São Paulo School of Advanced Science on Algorithms, Combinatorics and Optimization

The Perfect Matching Polytope, Solid Bricks and the Perfect Matching Lattice

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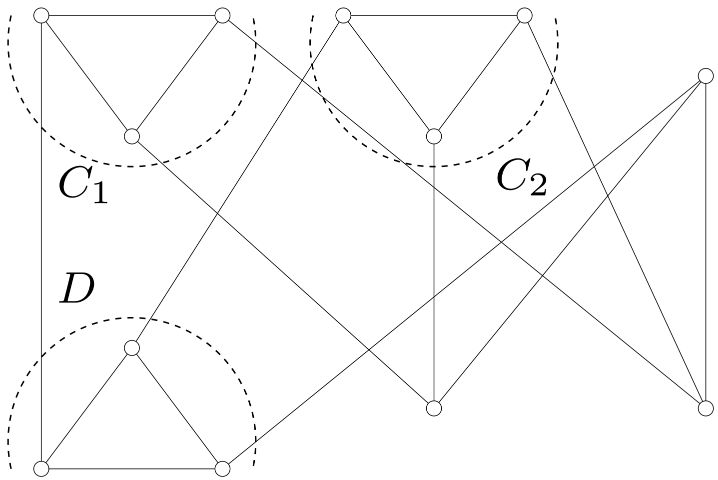
The Precedence Relation

- mc G, cut C <u>*precedes*</u> cut D ($C \preceq D$) if

$$|M \cap C| \leq |M \cap D| \quad \forall M \in \mathcal{M}$$

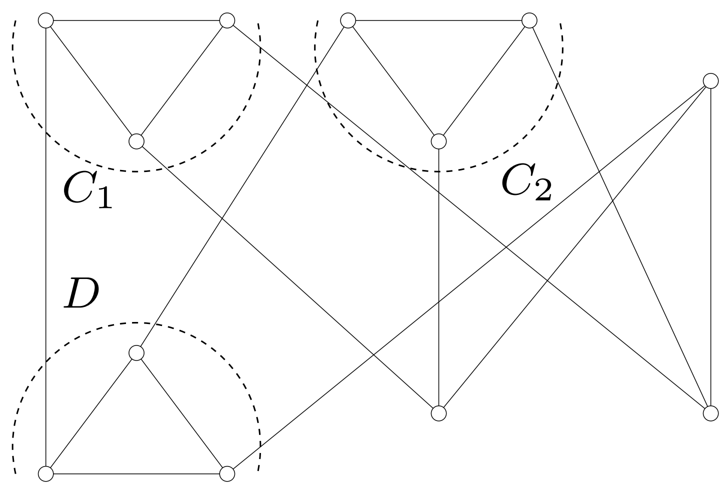


- $|M \cap D| + 2 = |M \cap C_1| + |M \cap C_2| + 1 \quad \forall M \in \mathcal{M}$
- $|M \cap D| + 2 \geq |M \cap C_1| + 2 \quad \forall M \in \mathcal{M}$
- $\therefore \quad C_1 \preceq D$

The Precedence Relation

- C *strictly precedes* D ($C \prec D$) if $C \preceq D$ and

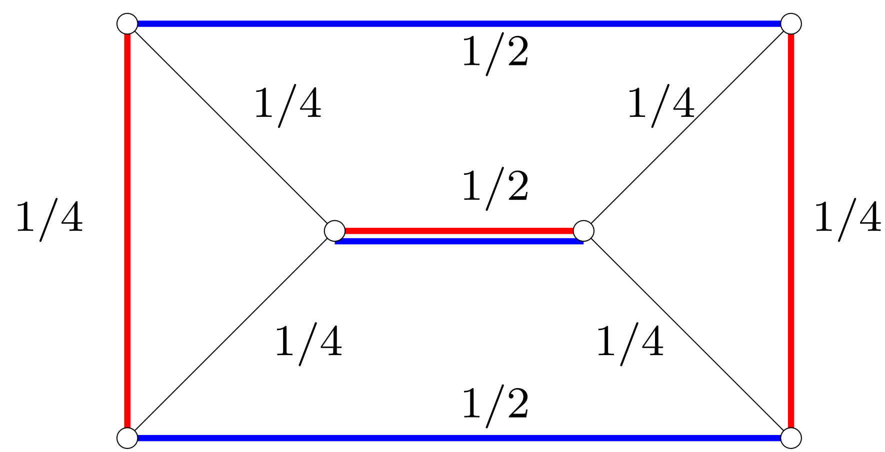
$$\exists M \in \mathcal{M} \quad |M \cap C| < |M \cap D|$$



- $|M \cap D| + 2 = |M \cap C_1| + |M \cap C_2| + 1 \quad \forall M \in \mathcal{M}$
- C_2 not tight $\Rightarrow \exists M : |M \cap D| + 2 > |M \cap C_1| + 2$
- $\therefore \quad C_1 \prec D$

The Perfect Matching Polytope

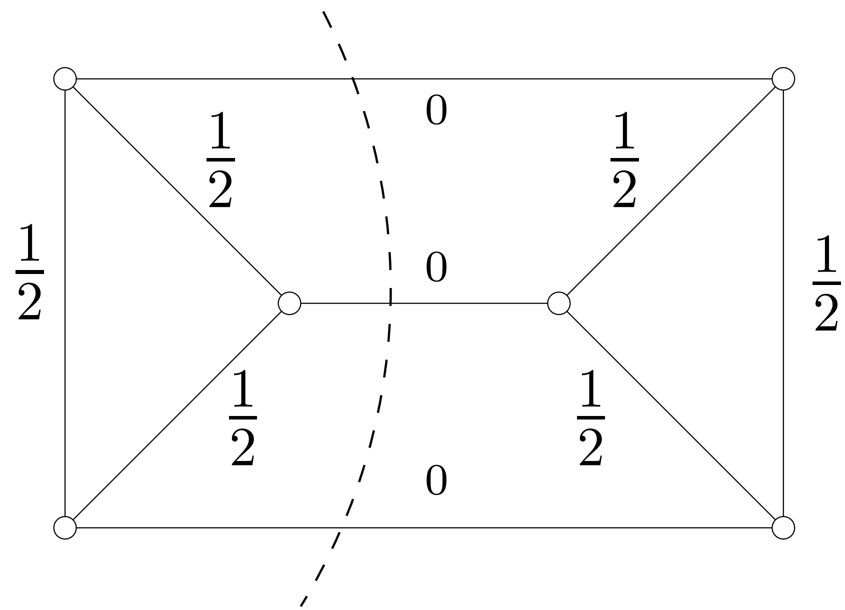
- $\mathcal{M}$: the set of pms of G
- $\chi^S \in 2^E$: the incidence vector of $S \subseteq E$
- $\mathcal{P}\text{oly}(G) := \sum_{M \in \mathcal{M}} \alpha_M \chi^M$
- $(\sum \alpha_M = 1, \alpha_M \in \mathbb{R}^+)$
- Example



- $\alpha_i = 1/4 \quad i = 1, 2, 3, 4$

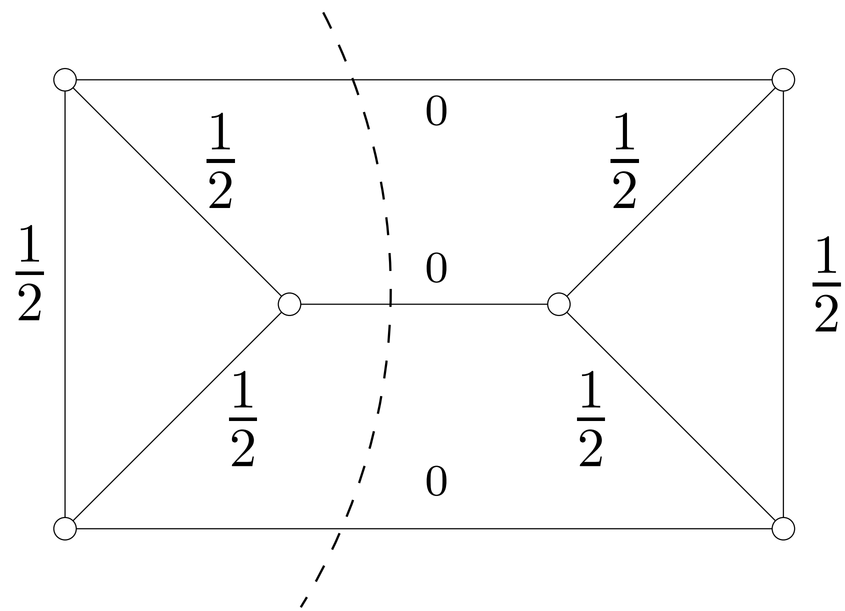
The Perfect Matching Polytope

- Theorem [Edmonds (1965)]
  A vector $\mathbf{x} \in \mathbb{R}^E \in \mathcal{P}\mathrm{oly}(G)$ *of a mc graph* G *iff:*
- $\mathbf{x} \geq \mathbf{0}$ (nonnegativity)
- $\mathbf{x}(\partial(v)) = 1, \forall v \in V$ (degree constraints)
- $\mathbf{x}(\partial(S)) \geq 1, \forall$ odd $S \subset V$ (odd set constraints)
- Example



Separating Cuts & The PM Polytope

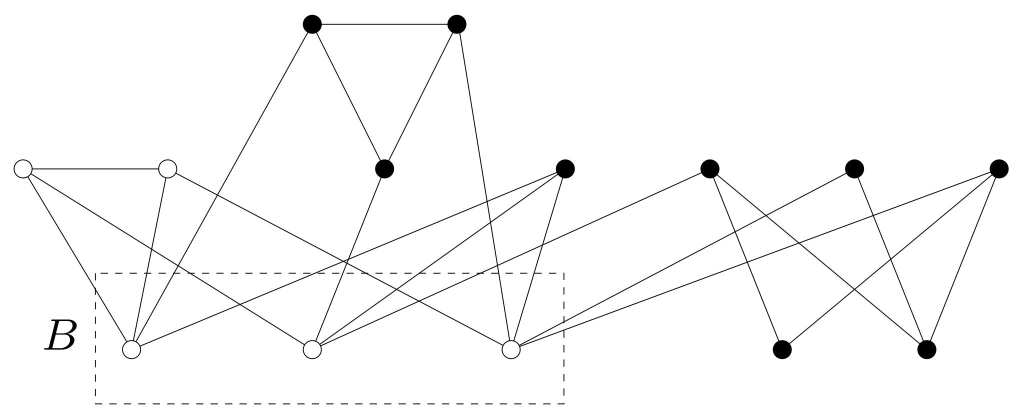
- Theorem [CLM (2004)]
  A vector $\mathbf{x} \in \mathbb{R}^E \in \mathcal{P}\mathrm{oly}(G)$ *of a mc graph* G *iff*:
- $\mathbf{x} \geq \mathbf{0}$ (nonnegativity)
- $\mathbf{x}(\partial(v)) = 1, \forall v \in V$ (degree constraints)
- $\mathbf{x}(C) \geq 1, \forall$ sep C (separating cut constraints)



Barriers and Admissible Edges

- G a matchable graph (ie G has a pm)
- A *barrier* is a set $B \subset V$ st:

$$|\mathcal{O}(G - B)| = |B|$$



- $e \in E$ is *admissible* if $\exists$ pm $M : e \in M$
- Lemma *e is admissible iff no barrier contains both ends of e*

Barriers and Admissible Edges

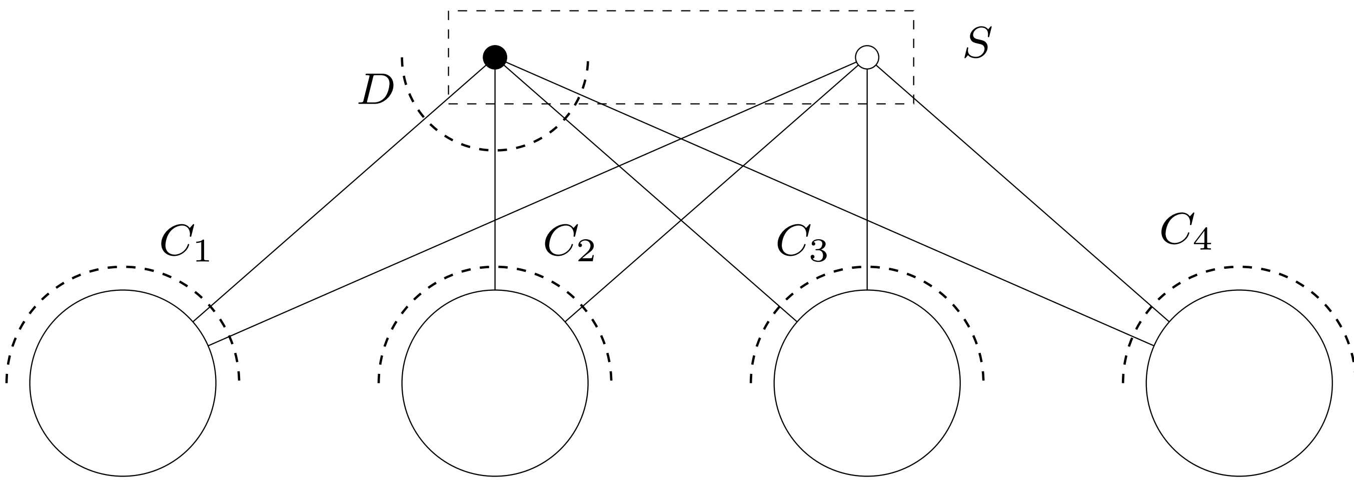
- <u>Lemma</u> *$e := uv$ is admissible iff no barrier contains both u and v*
- only if: $\{u, v\} \subseteq B \Rightarrow e$ not admissible
- converse: if $H := G - u - v$ matchable $\Rightarrow e$ adm
- H not matchable $\Rightarrow \exists S : |\mathcal{O}(H - S)| > |S|$
- parity: $|\mathcal{O}(H - S)| \geq |S| + 2$
- $B := S \cup \{u, v\}$
- $|\mathcal{O}(G - B)| \geq |B|$
- $\quad\therefore\quad B$ is a barrier

Separating Cuts & The PM Polytope

- Suppose $\mathbf{x}(C) < 1$ for some odd C
- $\mathcal{C} := \{\text{odd } D : \mathbf{x}(D) < 1, D \preceq C\}$
- take $D \in \mathcal{C}$ minimal wrt $\preceq$
- $\Rightarrow D$ is separating
- Suppose D not separating $\Rightarrow$ one D-contraction H not mc:
- either it has no pm or some edge $\notin$ pm

Separating Cuts & The PM Polytope

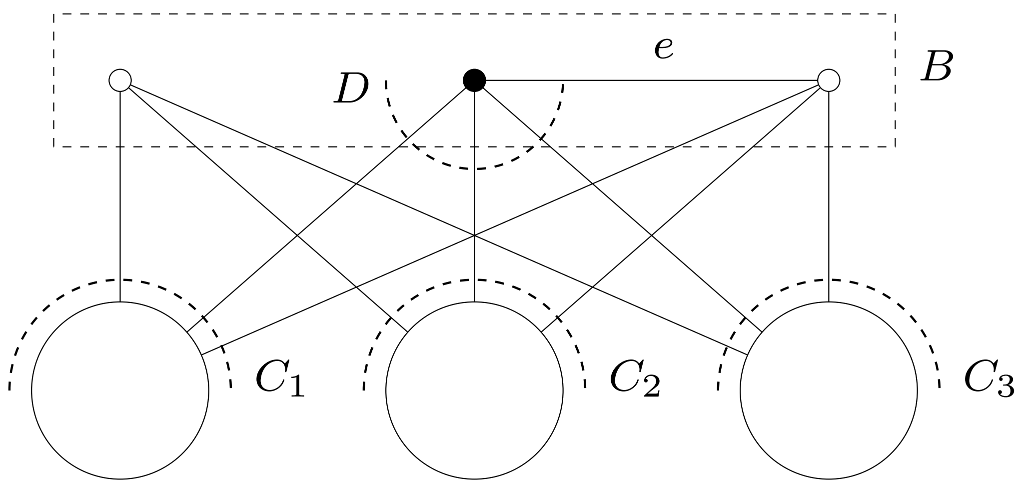
- H not matchable $\Rightarrow \exists S : |\mathcal{O}(H - S)| > |S|$
- G mc $\Rightarrow$ contraction vertex $\in S$



- $\mathbf{x}(D) + 1 \geq \sum \mathbf{x}(C_i)$
- $\quad\therefore\quad \exists i : 1 > \mathbf{x}(C_i)$, say $i = 1$
- $|M \cap D| + 1 = \sum |M \cap C_i| \geq |M \cap C_1| + 3 \qquad \forall M \in \mathcal{M}$
- $C_1 \prec D$, contradiction

Separating Cuts & The PM Polytope

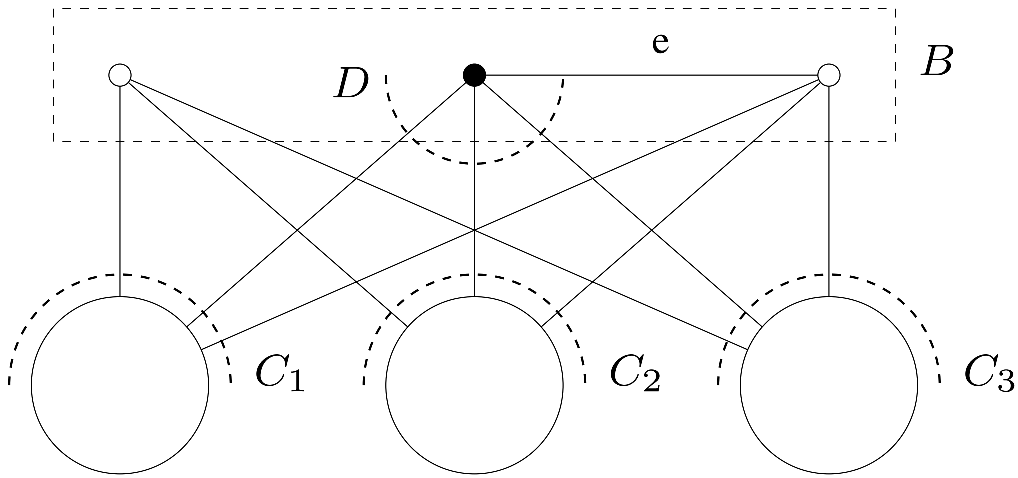
- H matchable, not mc $\Rightarrow \exists e := \{u, v\} : e$ not admissible
- $\exists$ barrier B: $\{u, v\} \subseteq B$
- G mc $\Rightarrow$ contraction vertex $\in B$



- $\mathbf{x}(D) + 2 \geq 2\mathbf{x}(e) + \sum \mathbf{x}(C_i) \geq \sum \mathbf{x}(C_i)$
- $\therefore \quad \exists i : 1 > \mathbf{x}(C_i)$, say $i = 1$

Separating Cuts & The PM Polytope

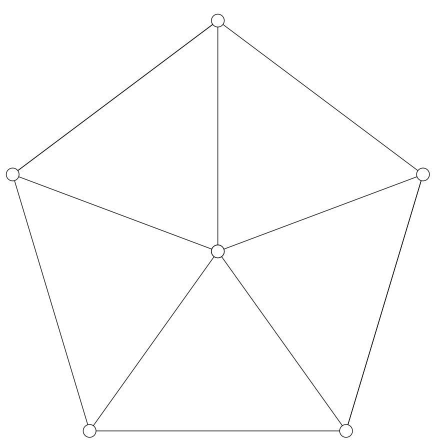
- continuation



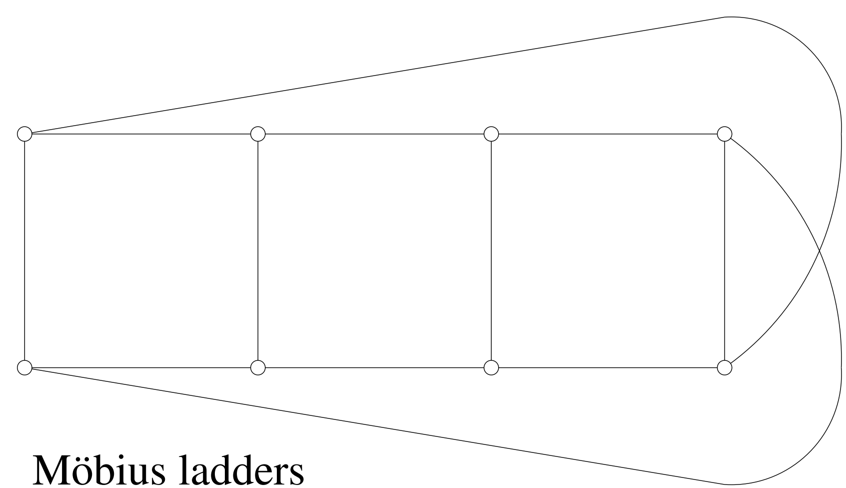
- $\exists i : 1 > \mathbf{x}(C_i)$, say $i = 1$
- $|M \cap D| + 2 \geq 2|M \cap \{e\}| + \sum |M \cap C_i| \quad \forall M \in \mathcal{M}$
- $|M \cap D| + 2 \geq 2|M \cap \{e\}| + |M \cap C_1| + 2$
- $C_1 \prec D$, contradiction

Building Blocks

- Bricks and braces are the building blocks of mc graphs
- In fact, we may also break some bricks, by cut-contractions of separating cuts
- A brick free of nontrivial separating cuts is *solid*
- Examples of Solid Bricks



wheels



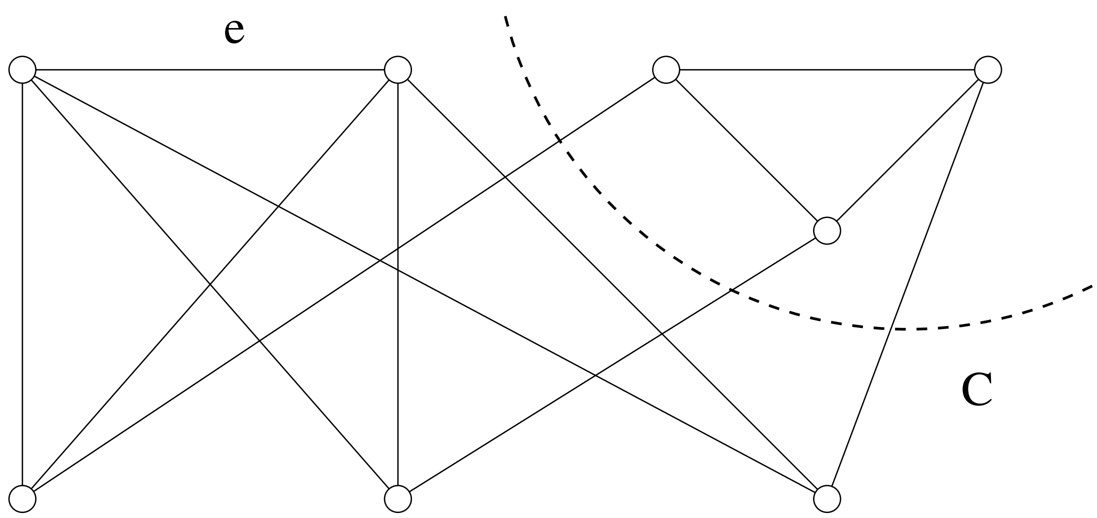
Möbius ladders

Properties of Solid Bricks

- Theorem [Reed and Wakabayshi (2003)]
  A brick G is nonsolid if and only if it has two disjoint odd cycles C_1 and C_2 such that $G - [V(C_1) \cup V(C_2)]$ has a perfect matching
- Corollary *Every odd intercyclic brick is solid*
- Theorem [CLM (2004)]
  A brick is solid if and only if its perfect matching polytope is characterized only by the degree constraints
  $$\forall \mathbf{x} \in \mathbb{R}^E, \mathbf{x} \geq \mathbf{0}:$$
  $$\mathbf{x} \in \mathcal{P}\text{oly} \Leftrightarrow \mathbf{x}(v) = 1 \quad (\forall v \in V(G))$$
- Solidity of Bricks is in co-NP
- Open Is Solidity of Bricks in P? in NP?

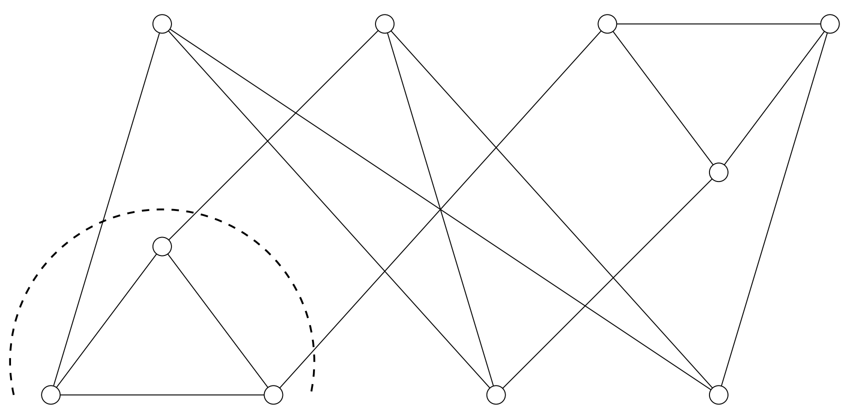
More Examples of Solid Bricks

- Möbius Ladders M_{4n}, $n \geq 1$ (odd intercylic)
- Odd Wheels W_{2n+1}, $n \geq 1$ (odd intercylic)
- Theorem [CLM (2006), Kothari and Murty (2015)]
  The odd wheels are the only planar solid bricks
- Murty's graph, a solid brick that is not odd intercyclic



Robust Cuts & The PM Polytope

- Theorem *A vector* $\mathbf{x} \in \mathbb{R}^E \in \mathcal{P}\mathrm{oly}(G)$ *of a brick* G *iff*:
- $\mathbf{x} \geq \mathbf{0}$ (nonnegativity)
- $\mathbf{x}(\partial(v)) = 1, \forall v \in V$ (degree constraints)
- $\mathbf{x}(C) \geq 1, \forall$ sep C (robust cut constraints)
- C is *robust* if both C-contractions are near-bricks
- G is a *near-brick* if $b(G) = 1$
- Example:

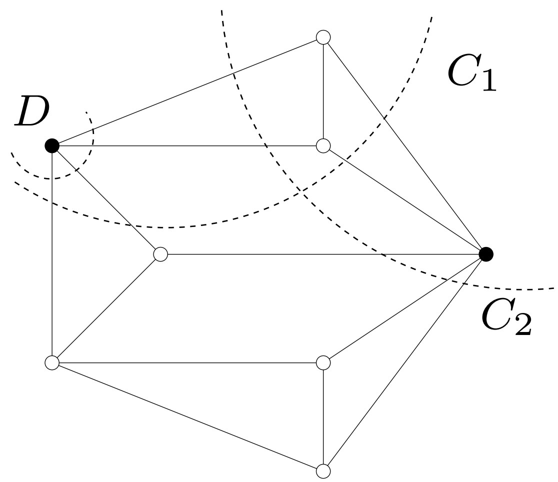


Getting Robust Cuts

- Suppose G is a brick and $\mathbf{x}(C) < 1$ for some odd C
- $\mathcal{C} := \{\text{odd } D : \mathbf{x}(D) < 1, D \preceq C\}$
- take $D \in \mathcal{C}$ minimal wrt $\preceq$
- $\Rightarrow$ D is robust
- previous reasoning: D is separating
- Suppose D not robust $\Rightarrow$ one D-contraction H not near-brick
- G brick $\Rightarrow b(H) > 0$ else D tight $\quad \therefore \quad b(H) \geq 2$
- H has nontrivial tight cuts
- ELP $\Rightarrow$ H has a nontrivial tight cut that is either a 2-separation cut or a barrier cut

Case: H Has a 2-Separation Cut S

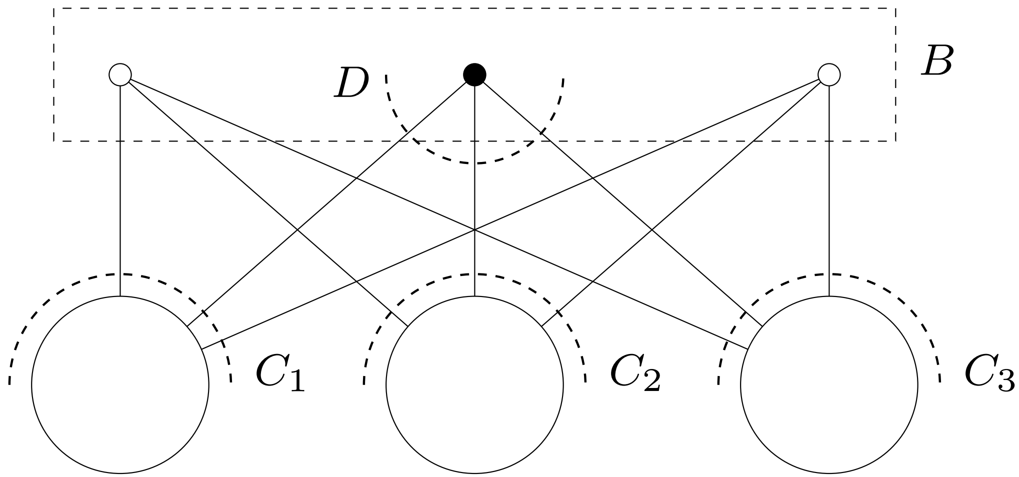
- G brick $\Rightarrow$ the contraction vertex $\in S$



- $\mathbf{x}(D) + 1 = \mathbf{x}(C_1) + \mathbf{x}(C_2)$
- $\therefore \quad \exists i : 1 > \mathbf{x}(C_i)$, say $i = 1$
- $|M \cap D| + 1 = |M \cap C_1| + |M \cap C_2| \quad \forall M \in \mathcal{M}$
- G brick $\Rightarrow$ C_2 not tight
- $C_1 \prec D$, contradiction

G has a nontrivial barrier B

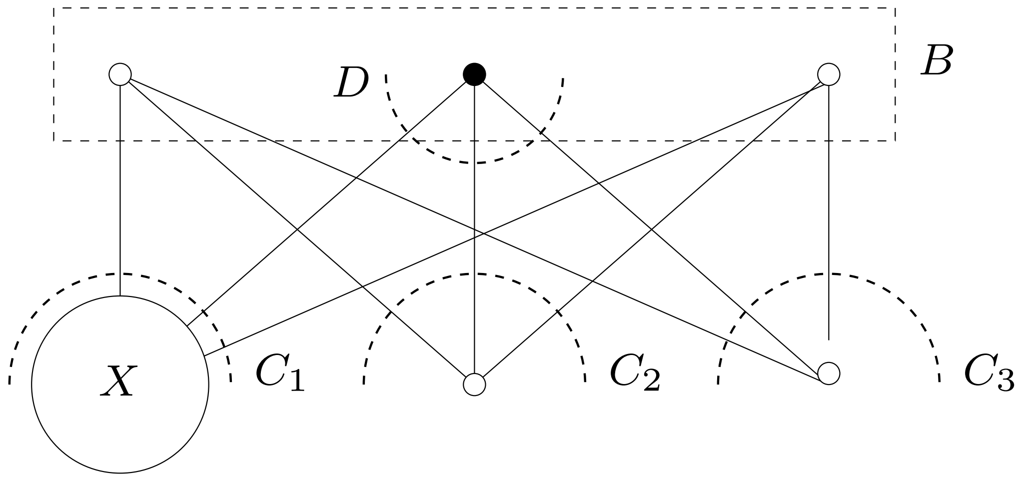
- G brick $\Rightarrow$ the contraction vertex $\in B$



- $\mathbf{x}(D) + 2 \geq \sum \mathbf{x}(C_i)$
- $\quad\therefore\quad \exists i : 1 > \mathbf{x}(C_i)$, say $i = 1$
- $|M \cap D| + 2 = \sum |M \cap C_i| \geq |M \cap C_1| + 2 \quad \forall M \in \mathcal{M}$
- $C_1 \preceq D$
- C_2 or C_3 nontrivial $\Rightarrow C_1 \prec D$, contradiction

Case: H has a nontrivial barrier B

- C_i is trivial ($\forall i \geq 2$)



- $|M \cap D| = |M \cap C_1|$
- C_1 and D are <u>*matching-equivalent*</u>
- $H_1 :=$ contract $\overline{X}$
- $b(H_1) = b(H) \geq 2$
- repeat the process with C_1 playing the role of D

Existence of Robust Cuts

- From the previous reasoning:
- Corollary [CLM (2002)]
  Every nonsolid brick has a robust cut