São Paulo School of Advanced Science on Algorithms, Combinatorics and Optimization

The Perfect Matching Polytope, Solid Bricks and the Perfect Matching Lattice

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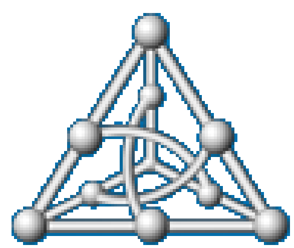
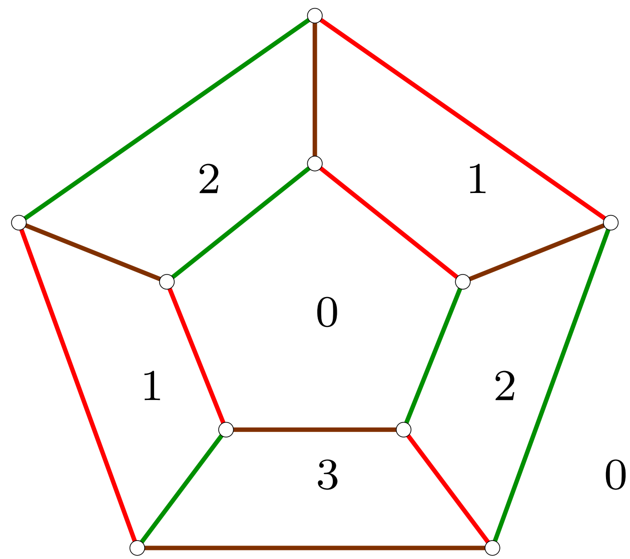


FACOM, UFMS, Brazil

Genesis

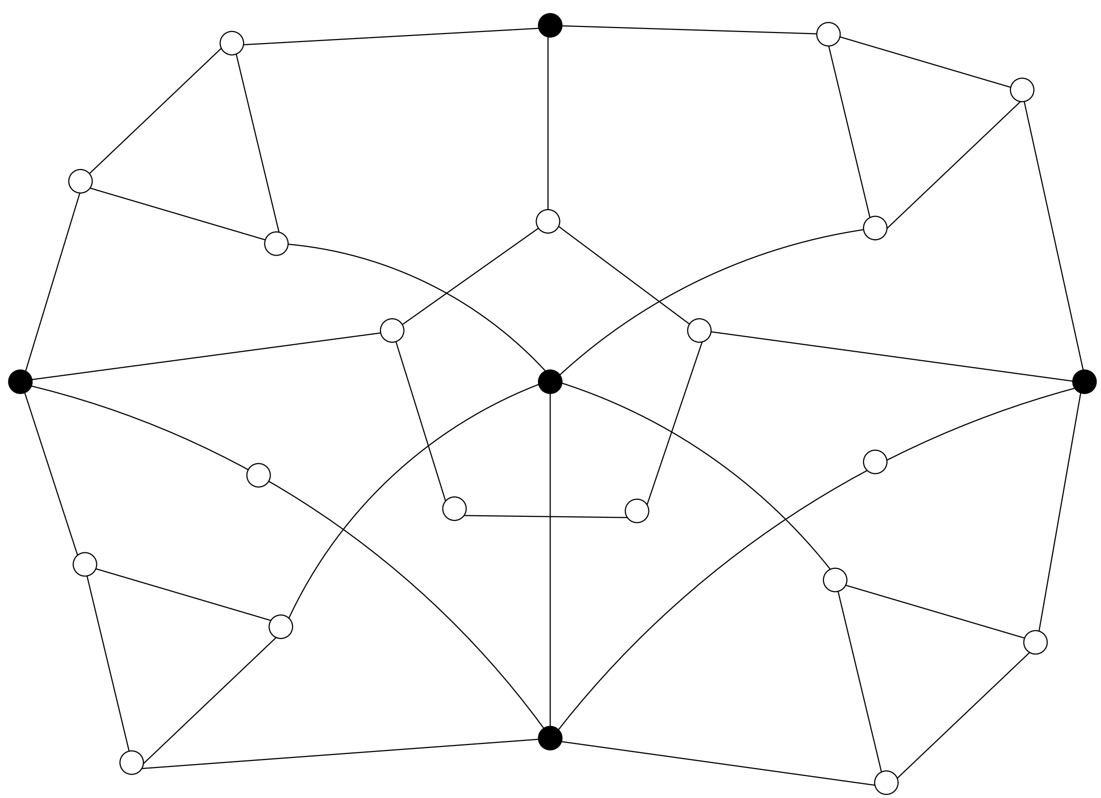
- <u>Theorem</u> [Tait (1880)]
  A 2-connected cubic graph is 4-face-colourable iff it is 3-edge-colourable



- <u>Theorem</u> [Petersen (1891)]
  Every 2-connected cubic graph has a perfect matching

Genesis

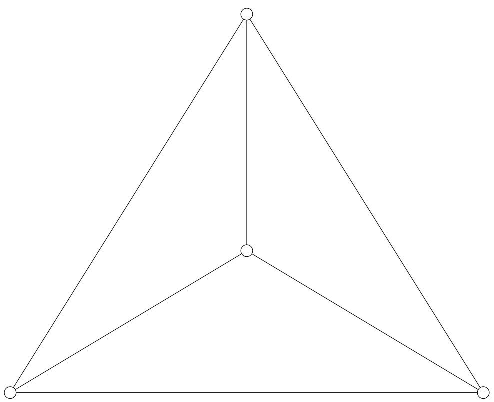
- Theorem [Tutte (1947)]
  A graph G admits a perfect matching iff
  $|\mathcal{O}(G - S)| \leq |S| \quad \forall S \subset V$

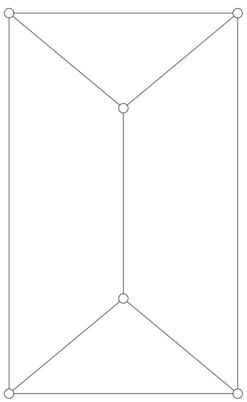


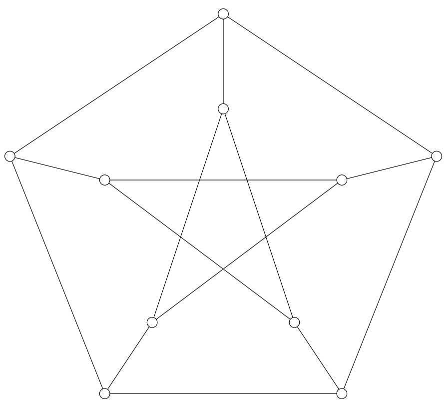
Matching Covered Graphs

- Corollary *Every edge of a 2-connected cubic graph is in a perfect matching*
- a *matching covered graph* is a connected nontrivial graph such that every edge is in a perfect matching
- Corollary *Every 2-connected cubic graph is mc*
- Lemma *Every mc graph G with $|V| \geq 4$ is 2-connected*

Illustrious Cubic Graphs

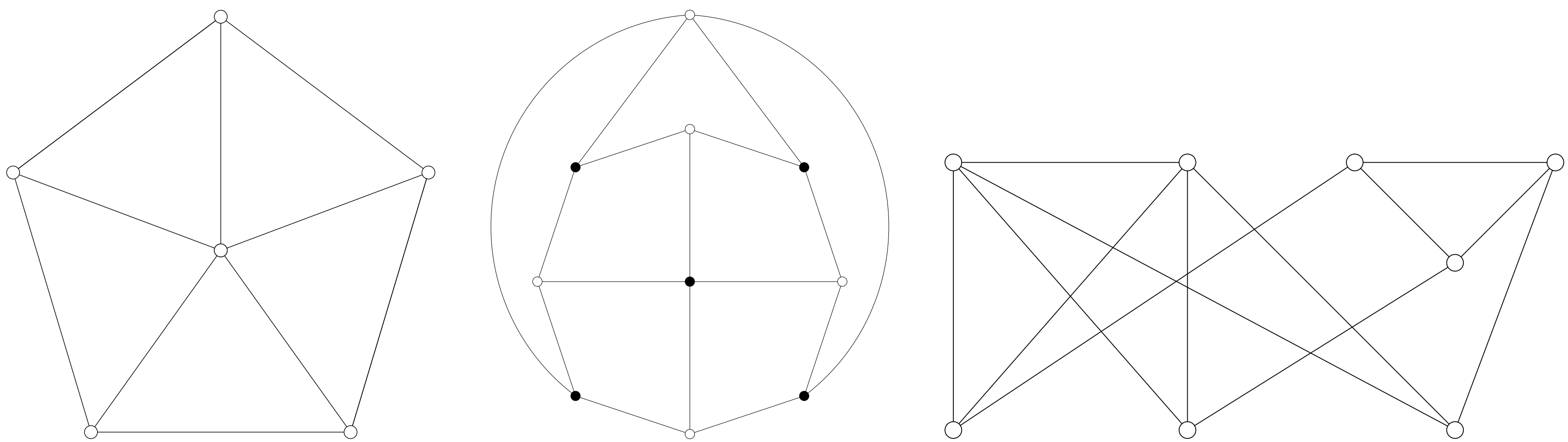






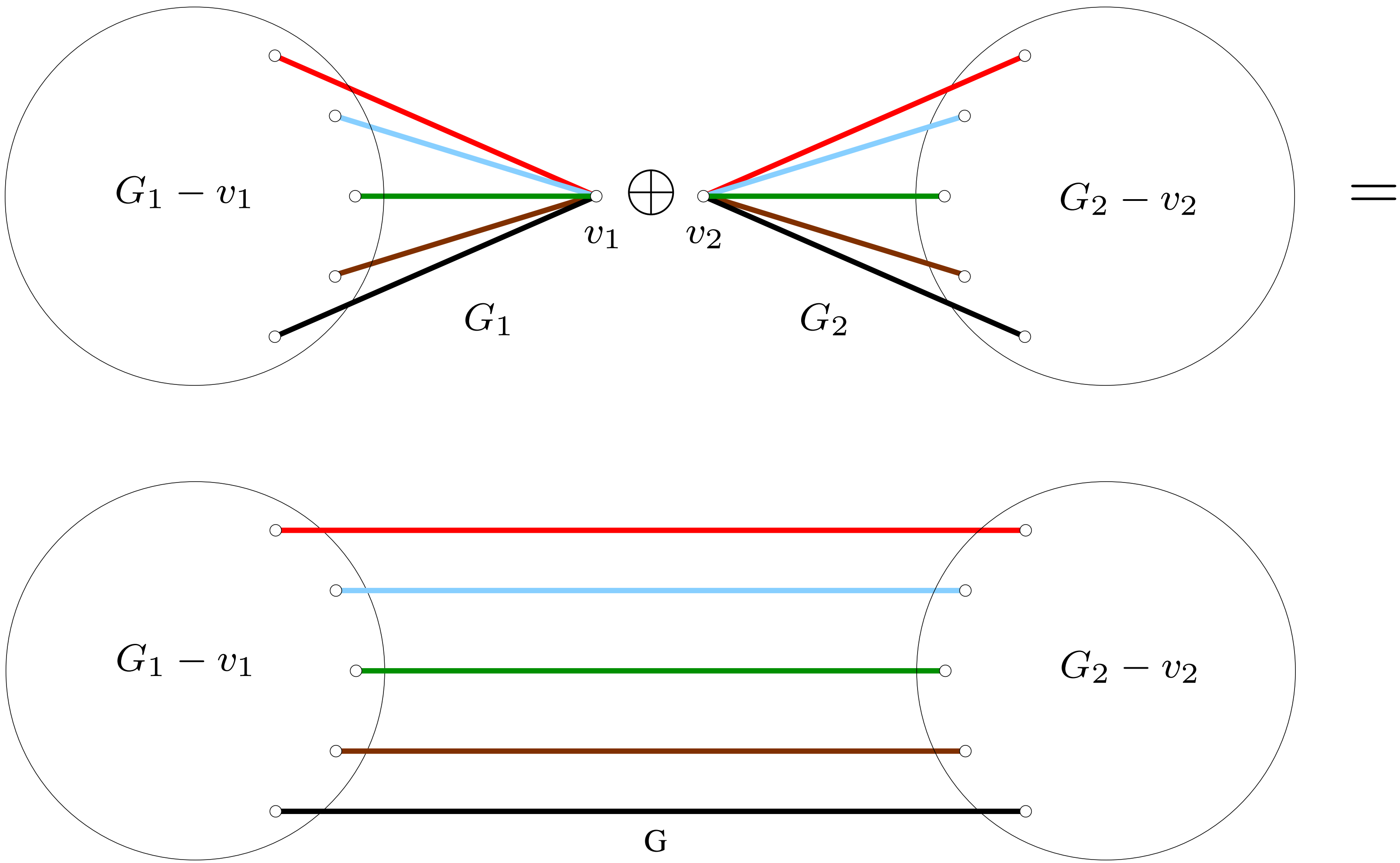
Noncubic mc graphs

- W_5, B_{10} and Murty's graph are examples of noncubic mc graphs:



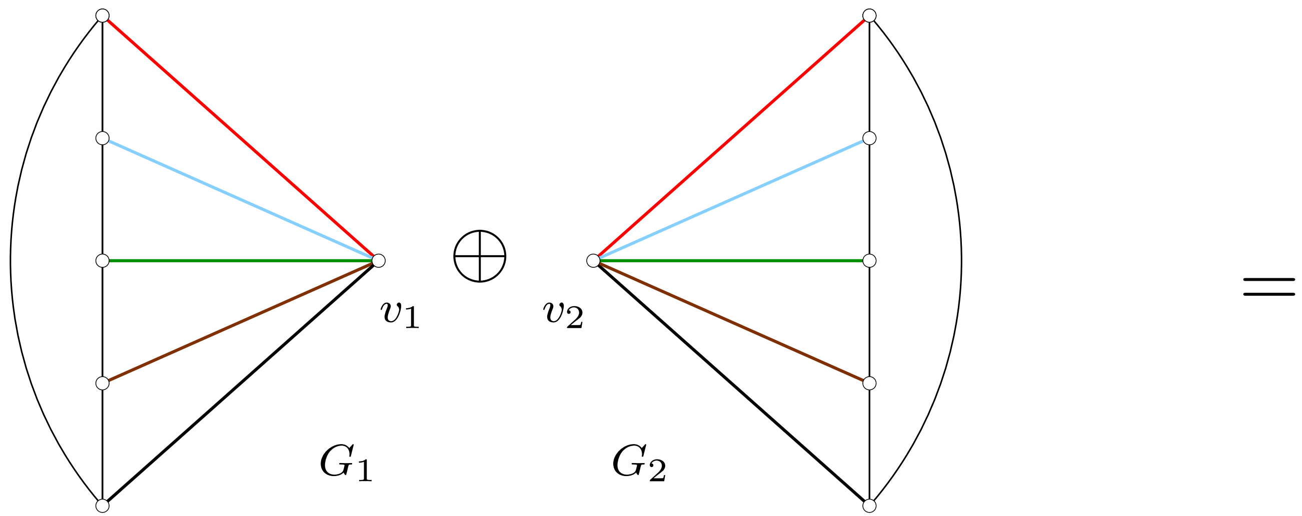
Building Blocks

- Splicing of two mc graphs yields another mc graph



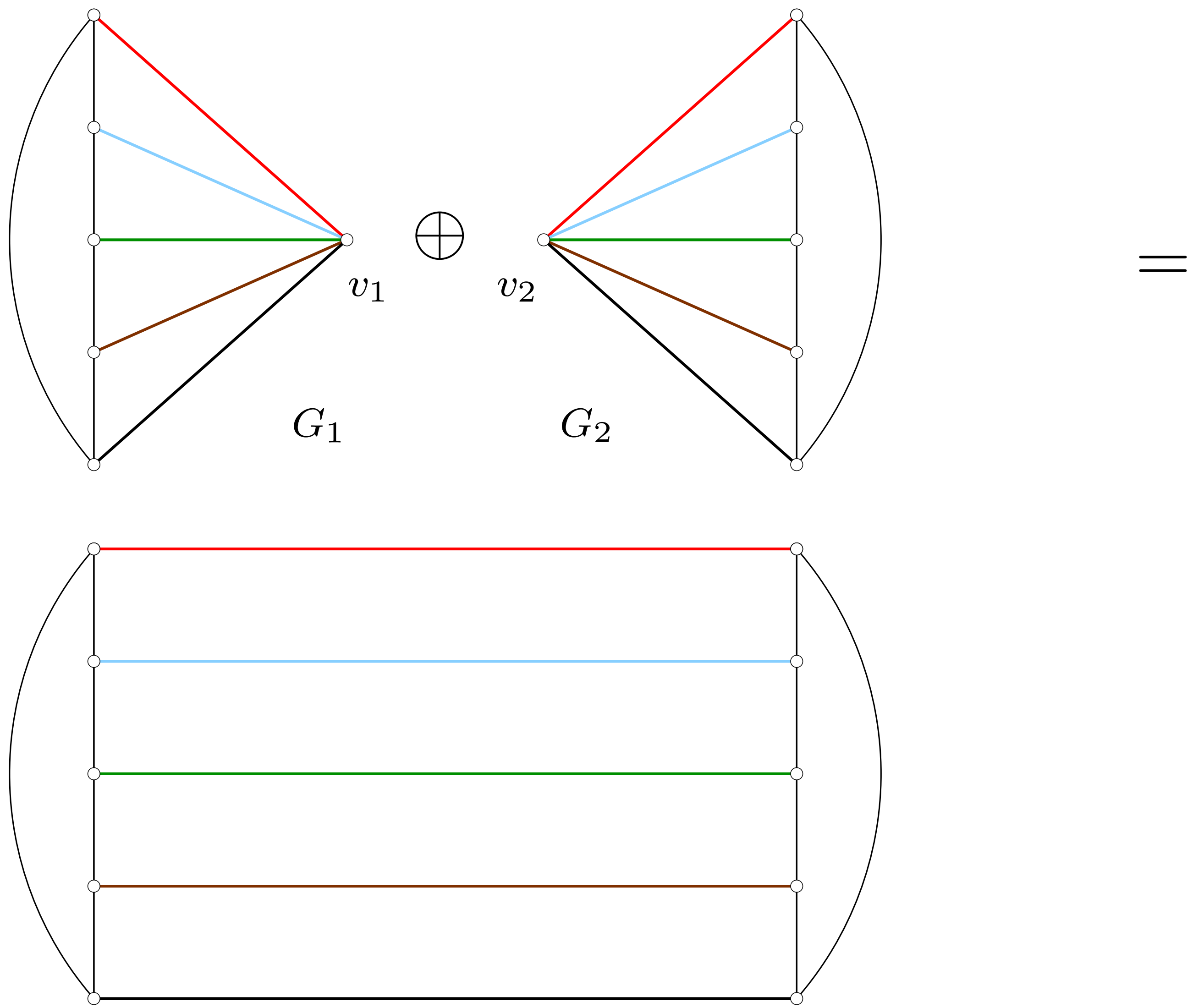
Building Blocks

- Splicing



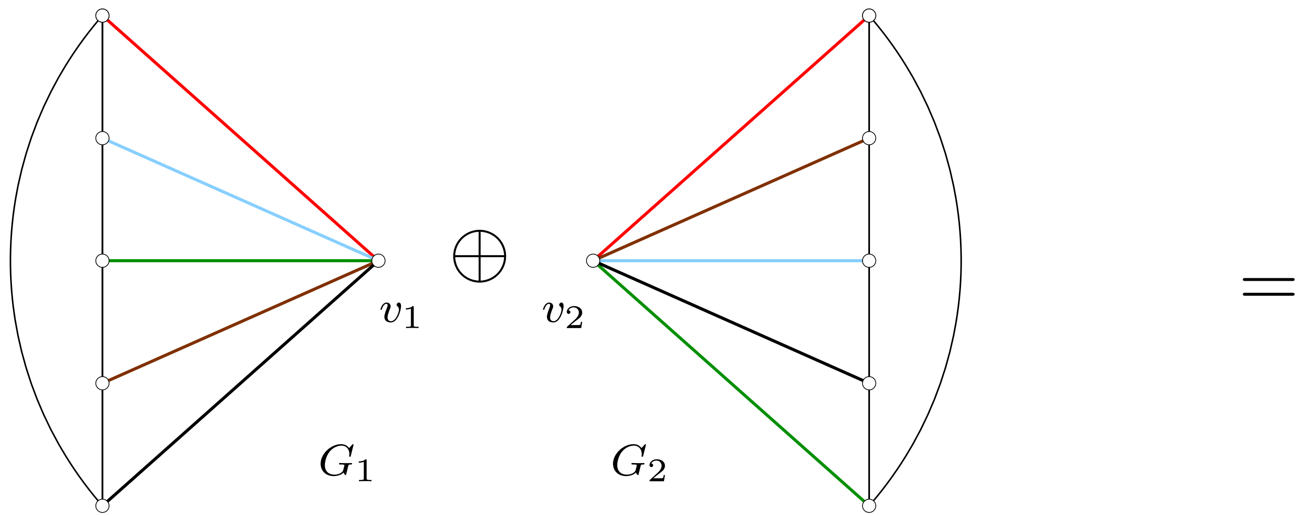
Building Blocks

- Splicing $\Rightarrow P_{10}$



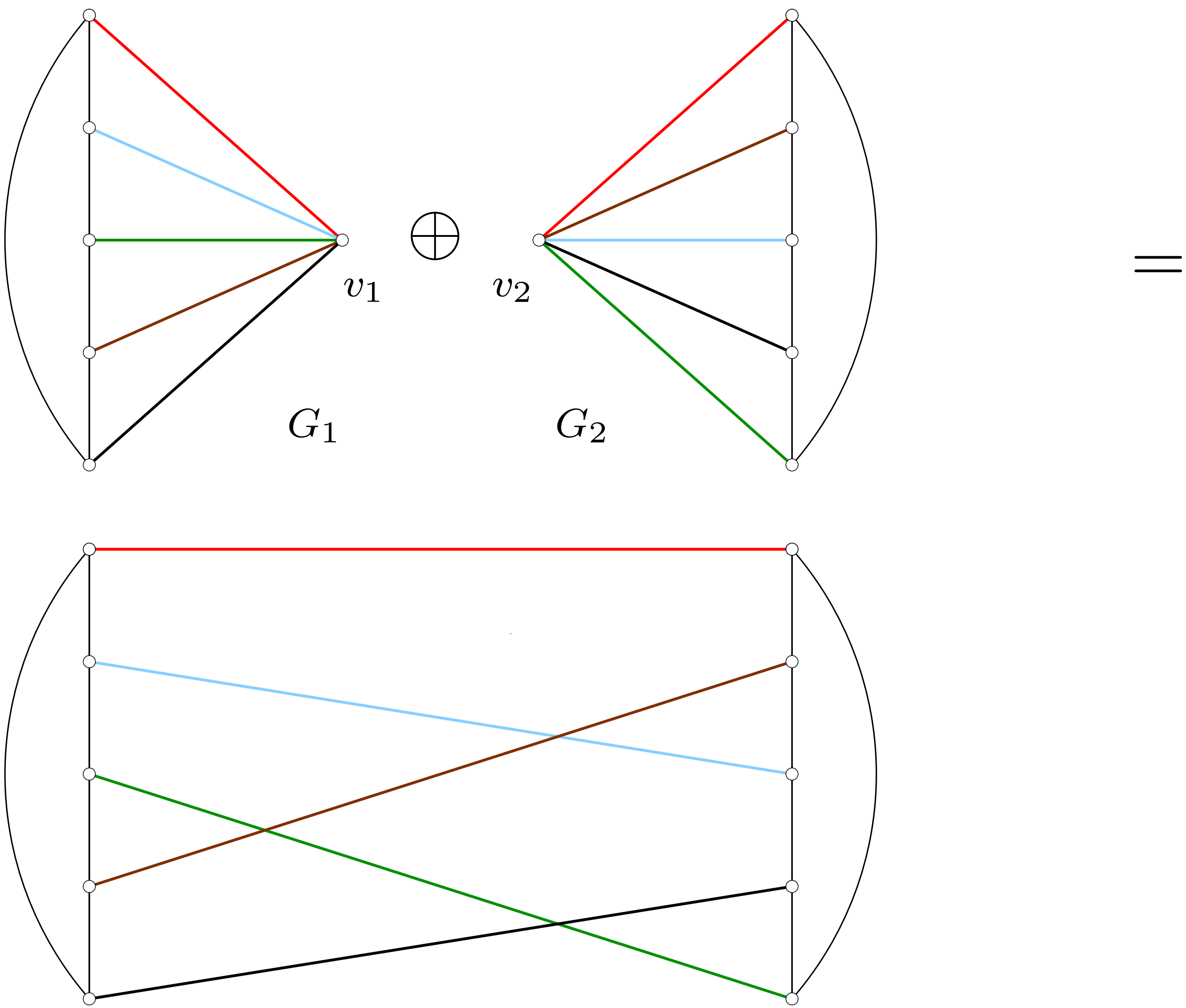
Building Blocks

- Splicing



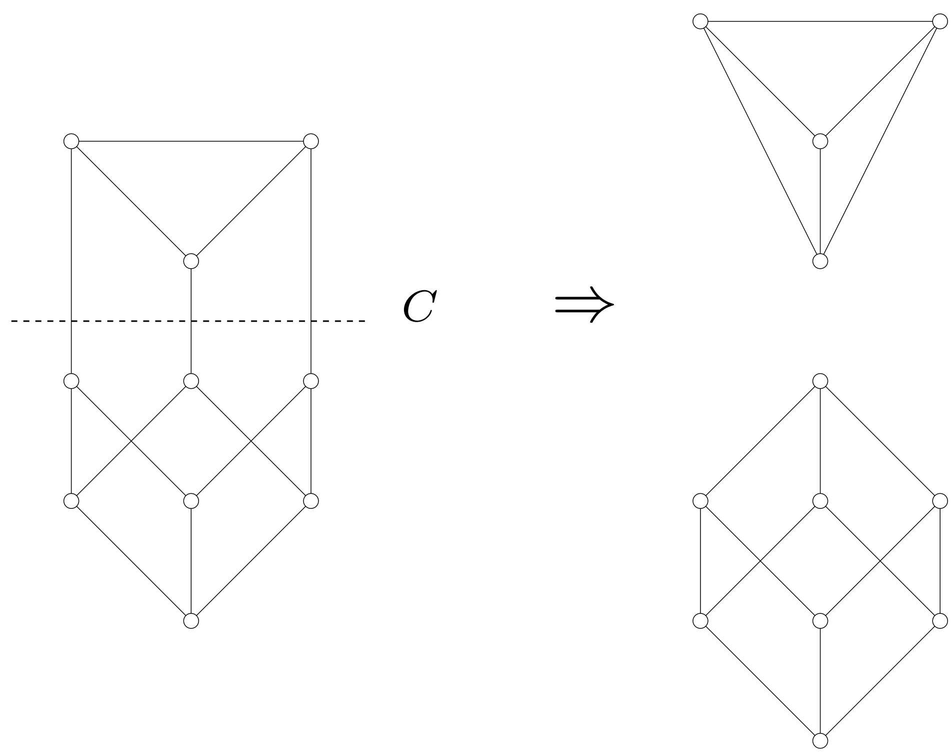
Building Blocks

- Splicing $\Rightarrow \mathbb{P}$



Separating Cuts

- Which mc graphs may be obtained by splicing two smaller mc graphs?
- Those mc graphs which have *separating cuts*
- *Cut-contraction* is the inverse of splicing

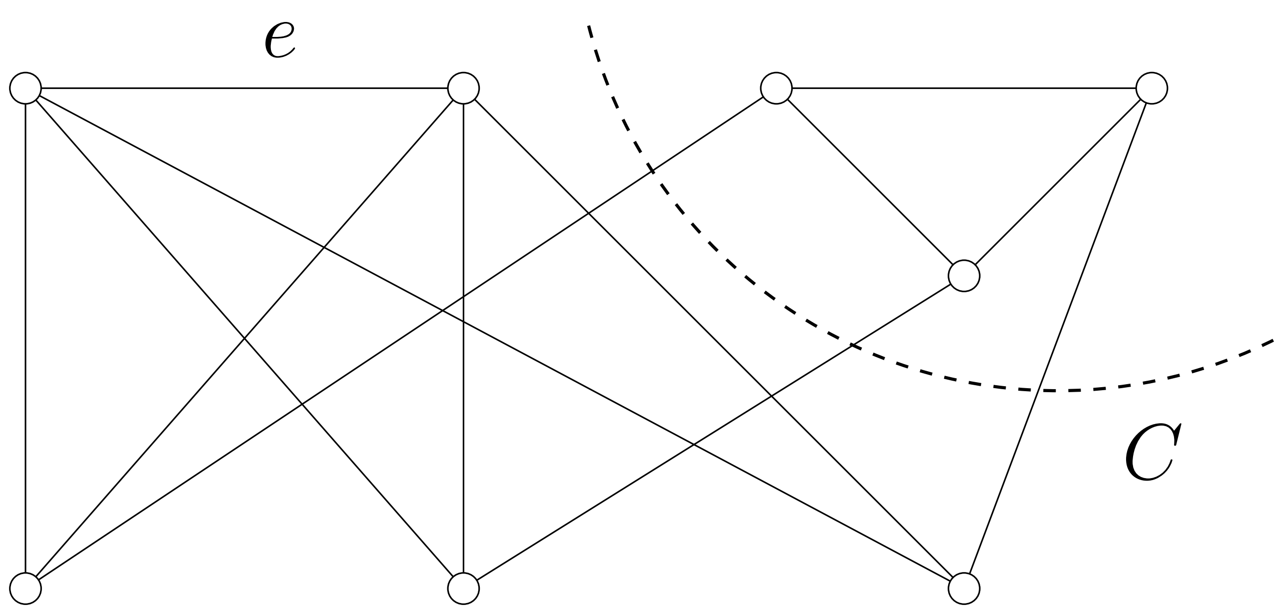


Separating Cuts

- a cut C of a mc G is *separating* if both C-contractions are mc
- Theorem *A cut C of mc G is* separating *iff*

$$\forall e \in E(G) \quad \exists \text{ pm } M : \quad e \in M, \quad |M \cap C| = 1$$

- A cut that is not separating

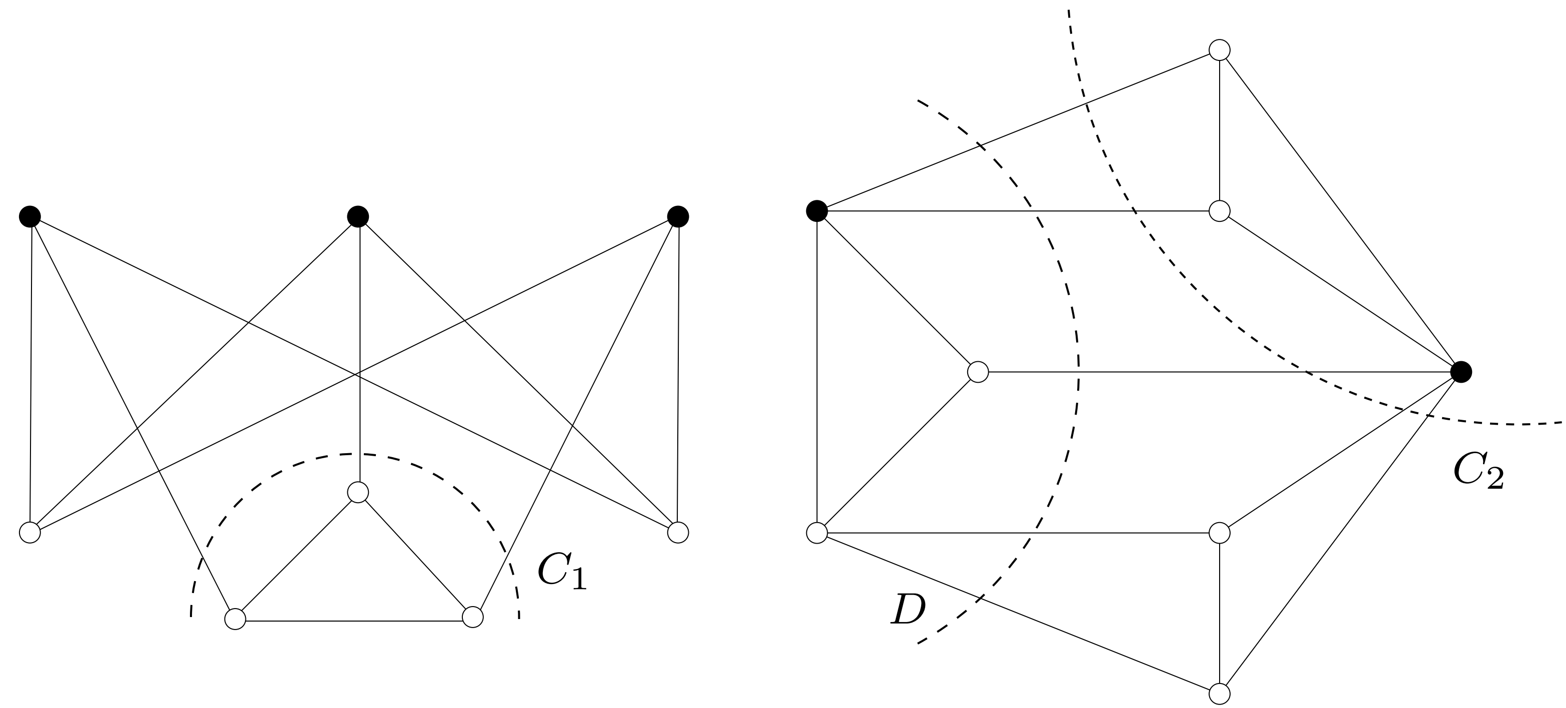


Tight Cuts

- A cut C of mc G is *tight* if $|M \cap C| = 1 \quad \forall M \in \mathcal{M}$
- Tight cuts are a special type of separating cuts
- mc graphs free of nontrivial tight cuts:
  - bipartite graphs: *braces*
  - nonbipartite graphs: *bricks*

Tight Cuts

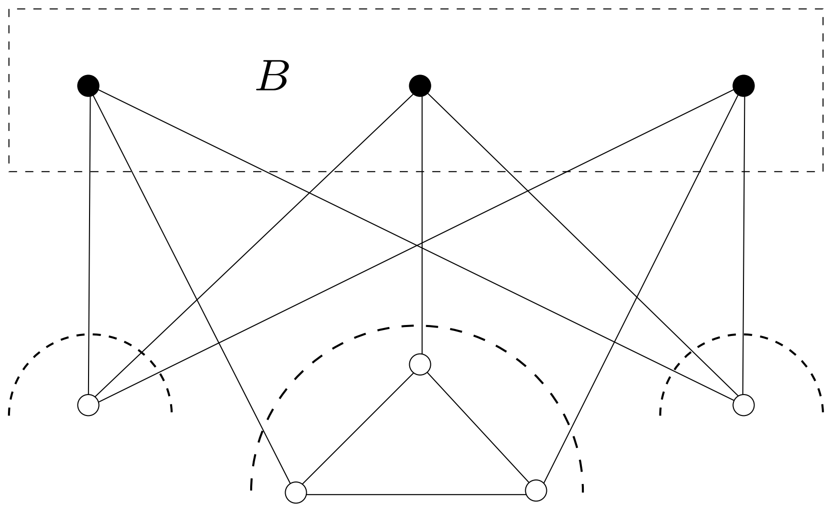
- special types of tight cuts



- C_1: a *barrier cut*
- C_2: a *2-separation cut*
- D: neither a barrier nor a 2-separation cut

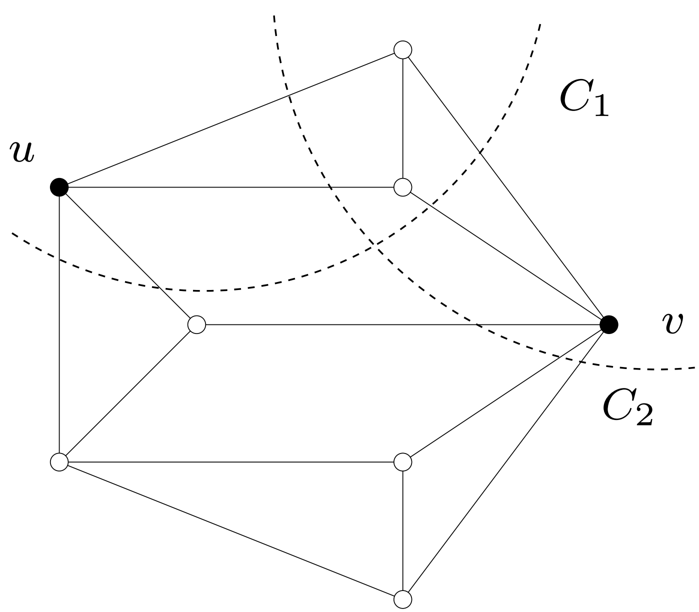
Barrier Cuts

- mc G, $B \subset V$ is a barrier if $|\mathcal{O}(G - B)| = |B|$
- given barrier B of mc G, and $K \in \mathcal{O}(G - B)$, $\partial(V(K))$ is a barrier cut



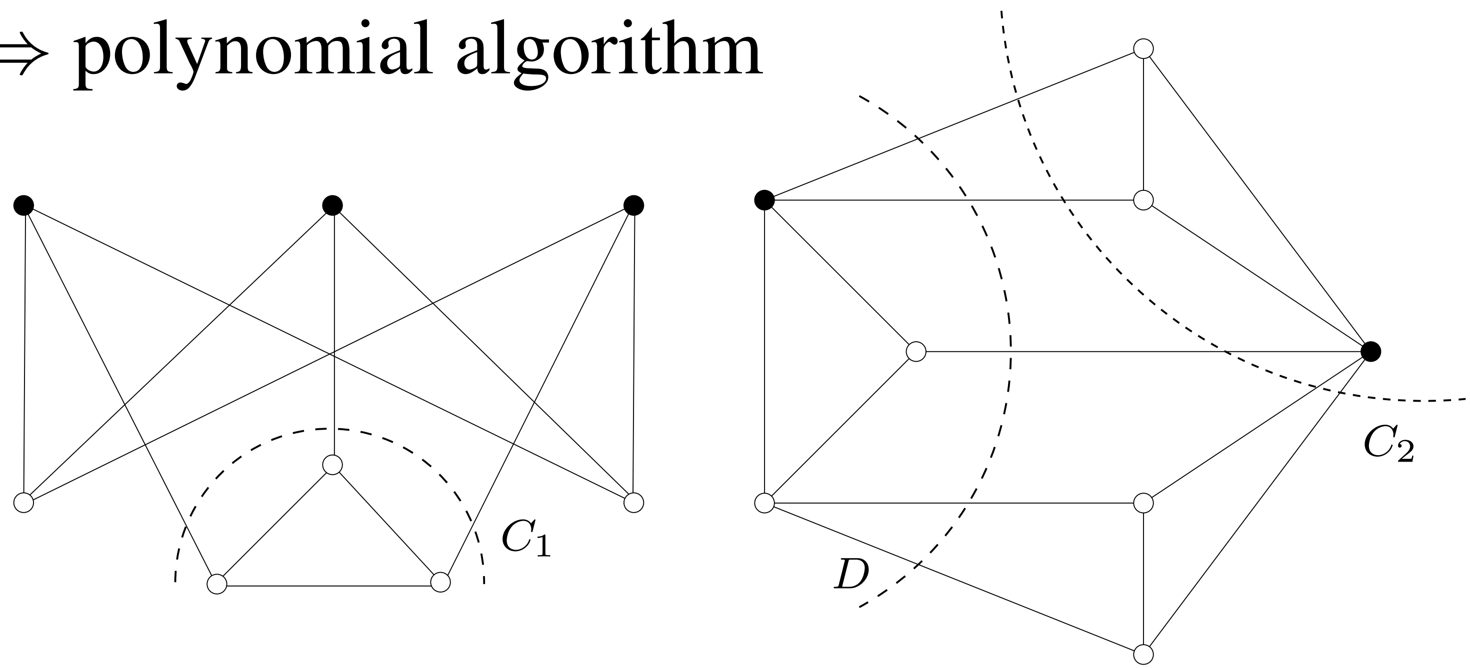
2-Separation Cuts

- mc G, a pair $S := \{u, v\} \subset V$ is a *2-separation* if
  - $G - S$ is not connected and
  - each component of $G - S$ is even
- 2-sep $\{u, v\}$ of mc G, component K of $G - u - v$, $\partial(\{u\} \cup V(K))$ and $\partial(\{v\} \cup V(K))$ are *2-sep cuts*



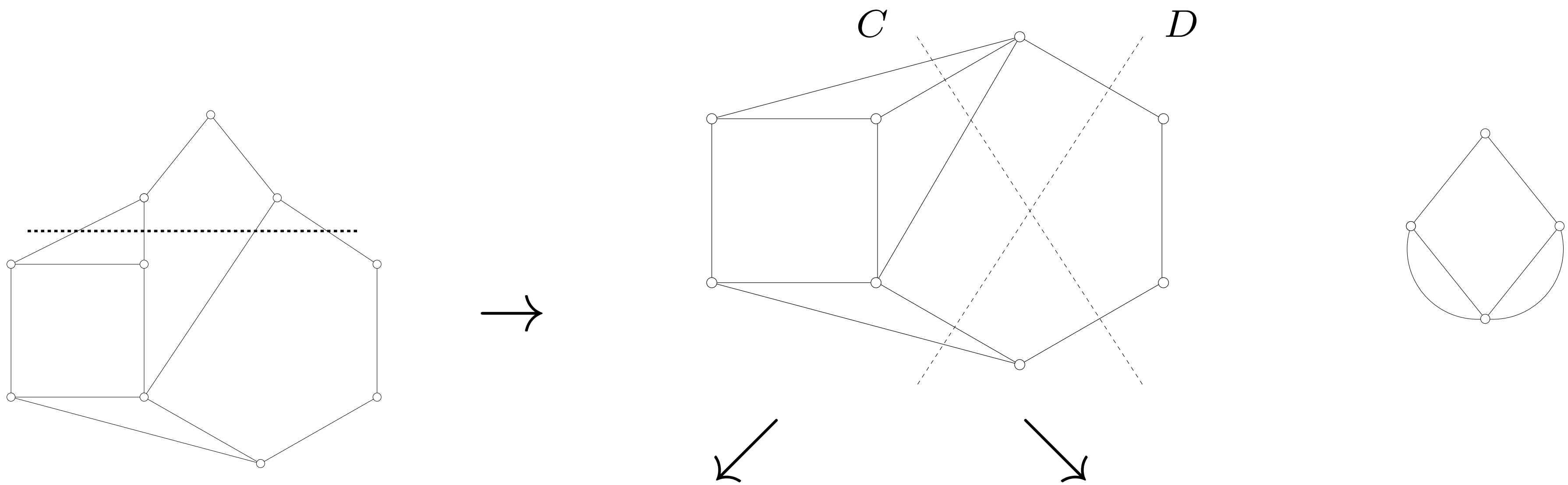
Tight Cuts

- *ELP cut*: nontrivial barrier cut or 2-sep cut
- Theorem [Edmonds, Lovász, Pulleyblank (1982)]
  If a mc graph has a nontrivial tight cut then it has an ELP cut
- $\Rightarrow$ polynomial algorithm

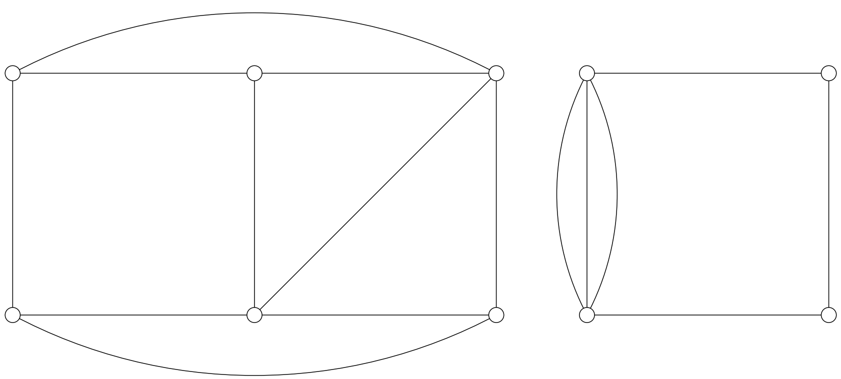


- C_1, C_2 are ELP, but D is not

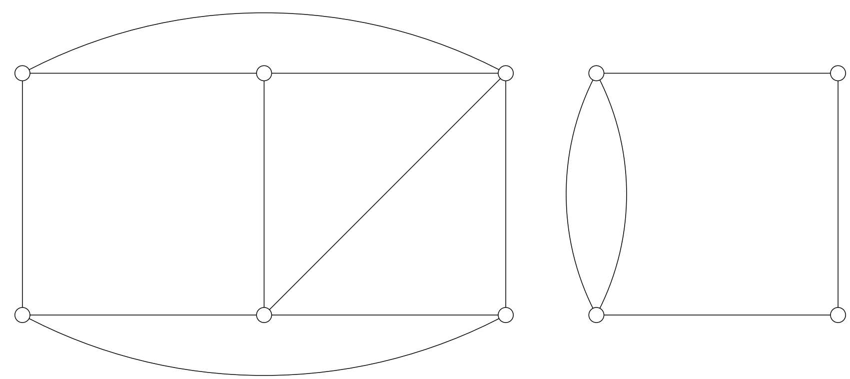
Tight Cut Decomposition



C-contractions



D-contractions

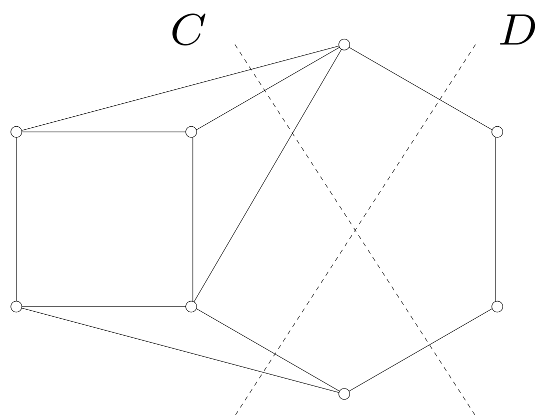


Tight Cut Decomposition

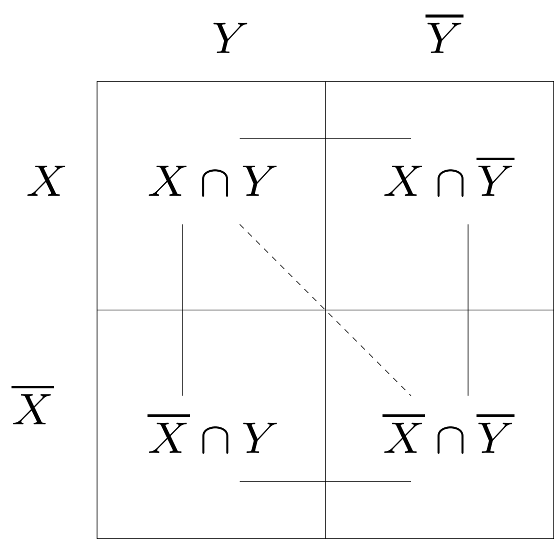
- Theorem [Lovász (1987)]
  Any two applications of the tight cut decomposition procedure produces the same collection of bricks and braces, up to multiple edges
- *proof* by induction on $|V|$

Crossing Cuts

- Crossing Cuts

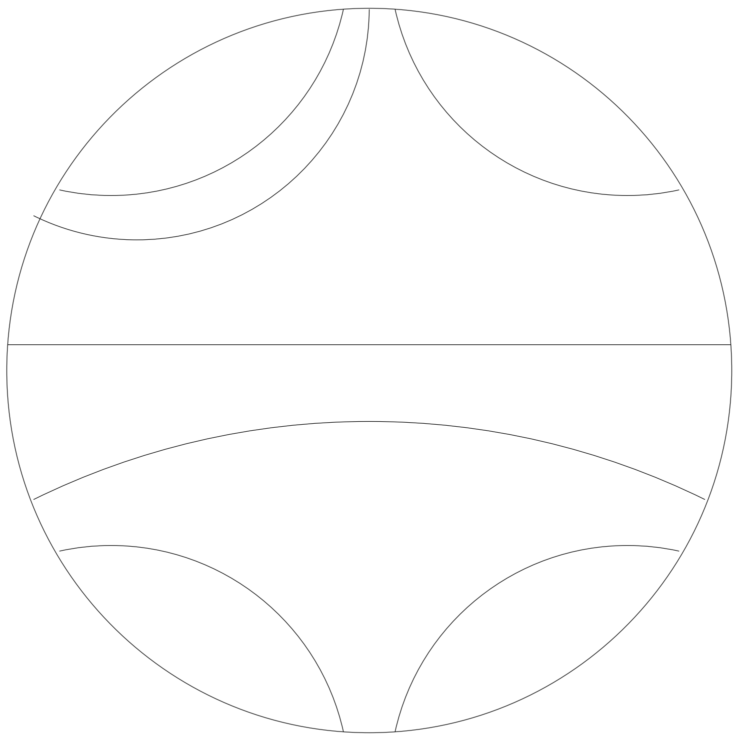


- $\partial(X)$ and $\partial(Y)$ ***cross***



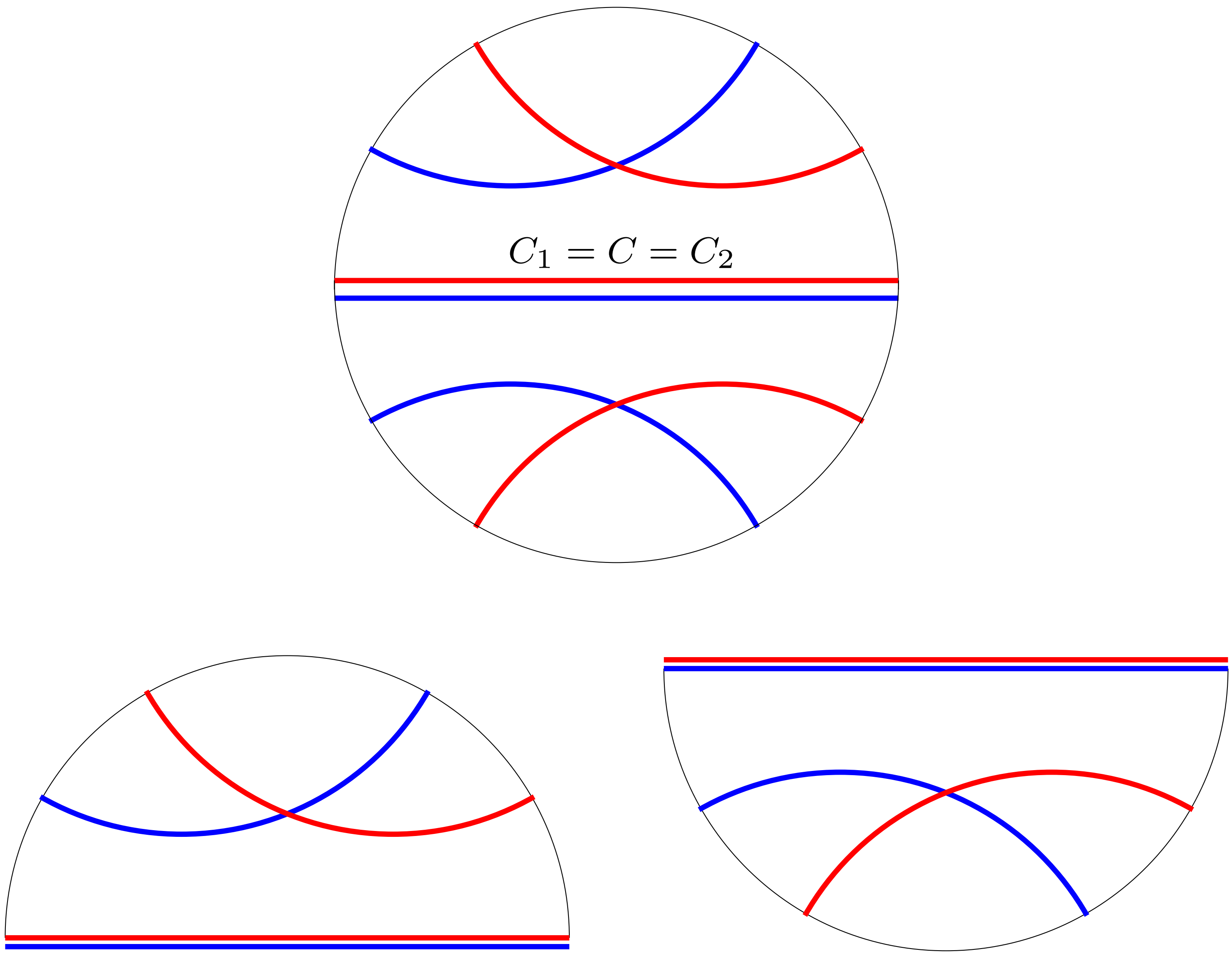
Tight Cut Decomposition

- Tight cut decomposition $\Leftrightarrow$ maximal laminar collection of nontrivial tight cuts

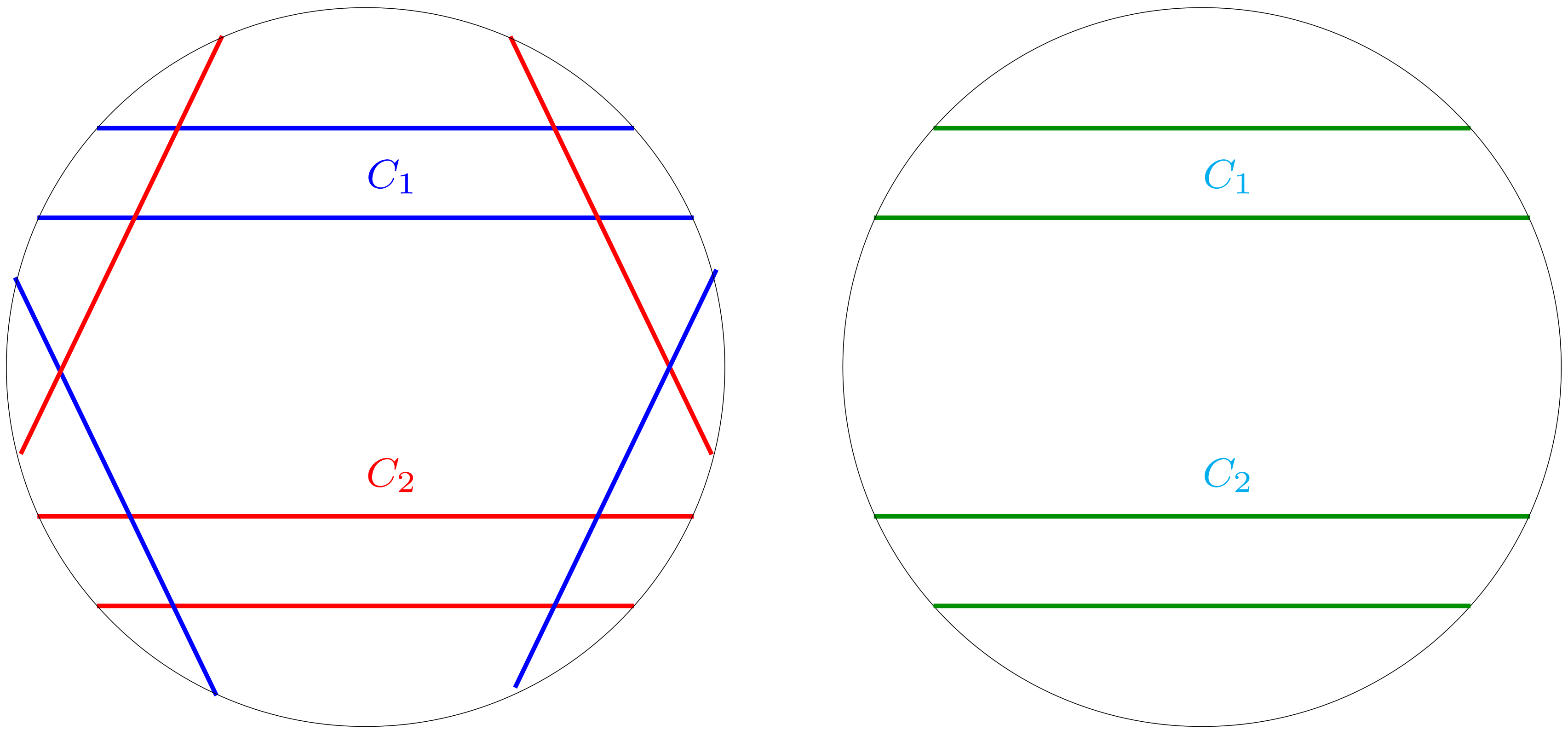


- laminar $\Leftrightarrow$ cuts do not cross

Common Cut C

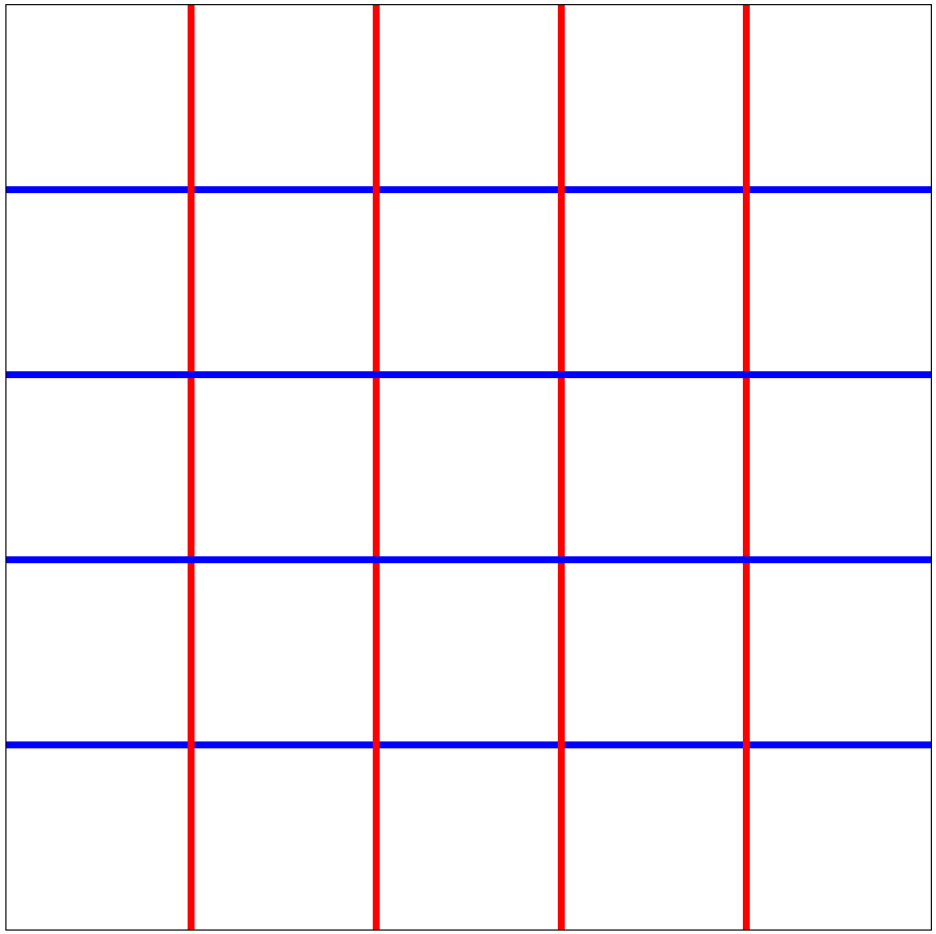


Blue C_1 and Red C_2 do not cross



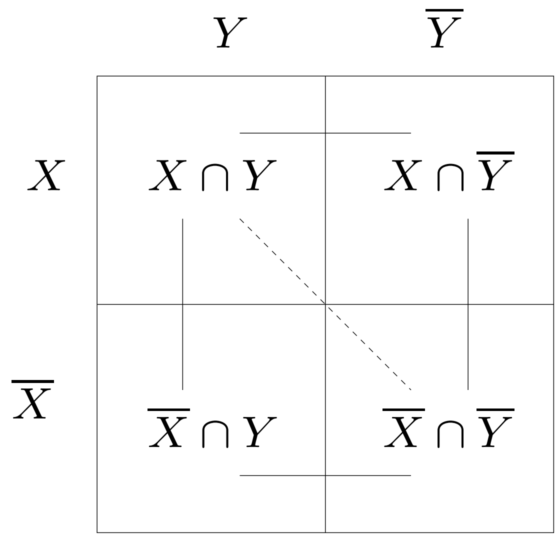
- Blue C_1 and green C_1 : previous case
- Red C_2 and green C_2 : previous case
- $\therefore$ Every blue C_1 and red C_2 cross

Every blue C_1 and red C_2 cross



Crossing Tight Cuts

- Lemma *If tight cuts* $\partial(X)$ *and* $\partial(Y)$ *cross, where* $|X \cap Y|$ *is odd, then no edge joins a vertex in* $X \cap \overline{Y}$ *to a vertex in* $\overline{X} \cap Y$



- Corollary $\forall S \subseteq E$

$$|S \cap \partial(X)| + |S \cap \partial(Y)| =$$

$$|S \cap \partial(X \cap Y)| + |S \cap \partial(\overline{X} \cap \overline{Y}|)$$

Crossing Tight Cuts

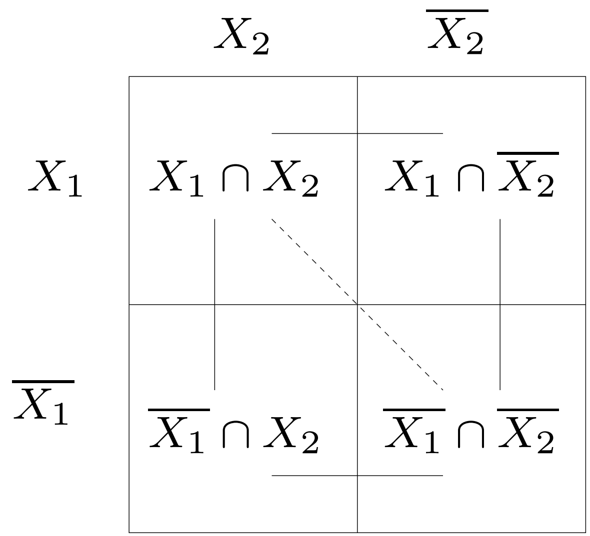
- Corollary *If tight cuts* $\partial(X)$ *and* $\partial(X)$ *cross, where* $|X \cap Y|$ *is odd,*

$$\forall S \subseteq E$$
$$|S \cap \partial(X)| + |S \cap \partial(Y)| =$$
$$|S \cap \partial(X \cap Y)| + |S \cap \partial(\overline{X} \cap \overline{Y}|)$$

- Corollary *If tight cuts* $\partial(X)$ *and* $\partial(Y)$ *cross, where* $|X \cap Y|$ *is odd, then* $\partial(X \cap Y)$ *and* $\partial(\overline{X} \cap \overline{Y})$ *are both tight*

$\partial(X_1)$, $\partial(X_2)$ cross, $|X_1 \cap X_2|$ odd, nontrivial

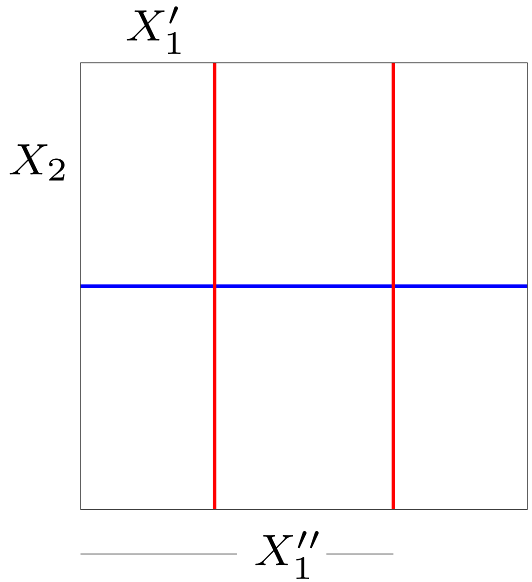
- $C_1 := \partial(X_1)$, $C_2 := \partial(X_2)$, $C_3 := \partial(X_1 \cap X_2)$ is tight



- green uses blue C_1 and C_3
- brown uses red C_2 and C_3
- previous case:
  - green $\sim$ blue (common C_1)
  - brown $\sim$ red (common C_2)
  - green $\sim$ brown (common C_3)

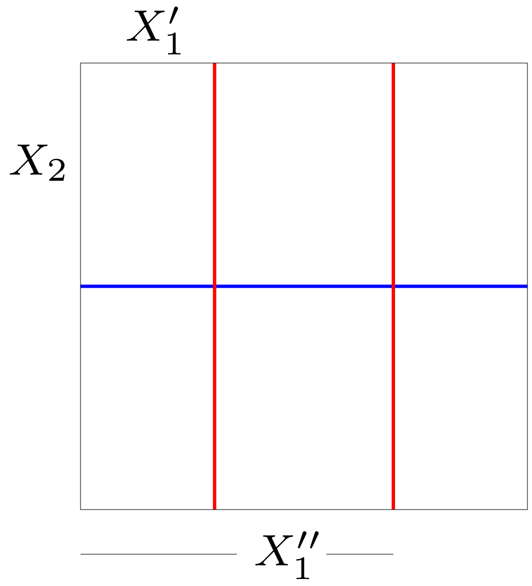
Last Case

- just one red cut
- assume two or more, $C_1' = \partial(X_1')$, $C_2' = \partial(X_1'')$, $X_1' \subset X_1''$



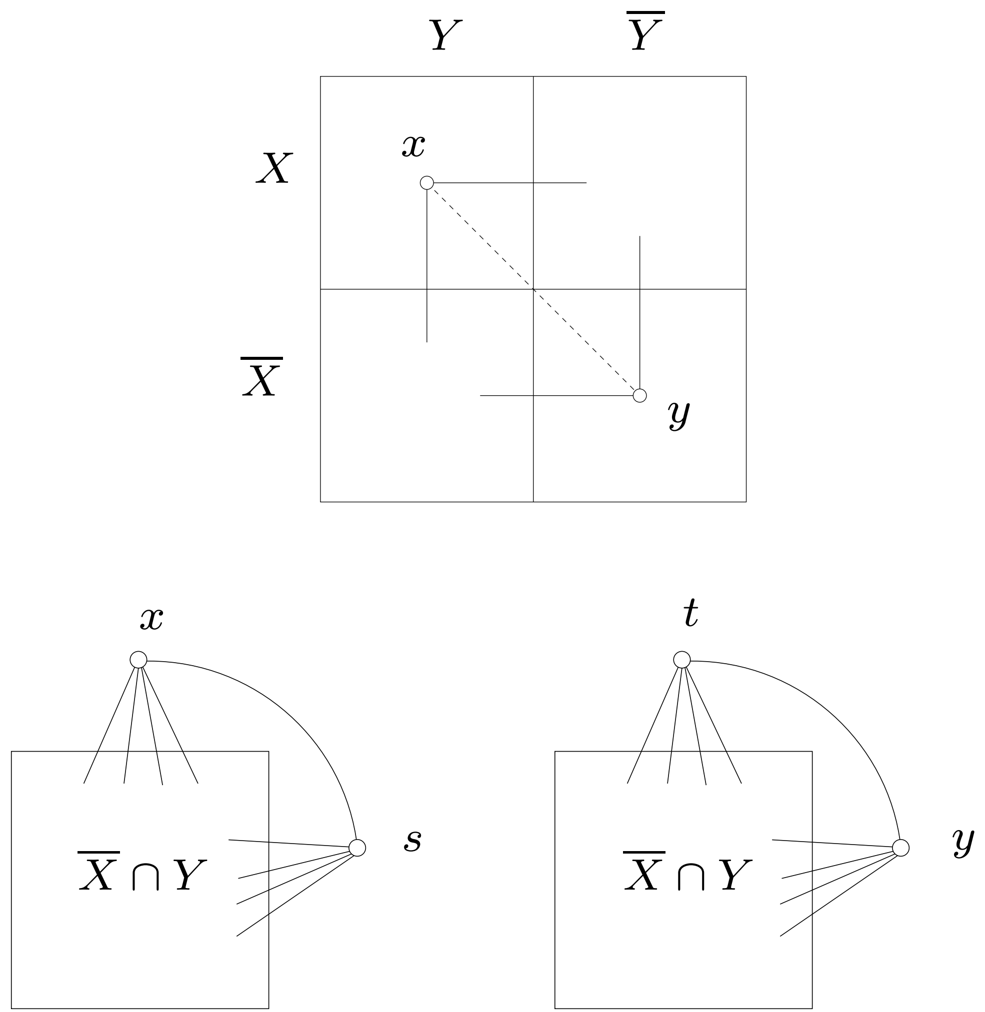
- assume $|X_1' \cap X_2|$ is odd $\Rightarrow |\overline{X_1'} \cap \overline{X_2}|$ odd
- if $|X_1' \cap X_2| > 1$ or $|\overline{X_1'} \cap \overline{X_2}| > 1$: previous case
- $\quad \therefore \quad X_1' \cap \overline{X_2} = X_1'' \cap \overline{X_2}$ (even)

Last Case



- $X_1' \cap \overline{X_2} = X_1'' \cap \overline{X_2}$ (even) $\Rightarrow X_1'' \cap X_2$ is odd
- if $|X_1'' \cap X_2| > 1$: previous case
- $\quad\therefore\quad X_1'' \cap X_2 = X_1' \cap X_2$
- $X_1'' = X_2'$, contradiction
- $\quad\therefore\quad$ only one blue, only one red

Last Case


Y
$\overline{Y}$
X
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x
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s
$\overline{X} \cap Y$
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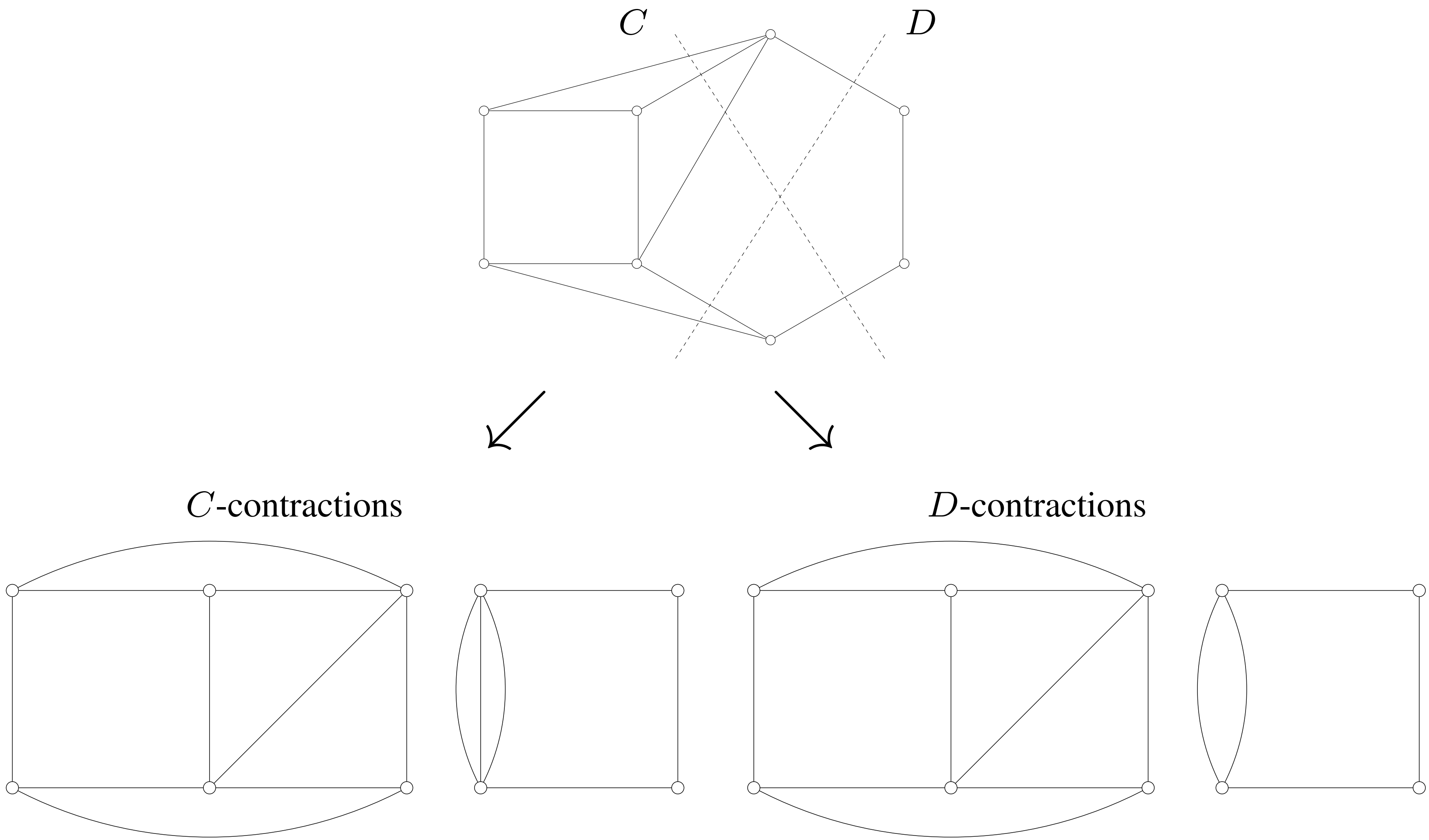
Last Case



Invariants b and $b+p$

- $b(G)$: the number of bricks of mc graph G
- $p(G)$: the number of Petersen bricks of mc graph G
- G is a *Petersen brick* if its underlying simple graph is $\mathbb{P}$
- $(b+p)(G) := b(G) + p(G)$
- b and $b+p$ are important invariants