A Proof for a Conjecture of Gorgol

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1 Introduction

The *Turán number* $\mathrm{ex}(n,F)$ is the maximum number of edges in a graph on n vertices which does not contain H as a subgraph. For a general graph F, the value of $\mathrm{ex}(n,F)$ is asymptotically well known and dependent of the *chromatic number* $\chi(F)$. Erdős, Stone and Simonovits' Theorem [1] state that

$$\lim_{n\to\infty} \frac{ex(n,F)}{n^2} = \frac{\chi(F)-2}{2\chi(F)-2}.$$

Although this theorem gives a lot of information on the asymptotic growth of $\mathrm{ex}(n,F)$, it should be noted that it is only of interest for nonbipartite graphs. If F is bipartite, it asserts merely that $\mathrm{ex}(n,F) = \mathrm{o}(n^2)$. Therefore, we focus on the Turán number for bipartite graphs. In particular, we consider when $F = kP_3$, where P_r is a path on r vertices and kG consists of k disjoint copies of the graph G.

For graphs G and F, let $G+F$ and $G \vee F$ be the *disjoint union* and *join* of G and F, respectively. The join of G and F is the graph obtained from $G+F$ by adding edges from all vertices of G to all vertices of F. Let $\mathrm{H}_{\mathrm{ex}}(n,G)$ represent an *extremal graph* on n vertices without G as subgraph with $\mathrm{ex}(n,G)$ edges.

Gorgol [2] gave upper and lower bounds for the Turán number forbidding kF, for a graph F on r vertices. The lower bound was obtained by noting that neither

$$G_1(n,kF) = K_{kr-1} + \mathrm{H}_{\mathrm{ex}}(n-kr+1,F)$$
$$G_2(n,kF) = K_{k-1} \vee \mathrm{H}_{\mathrm{ex}}(n-k+1,F)$$

contain k disjoing copies of F. Indeed, $G_1(n,kF)$ contains only $k-1$ copies of F in K_{kr-1} and any copy of F in $G_2(n,kF)$ must contain at least one vertex in K_{k-1}. Gorgol's lower bound is, therefore,

$$\mathrm{ex}(n,kF) \geq \max\{e(G_1(n,kF)), e(G_2(n,kF))\}$$

where $e(G)$ denotes the number of edges in a graph G.

2 Disjoint copies of P_3

We consider the case when $F = P_3$. Let

$$g_1(n,k) = e(G_1(n,kP_3)),$$
$$g_2(n,k) = e(G_2(n,kP_3)) \text{ and}$$
$$\mathrm{Gorgol}(n,k) = \max\{g_1(n,k), g_2(n,k)\}.$$

Note that $\mathrm{H}_{\mathrm{ex}}(n,P_3) = M_n$, where M_n is a *near perfect matching* on n vertices. If n is even, then M_n consists of a matching on n vertices and, if n is odd, then $M_n = M_{n-1} + K_1$. Thus, $e(M_n) = \lfloor n/2 \rfloor$ and we have

$$g_1(n,k) = \binom{3k-1}{2} + \left\lfloor \frac{n-3k+1}{2} \right\rfloor;$$
$$g_2(n,k) = \binom{k-1}{2} + (k-1)(n-k+1) + \left\lfloor \frac{n-k+1}{2} \right\rfloor;$$
$$\mathrm{Gorgol}(n,k) = \begin{cases} \mathrm{g}_1(n,k), & \text{for } 3k \leq n \leq 5k, \\ \mathrm{g}_2(n,k), & \text{for } n \geq 5k. \end{cases}$$

Gorgol's Conjecture states that, when $n \geq 3k$,

$$\mathrm{ex}(n,kP_3) = \mathrm{Gorgol}(n,K).$$

In the same paper, Gorgol proved this is true when $k \in \{2,3\}$. Bushaw and Kettle [3] also proved the conjecture is true for $n \geq 7k$.

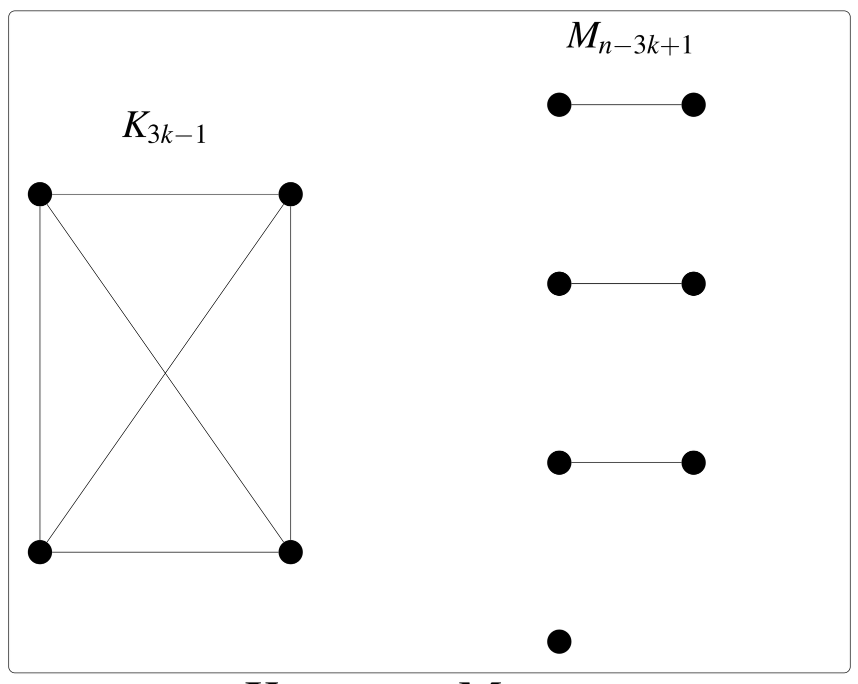
In this work, we give a constructive proof for Gorgol's Conjecture for all values of n and k. In particular, we give an algorithm that finds a set of disjoint copies of P_3 given a graph G as input. We prove Gorgol's Conjecture by showing that, if G has n vertices and more than $\mathrm{Gorgol}(n,k)$ edges, then this algorithm returns at least k disjoint copies of P_3 in G.

3 Algorithm

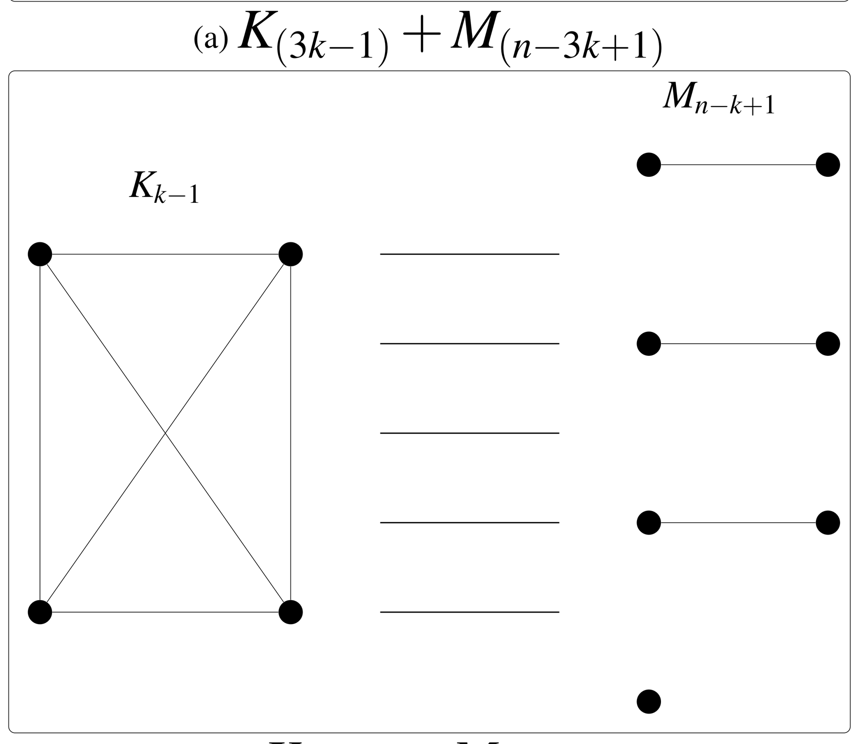
The algorithm receives as input a graph $G = (V,E)$ and an integer $k \leq n/3$.
Let $\mathcal{Q} = \{Q_1, Q_2, \cdots Q_s\}$ be a collection of subsets of $V(G)$ such that:

- $Q_i \cap Q_j = \emptyset$.
- $|Q_i| = 3$.
- $G[Q_i]$ contains a copy of P_3 as subgraph.

We start with an empty collection $\mathcal{Q}$ and iteratively look for an improvement $\mathcal{Q}'$ for $\mathcal{Q}$. We say that a collection $\mathcal{Q}'$ is an improvement of $\mathcal{Q}$ if either $|\mathcal{Q}'| > |\mathcal{Q}|$ or $|\mathcal{Q}'| = |\mathcal{Q}|$ and $\mathcal{Q}'$ has more triangles than $\mathcal{Q}$.



(a) $K_{(3k-1)} + M_{(n-3k+1)}$



(b) $K_{(k-1)} \vee M_{(n-k+1)}$

The given algorithm is then as follows:

Algorithm 1 Disjoint copies of P_3.
Input: (G,k).

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1: Q ← ∅
2: while ∃ improvement Q' of Q do
3:     Q ← Q'
4: end while
5: return Q
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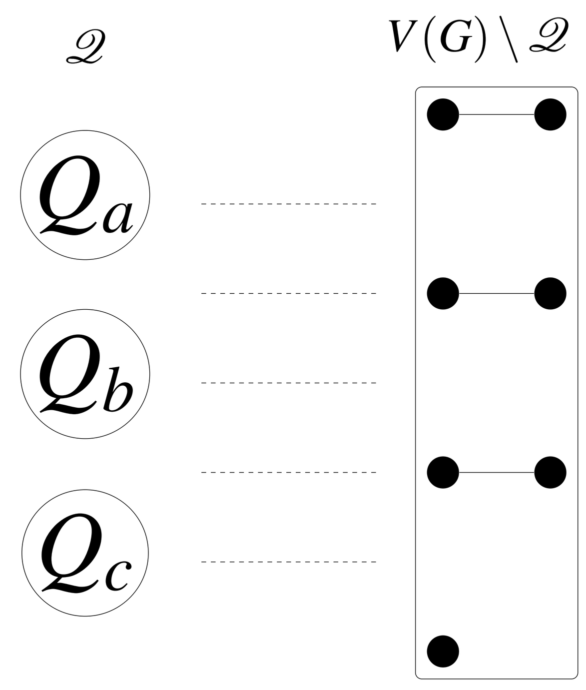
To find improvements for our current collection $\mathcal{Q}$, we need to search only in a small subgraph of G.



With local improvements only we can find the desired k disjoint copies of P_3, proving the conjecture. The naive complexity of the algorithm is $\mathrm{O}(n^{12}k^5)$. With a subset of local improvements and good use of data structures, we find a collection of k disjoint P_3 in $\mathrm{O}(k|E|)$ time. Our main result, which proves Gorgol's Conjecture, can then be stated as

Theorem. *If a graph G has n vertices and $e(G) \geq \mathrm{Gorgol}(n,k)$ edges, then the provided algorithm will find a collection $\mathcal{Q}$ of size at least k.*

References

[1] Bondy, J. A. and Murty, U.S.R. *Graph Theory* Graduate Texts in Mathematics 244, 2008, pp.317-320

[2] Gorgol, I. *Turán Numbers fo disjoints copies of graphs.* Graphs and Combinatories (4 January 2011), pp. 1-7.

[3] Bushaw, N. and Kettle, N. *Turán Numbers of multiple paths and equibipartite forests.* Combinatorics, Probability and Computing, 2011, Vol.20(6), pp.837-853