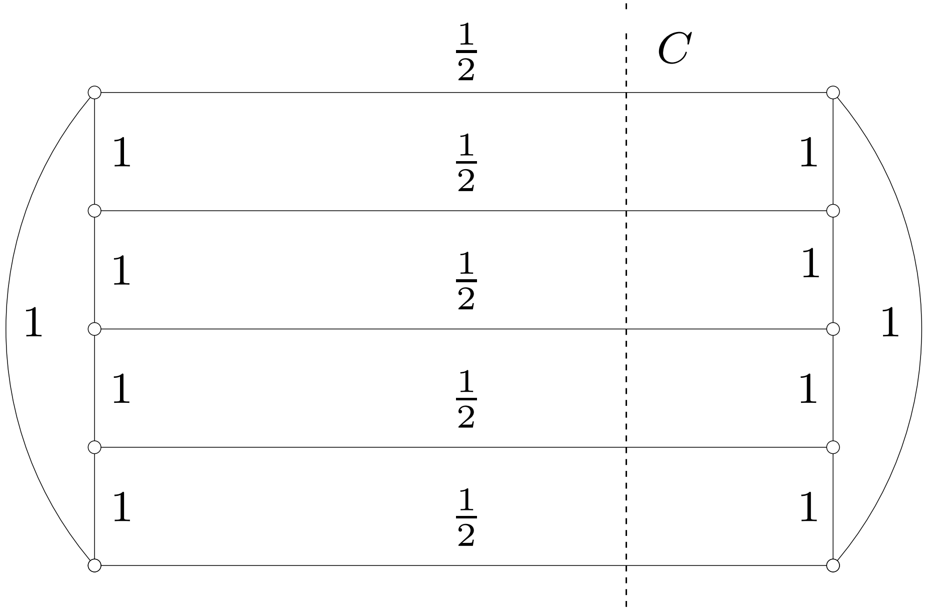


$\mathbf{y} = \mathbf{x} - \beta \chi^M$

$s = r - \beta = \frac{5}{2}$

$\mathbf{y}(C) = \frac{5}{2}$

Merger - Robust Cuts

- let $\beta := \frac{\mathbf{x}(C)-r}{|M\cap C|-1}$
- $M \in \mathcal{M}$: $|M \cap C| > 1$
- let $s := r - \beta$, let $\mathbf{y} := \mathbf{x} - \beta\chi^M$
- $\mathbf{y}$ is s-regular
- $\mathbf{y}(C) = \mathbf{x}(C) - |M \cap C| \cdot \beta = r - \beta = s$
- now, the restrictions of $\mathbf{y}$ to the C-contractions of G are s-regular!
- proceed as in the case of tight cuts

Relaxations of Tutte's Conjecture

- Theorem [Seymour (1979)]
  For every 2-connected cubic graph G free of Petersen minors, $\mathbf{1} \in \mathcal{L}\mathrm{at}(G)$
- Theorem [Lovász (1987)]
  For every mc graph G free of Petersen bricks, every regular vector in $\mathbb{Z}^E$ lies in $\mathcal{L}\mathrm{at}(G)$
- If G is 2-connected and cubic, then $\forall$ tight C, $|C| = 3$
- Corollary *For every 2-connected cubic G free of Petersen bricks*, $\mathbf{1} \in \mathcal{L}\mathrm{at}(G)$

Vectors in the Matching Lattice

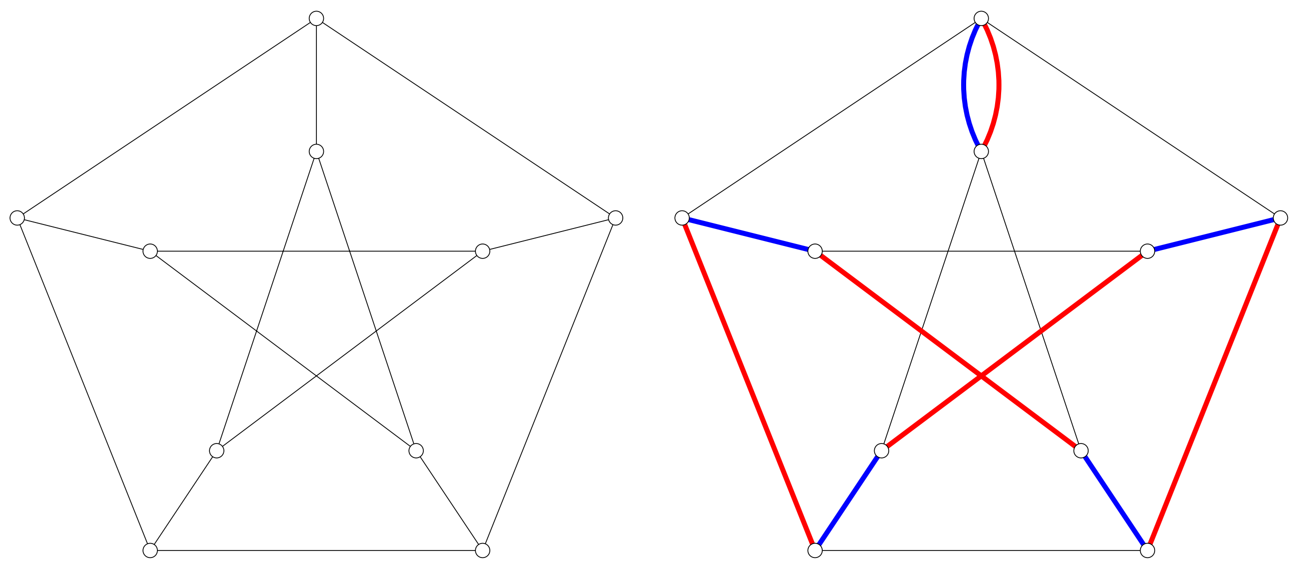
- Theorem [Lovász (1987)]
  $\forall$ mc graph G free of Petersen bricks, every r-regular vector in $\mathbb{Z}^E$ lies in $\mathcal{L}\mathrm{at}(G)$

$$\mathcal{L}\mathrm{at}(G) := \sum_{\alpha_M \in \mathbb{Z}} \alpha_M \chi^M \quad \mathcal{L}\mathrm{in}(G) := \sum_{\alpha_M \in \mathbb{R}} \alpha_M \chi^M$$

- A brick is a *Petersen brick* if its underlying simple graph is $\mathbb{P}$
- A mc G is *free of Petersen bricks* if its tight cut decomposition has no Petersen brick.

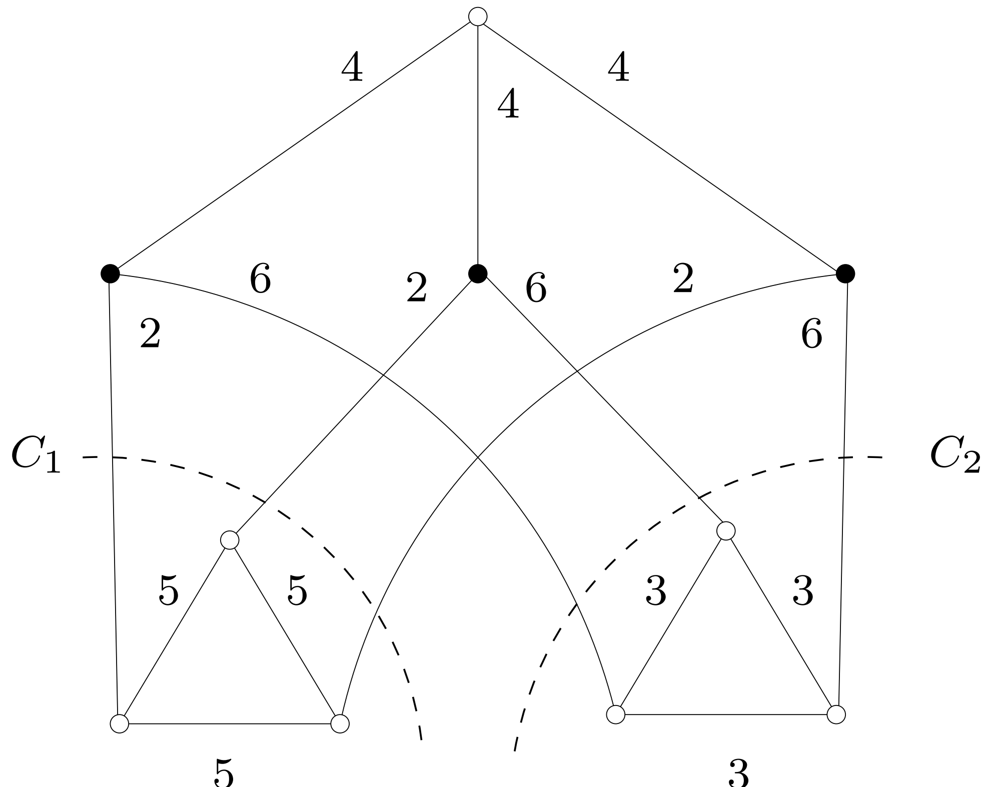
An example: $\mathbf{2} \in \mathcal{L}\mathrm{at}(P)$

- $\mathbb{P}$ has six perfect matchings



- one M_0 and five $M_1, \cdots, M_5$
- every edge in precisely two pms
- $\mathbf{2} \in \mathcal{L}\mathrm{at}(\mathbb{P}) : \mathbf{2} = \sum_{i=1}^{6} \chi^{M_i}$
- $\mathbf{1} \notin \mathcal{L}\mathrm{at}(\mathbb{P})$

Regular in Every Tight Cut



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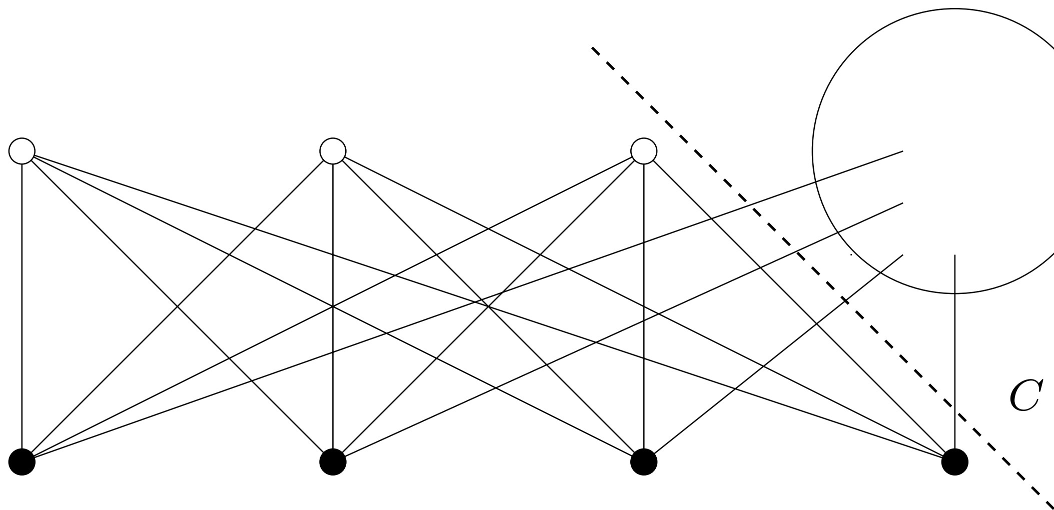

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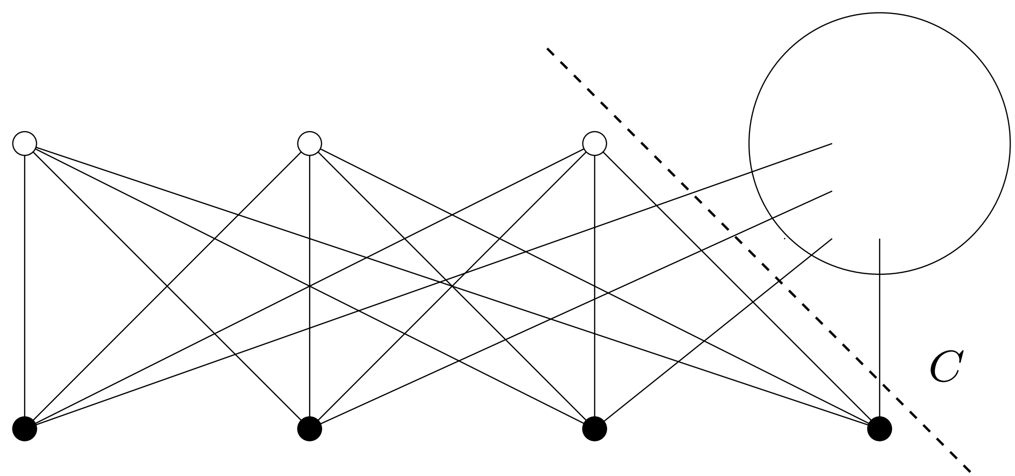

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Merger for Tight Cuts

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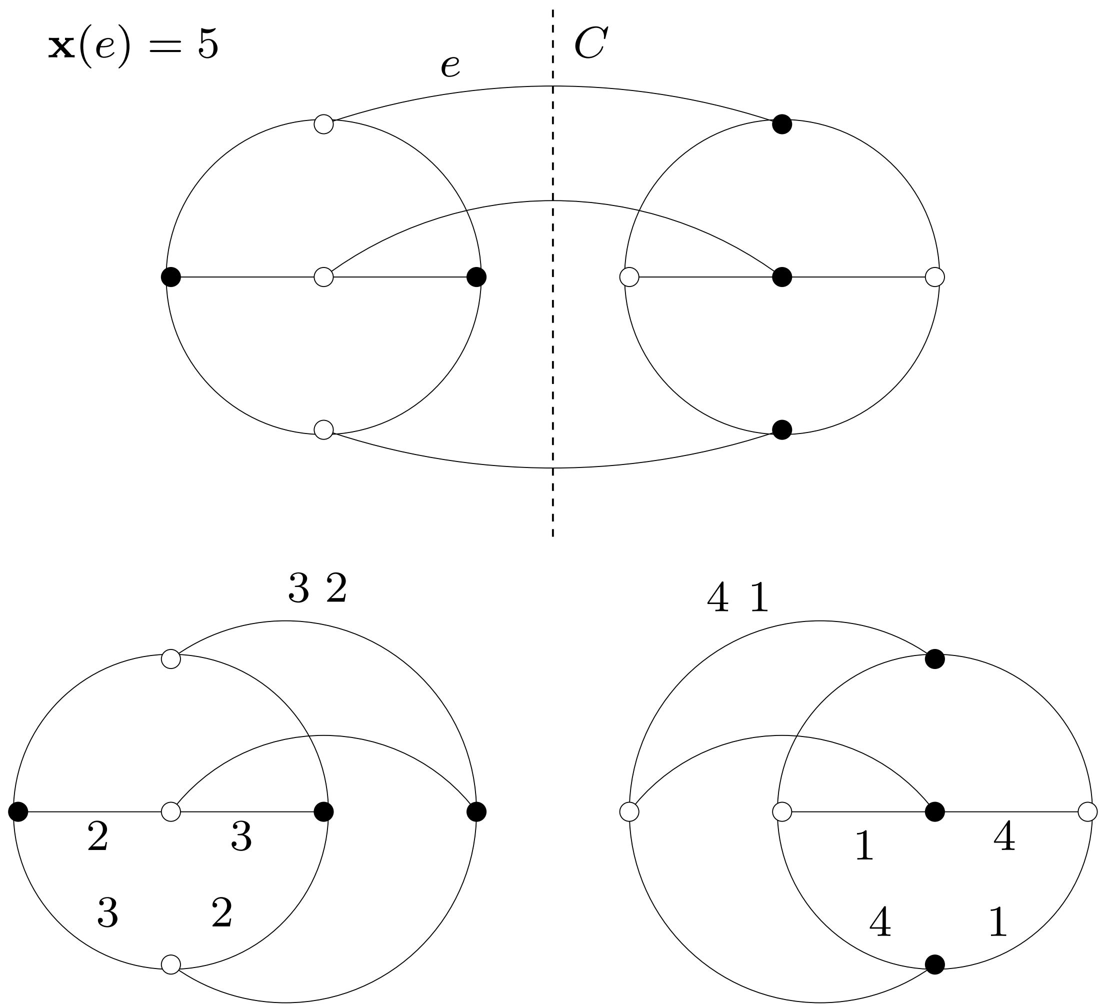

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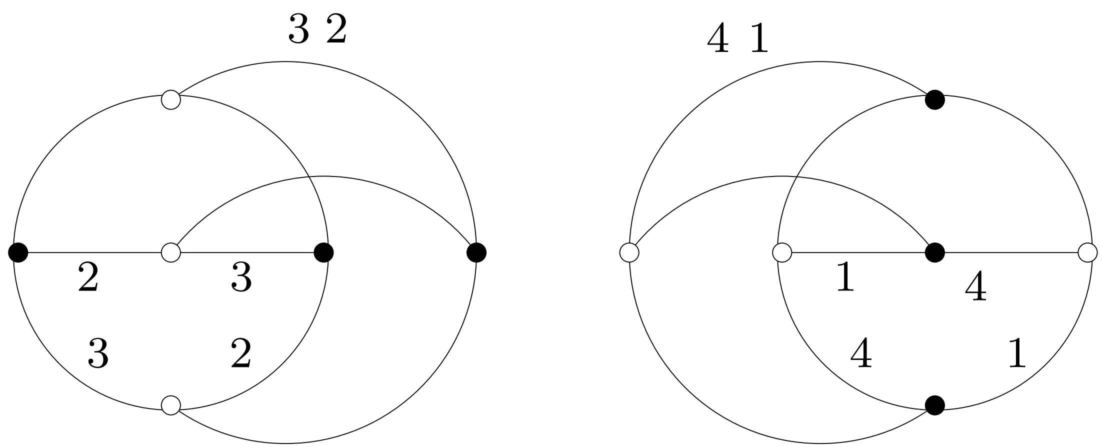

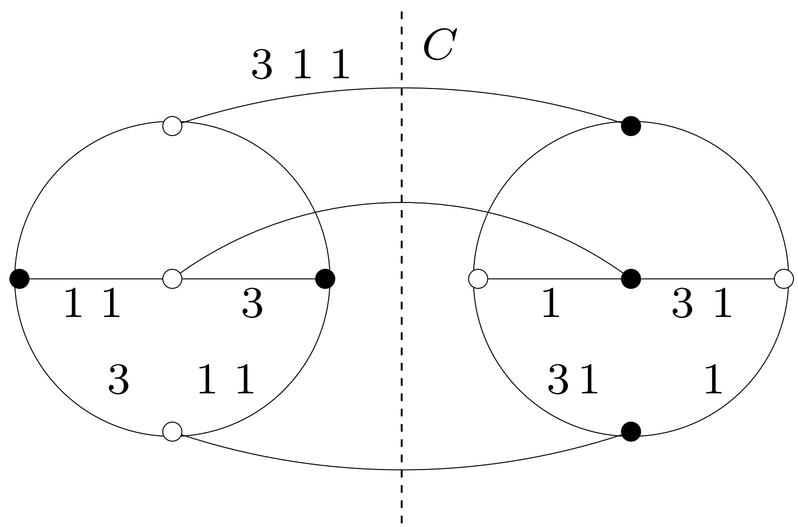

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Induction Step for Solid Bricks

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- Theorem [CLM (2002)]
  e removable in solid brick $G \Rightarrow G - e$ is a solid *near-brick*
- i.e. the brick of $G - e$ is solid
- $\mathbb{P}$ is not solid
- $\therefore \quad G - e$ free of $\mathbb{P}$

Induction Step for Solid Bricks

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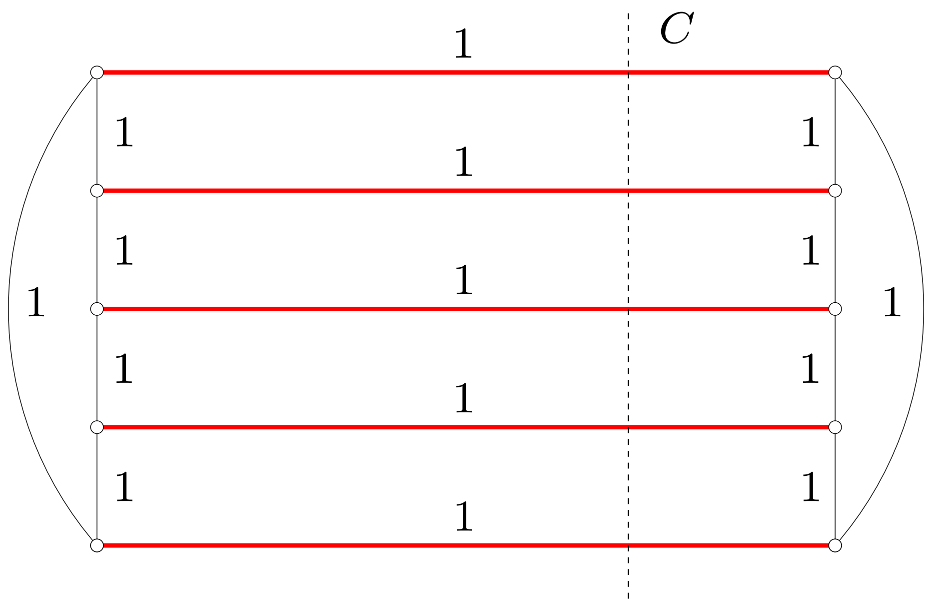
The Merger - Robust Cuts

- C: robust cut of brick G free of $\mathbb{P}$
- $\mathbf{x} \in \mathbb{Z}^E$: r-regular,
- $\mathbf{x}(C) \neq r$
- restrictions of $\mathbf{x}$ to C-contractions of G not regular!

Merger - Robust Cuts Does Not Work



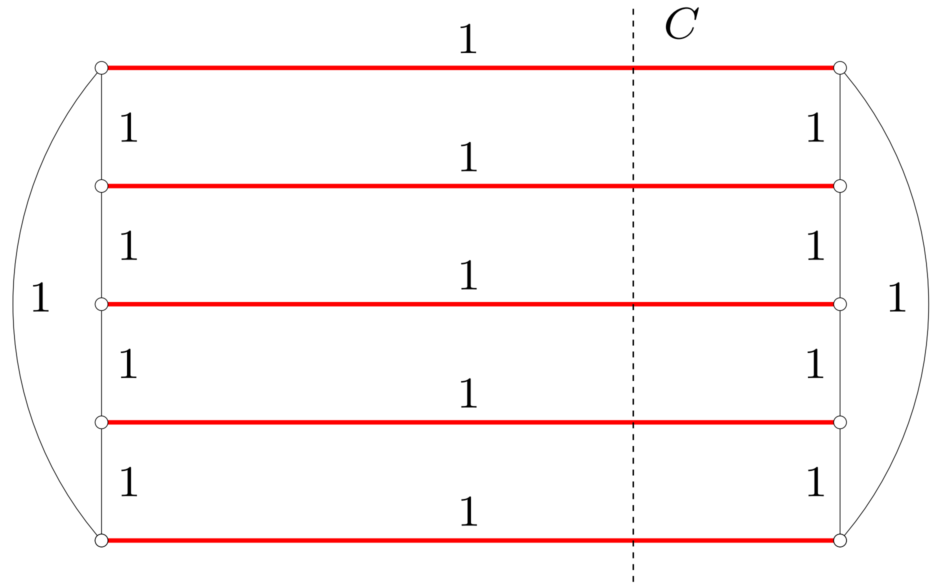
$r = 3$

$\mathbf{x}(C) = 5$

$\beta = \frac{\mathbf{x}(C) - r}{|M \cap C| - 1} = \frac{1}{2}$

M

Merger - Robust Cuts Does Not Work



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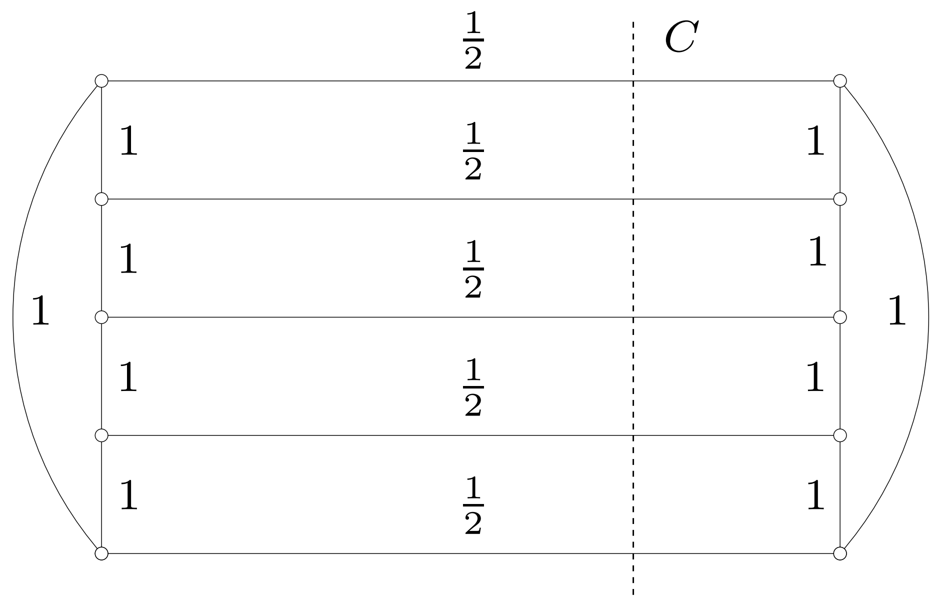


$\mathbf{y} = \mathbf{x} - \beta\chi^M$

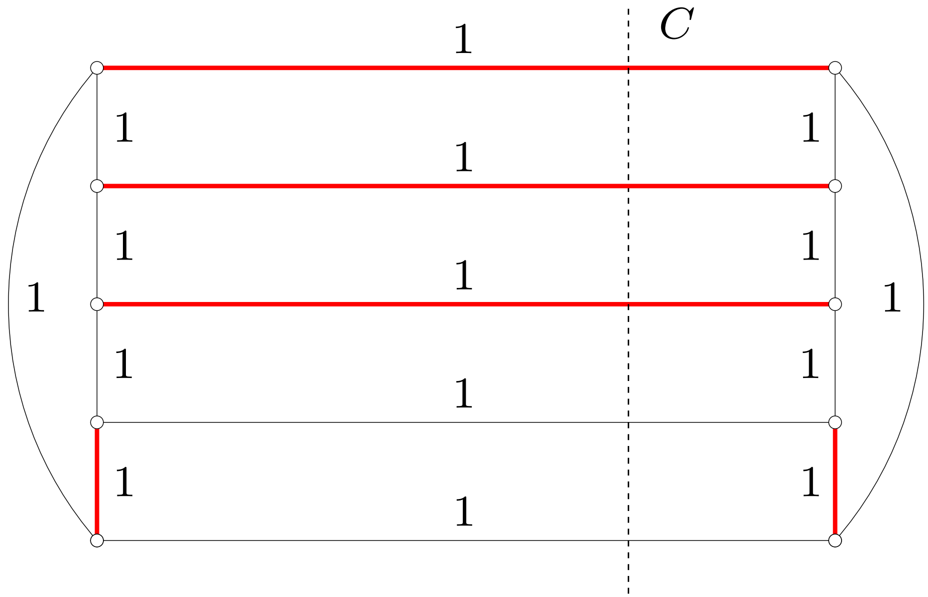
$s = r - \beta = \frac{5}{2}$

$\mathbf{y}(C) = \frac{5}{2}$

The Characteristic of $\mathbb{P}$

- A robust cut of mc G is <u>λ-*robust*</u> is G has a pm $M : |M \cap C| = \lambda$
- <u>Theorem</u> [CLM (2002)]
  Let G be a brick $\neq \mathbb{P}$. Then G has a 3-robust cut C s.t. both C-contractions of G are free of $\mathbb{P}$

Merger - 3-Robust Cuts



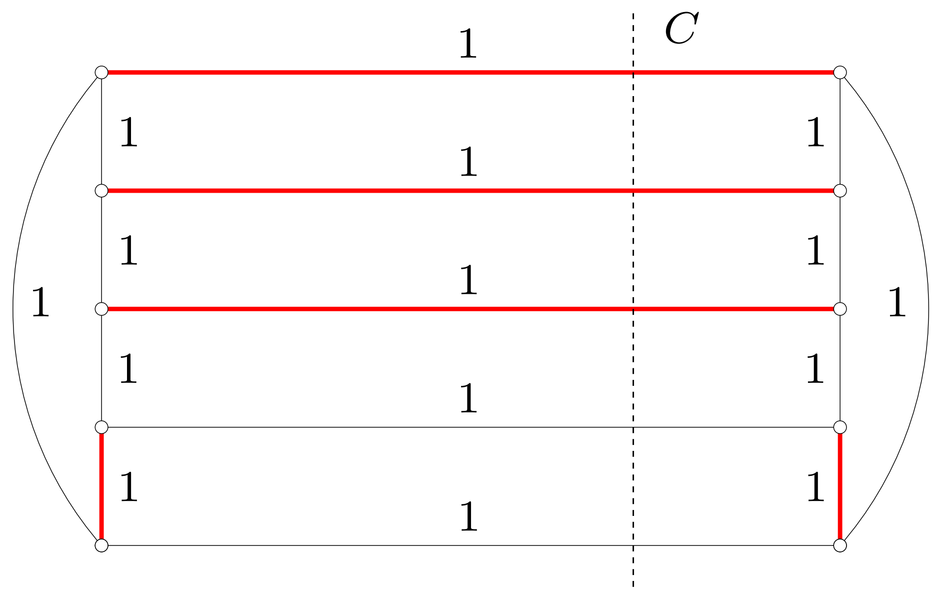
$r = 3$

$\mathbf{x}(C) = 5$

$\beta = \frac{\mathbf{x}(C) - r}{|M \cap C| - 1} = 1$

M

Merger - 3-Robust Cuts

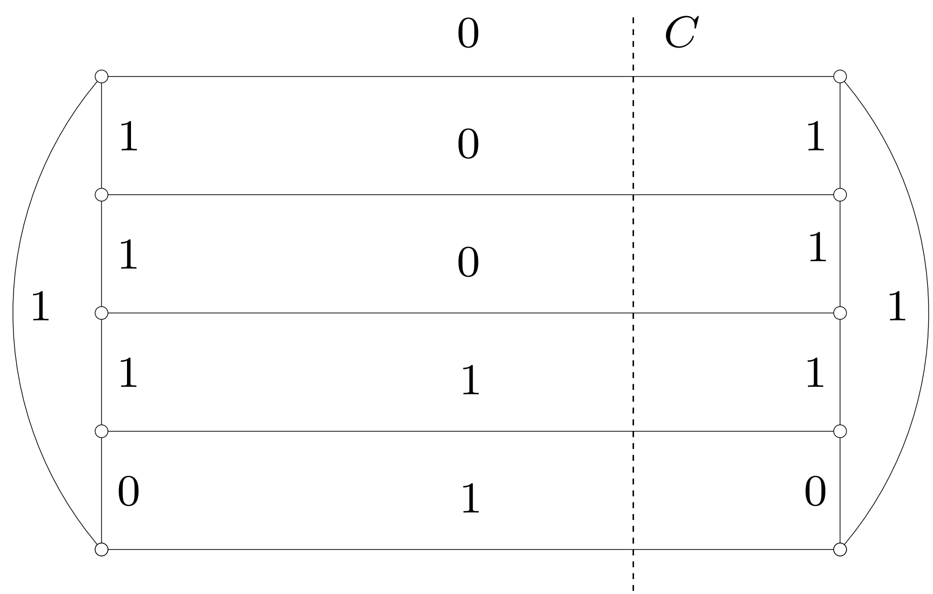


$r = 3$

$\mathbf{x}(C) = 5$

$\beta = \frac{\mathbf{x}(C) - r}{|M \cap C| - 1} = 1$

M

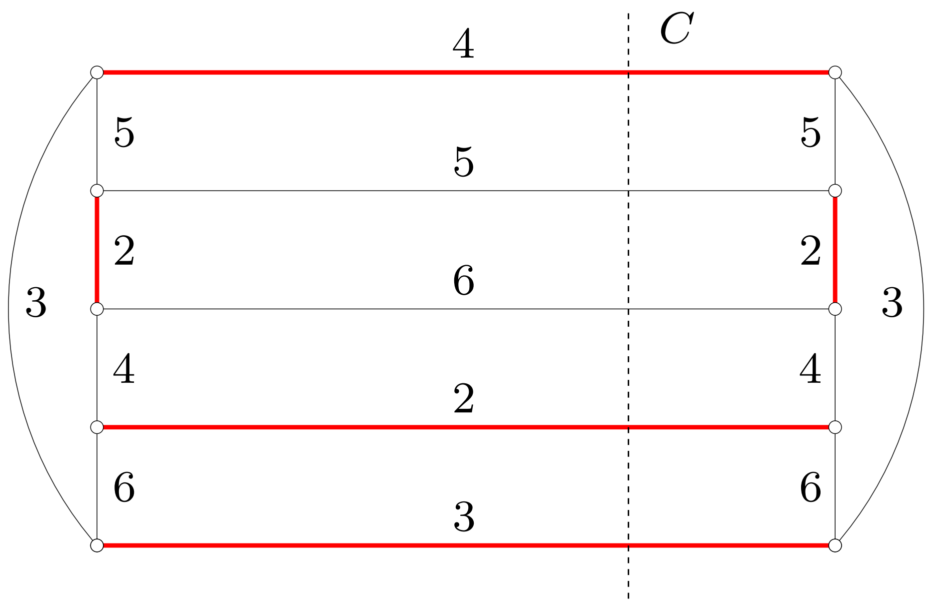


$\mathbf{y} = \mathbf{x} - \beta \chi^M$

$s = r - \beta = 2$

$\mathbf{y}(C) = 2$

Merger - 3-Robust Cuts



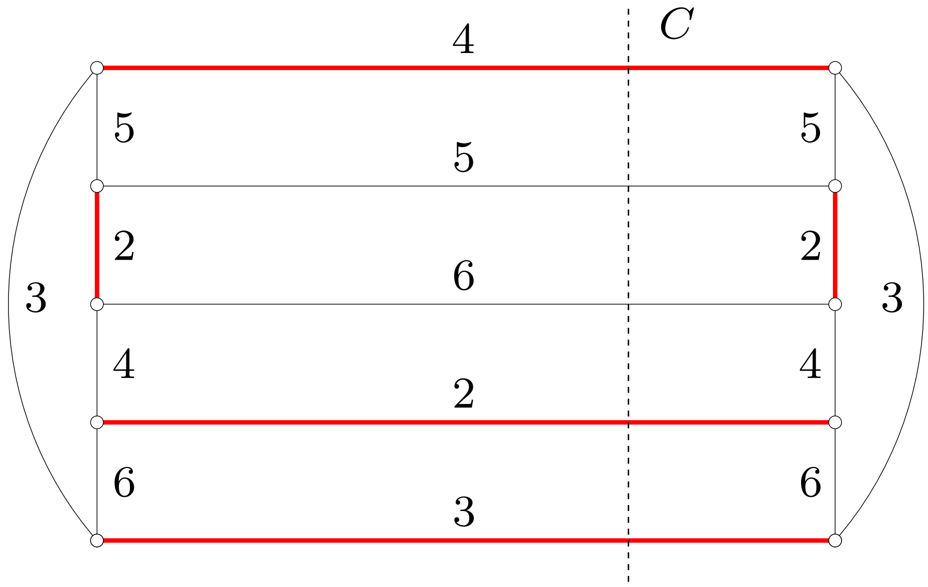
$r = 12$

$\mathbf{x}(C) = 20$

$\beta = \frac{\mathbf{x}(C)-r}{2} = 4$

M

Merger - 3-Robust Cuts

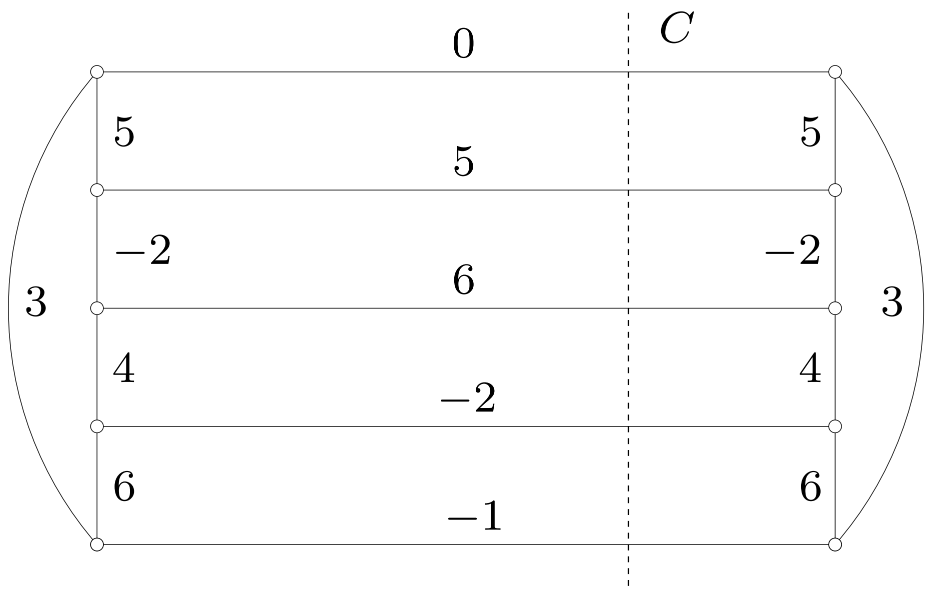


$r = 12$

$\mathbf{x}(C) = 20$

$\beta = \frac{\mathbf{x}(C) - r}{2} = 4$

M



$\mathbf{y} = \mathbf{x} - \beta\chi^M$

$s = r - \beta = 8$

$\mathbf{y}(C) = 8$

Merger - 3-Robust Cuts

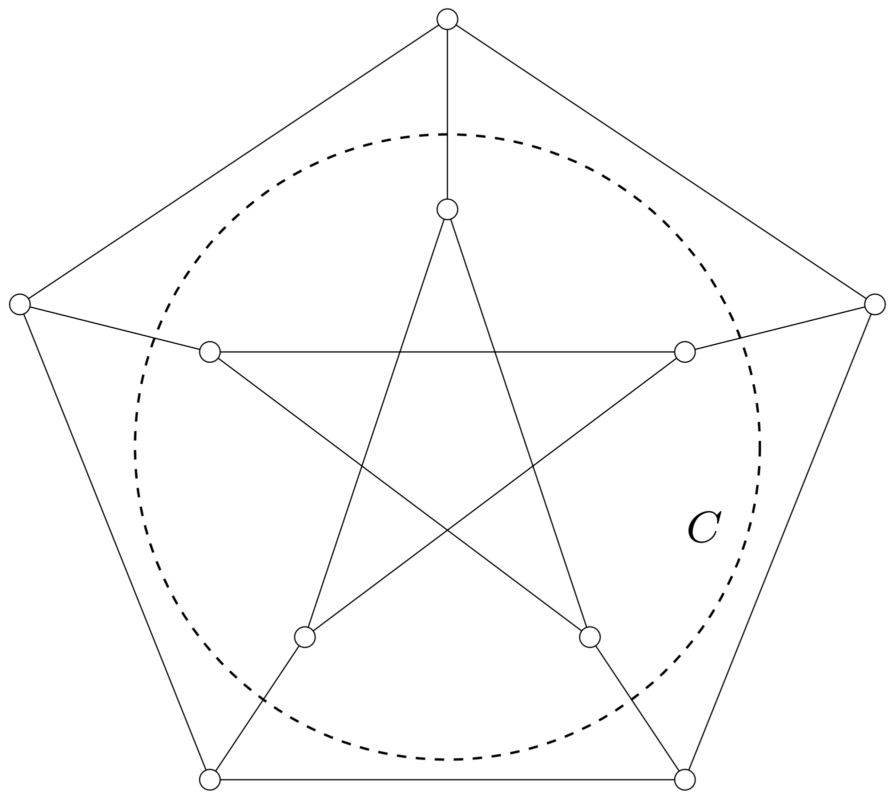
- $M \in \mathcal{M}$: $|M \cap C| = 3$
- let $\beta := \frac{\mathbf{x}(C)-r}{|M\cap C|-1} = \frac{\mathbf{x}(C)-r}{2}$
- $\mathbf{x}(C) \equiv r \pmod 2$
- $\therefore \quad \beta \in \mathbb{Z}$!!!

Merger - 3-Robust Cuts

- let $\beta := \frac{\mathbf{x}(C)-r}{|M \cap C|-1} = \frac{\mathbf{x}(C)-r}{2}$
- $\beta \in \mathbb{Z}$!!!
- let $s := r - \beta$, let $\mathbf{y} := \mathbf{x} - \beta\chi^M$
- $\mathbf{y}$ is s-regular, $\mathbf{y} \in \mathbb{Z}^E$
- $\mathbf{y}(C) = \mathbf{x}(C) - 3\beta = r - \beta = s$
- now, the restrictions of $\mathbf{y}$ to the C-contractions of G are s-regular!
- both C-contractions are free of $\mathbb{P}$
- proceed as in the case of tight cuts

The Matching Lattice of $\mathbb{P}$

- which r-regular $\mathbf{x} \in \mathbb{Z}^E$ lie in $\mathcal{L}\mathrm{at}(\mathbb{P})$?
- both shores of every robust cut C of $\mathbb{P}$ induce a pentagon

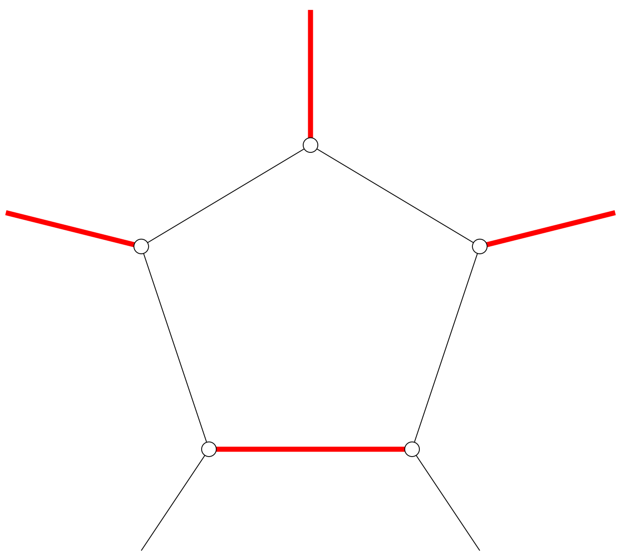


The Matching Lattice of $\mathbb{P}$

- let Q be a pentagon in a cubic graph, $M \in \mathcal{M}$
- $|M \cap \partial(V(Q))| = 3 \Rightarrow$ the 3 edges of $M \cap \partial(Q)$ are "cyclically consecutive" in S

The Matching Lattice of $\mathbb{P}$

- let Q be a pentagon in a cubic graph, $M \in \mathcal{M}$
- $|M \cap \partial(V(Q))| = 3 \Rightarrow$ the 3 edges of $M \cap \partial(Q)$ are "cyclically consecutive" in S



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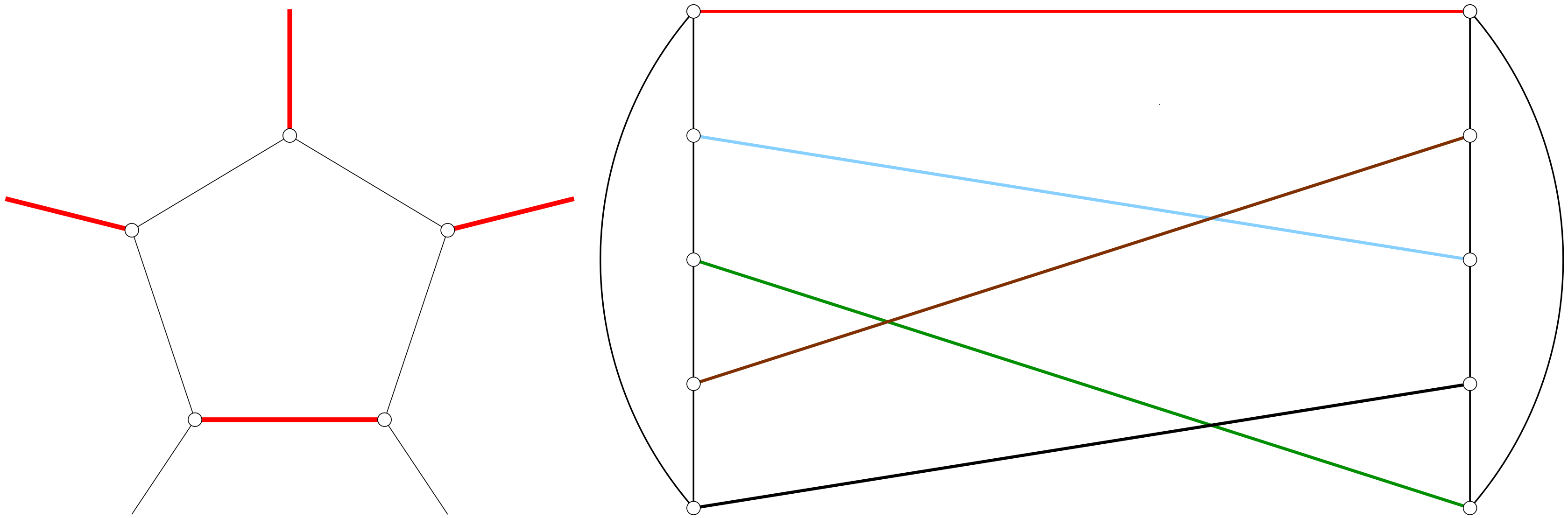

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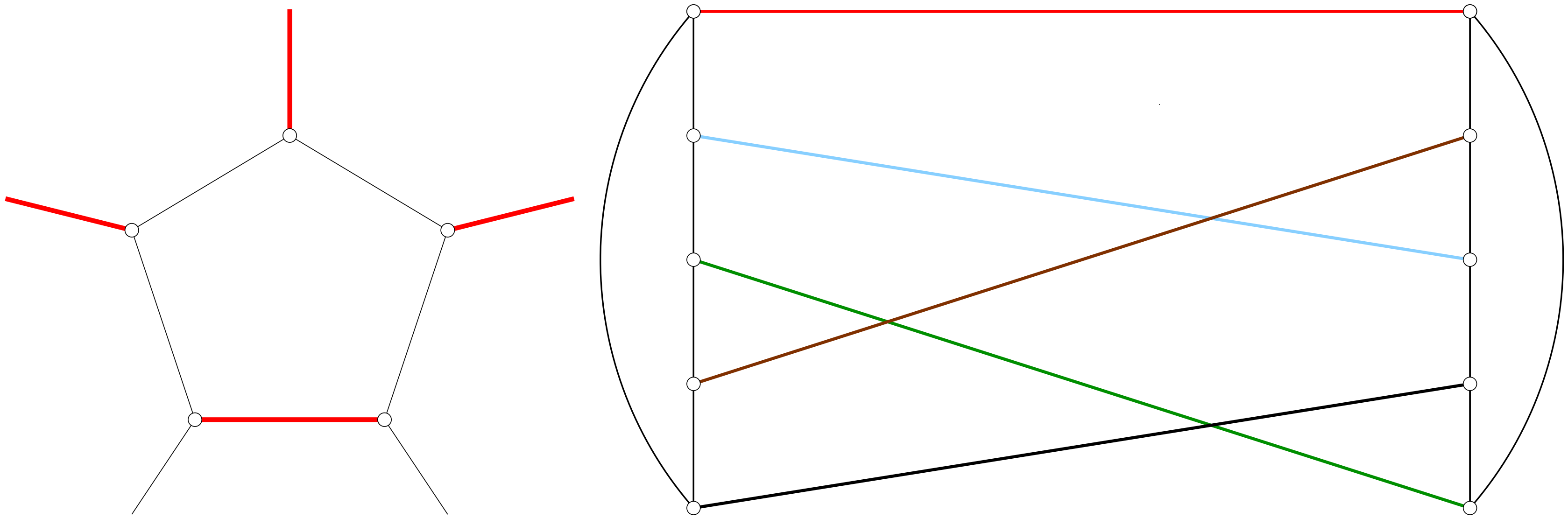


- $\lambda(\mathbb{P}) = 5$

The Matching Lattice of $\mathbb{P}$

- let Q be any pentagon of $\mathbb{P}$
- $C := \partial(V(Q))$ is robust
- five perfect matchings M_i, $i = 1, \cdots, 5$: $|M_i \cap C| = 1$
- one perfect matching, M_0: $|M_0 \cap C| = 5$
- let $\mathbf{x} \in \mathcal{L}\mathrm{at}(P)$
- $\mathbf{x} = \alpha_0 \chi_0^M + \sum_{i=1}^{5} \alpha_i \chi_i^M \quad \alpha_i \in \mathbb{Z}$
- $\mathbf{x}$ is r-regular, $r = \sum \alpha_i$
- $\mathbf{x}(C) = 5\alpha_0 + \sum_{i=1}^{5} \alpha_i = r + 4\alpha_0$
- necessary condition: $\frac{\mathbf{x}(C) - r}{4} \in \mathbb{Z}$

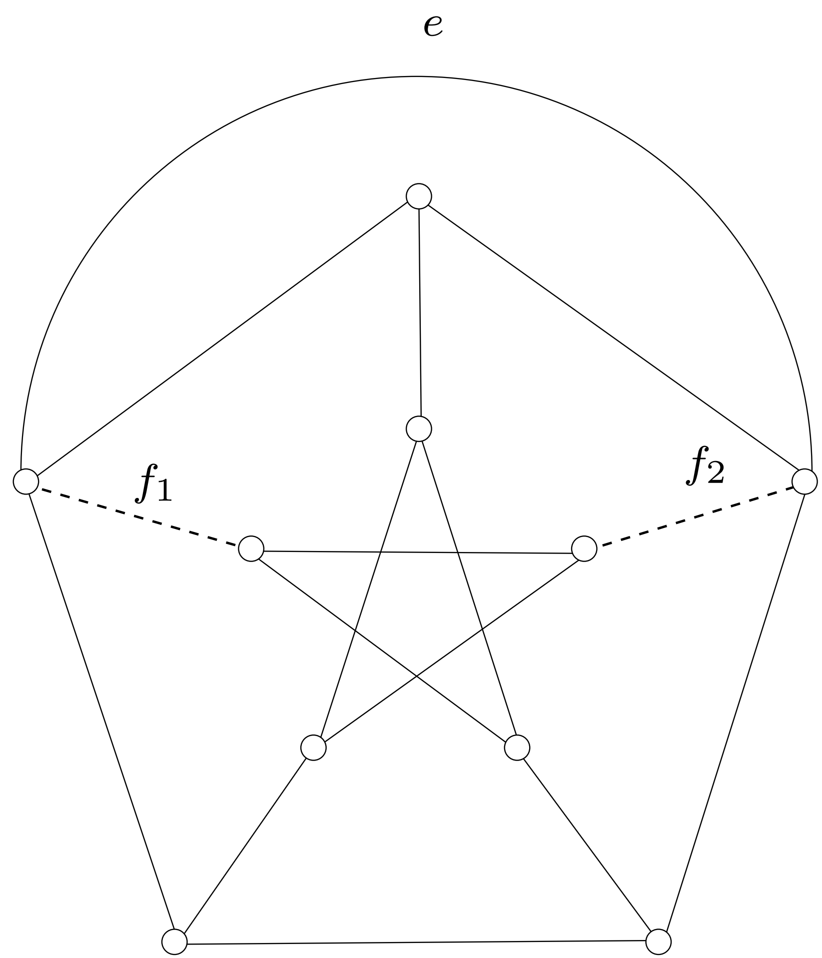
The Matching Lattice of $\mathbb{P}$

- let Q be any pentagon of $\mathbb{P}$
- $C := \partial(V(Q))$ is robust
- necessary condition: $\frac{\mathbf{x}(C)-r}{4} \in \mathbb{Z}$
- merger $\Rightarrow$ it is also sufficient
- An r-regular vector $\mathbf{x} \in \mathbb{Z}^E$ lies in $\mathcal{L}\mathrm{at}(\mathcal{P})$ if and only if
$$\mathbf{x}(C) \equiv r \pmod 4$$
- $\mathbf{2} \in \mathcal{L}\mathrm{at}(\mathbb{P})$: $\mathbf{2}(C) = 10 \equiv 6 \pmod 4$
- $\mathbf{1} \notin \mathcal{L}\mathrm{at}(\mathbb{P})$: $\mathbf{1}(C) = 5 \not\equiv 3 \pmod 4$

The Characteristic of $\mathbb{P}$

- Theorem [CLM (2002)]
  The characteristic $\lambda(G)$ of every nonsolid brick G lies in $\{3, 5\}$. Moreover, $\lambda(G) = 5$ iff G is a Petersen brick
- Theorem [CLM (2002)]
  Every brick $G \notin \{K_4, \overline{C_6}\}$ has a $(b+p)$-invariant edge
- An edge e of a mc graph G is *$(b+p)$-invariant* if e is removable and $(b+p)(G-e) = (b+p)(G)$
- An edge e of a mc graph G is *removable* if $G - e$ is mc

Example



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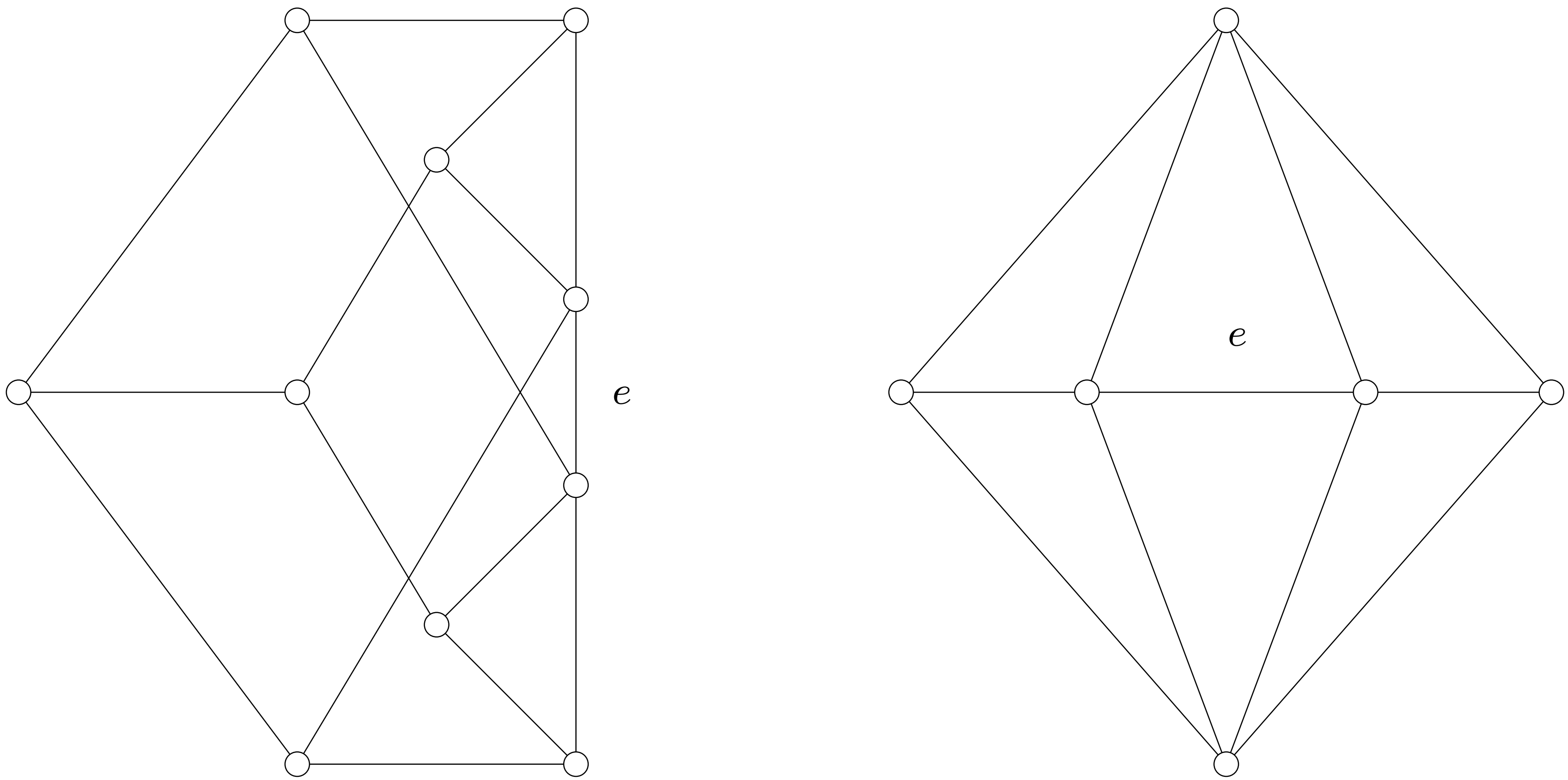

A Polynomial Algorithm

- Theorem [CLM (2002)]
  Every brick $G \notin \{K_4, \overline{C_6}\}$ *has a* $(b+p)$*-invariant edge*
- Corollary [CLM (2002)]
  Every simple brick G *not in* $\{K_4, \overline{C_6}, \mathbb{P}\}$ *has a removable edge* e *such that* $G - e$ *is a near-brick and the brick of* $G - e$ *is not* $\mathbb{P}$
- $\Rightarrow$ polynomial algorithm for the terms of Lovász's Theorem