A characterization of functions with vanishing averages over products of disjoint sets

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Basic Analytic Questions

- Which function $f : [0,1] \to \mathbb{R}$ satisfies
$$\int_0^1 f\, dx = 0$$
and is continuous, non-negative?
- Which (Lebesgue integrable) function $f : [0,1]^m \to \mathbb{C}$ satisfies
$$\int_{A_1 \times A_2 \times \cdots \times A_m} f\, dx_1 \cdots dx_m = 0$$
for all collections of disjoint measurable sets $A_1, ..., A_m \subseteq [0,1]$?

Answer: $f = 0$ a.e.

Analysis Problem [Janson-Sós'14]

Given $0 < \alpha_1, \dots, \alpha_m < 1$ and $\sum_i \alpha_i \le 1$. Which functions $f : [0,1]^m \to \mathbb{C}$ satisfy
$$\int_{A_1 \times \ldots \times A_m} f(x_1, \dots, x_m) dx_1 \dots dx_m = 0, \qquad (*)$$
for all collection of disjoint subsets $A_1, \dots, A_m \subseteq [0,1]$, and of respective measures $\alpha_1, \dots, \alpha_m$?

Motivation: study of quasi-random property of graph sequence.

Walsh Expansion

The generalized Walsh expansion of an integrable function $f : [0,1]^m \to \mathbb{C}$ is a unique decomposition $f = \Sigma_{S \subseteq [m]} F_S$ such that

- F_S depends only on coordinates in S.
- $\int F_S(x_1, \dots, x_m) dx_i = 0$ for every $i \in S$.

Example: $f : [0,1]^2 \to \mathbb{R}$ gives

- $F_\emptyset = \int f(x_1, x_2) dx_1 dx_2$;
- $F_{\{1\}} = \int f(x_1, x_2) dx_2 - F_\emptyset$;
- $F_{\{2\}} = \int f(x_1, x_2) dx_1 - F_\emptyset$;
- $F_{\{1,2\}} = f - \int f(x_1, x_2) dx_2 - \int f(x_1, x_2) dx_1 + F_\emptyset$.

Answer: A Characterization

We answered Problem [Janson-Sós'14] by giving the following complete characterization theorem in terms of f's Walsh expansion.

Main Theorem [HHL'15]

Let $\sum \alpha_i = 1$. An integrable function $f : [0,1]^m \to \mathbb{C}$ with Walsh expansion $f = \sum F_S$ satisfies
$$\int_{A_1 \times \cdots \times A_m} f(x_1, ..., x_m) = 0$$
for all disjoint subsets $A_1, ..., A_m \subseteq [0,1]$ with $\mu(A_i) = \alpha_i$, if and only if

(i) $F_\emptyset = 0$;

(ii) F_S is an alternating function, when $|S| \ge 2$;

(iii) For $S \subseteq [m-1]$,
$$\frac{1}{\prod_{i \in S} \alpha_i} F_S(x) = \sum_{i \in S} \frac{1}{\prod_{j \in S_i} \alpha_j} F_{S \cup \{m\} \setminus \{i\}}(x^{(i)}),$$
where $x^{(i)} = (x_1, ..., x_{i-1}, x_m, x_{i+1}, ..., x_{m-1}, x_i)$.

Example [HHL, m=3]

$f : [0,1]^3 \to \mathbb{C}$ satisfies $(*)$, if and only if f is of the following form

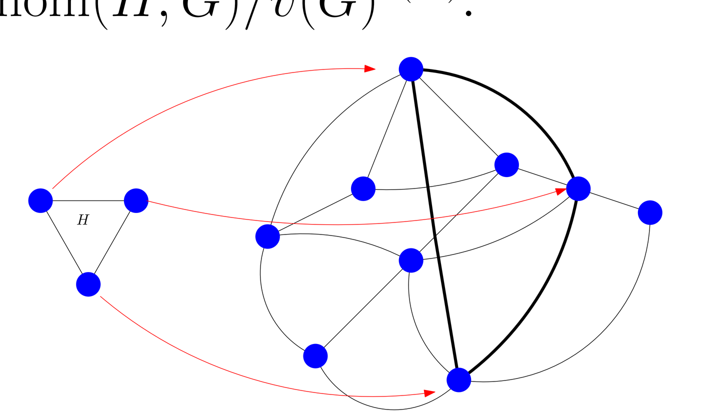
(i) $F_\emptyset = 0$;

(ii) $F_{\{1,2\}}(x_1, x_2) = -F_{\{1,2\}}(x_2, x_1)$, ...;

(iii)
$$\frac{1}{\alpha_1} F_{\{1\}}(x) = \frac{1}{\alpha_2} F_{\{2\}}(x) = \frac{1}{\alpha_3} F_{\{3\}}(x)$$
and
$$\begin{aligned}\frac{1}{\alpha_1 \alpha_2} & F_{\{1,2\}}(x_1, x_2) \\ &= \frac{1}{\alpha_1 \alpha_3} F_{\{1,3\}}(x_1, x_2) + \frac{1}{\alpha_2 \alpha_3} F_{\{2,3\}}(x_2, x_1) \\ &= \frac{1}{\alpha_1 \alpha_3} F_{\{1,3\}}(x_1, x_2) - \frac{1}{\alpha_2 \alpha_3} F_{\{2,3\}}(x_1, x_2).\end{aligned}$$

Quasi-randomness of graphs

Homomorphism density of H in G,
$$\begin{aligned} t_H(G) &:= \Pr[h \colon V(H) \to V(G) \text{ is edge preserving}] \\ &= \hom(H, G)/v(G)^{v(H)}. \end{aligned}$$



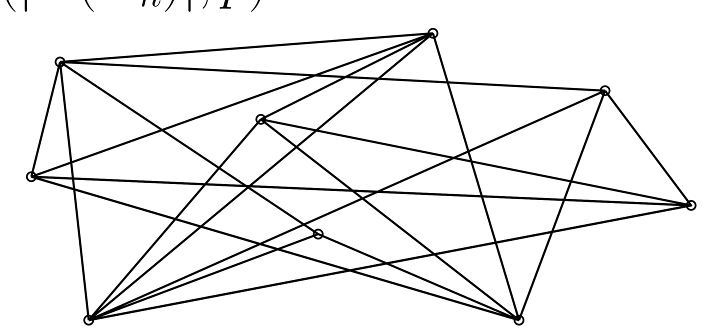
Quasi-randomness of graphs

A graphon $W : [0,1]^2 \to [0,1]$ is a symmetric measurable function. Graphons can be thought of as continuous version of graphs. Homomorphism density of H in a graphon W is defined as,
$$t_H(W) := \int_{[0,1]^{v(H)}} \prod_{ij \in E(H)} W(x_i, x_j).$$

[Lovasz-Szegedy'06] The limit of a graph sequence is a graphon. Formally, a graph sequence $\{G_n\}_{n=1}^{\infty}$ converges $\Leftrightarrow$ there exists a graphon W such that
$$t_H(G_n) \to t_H(W), \quad \forall\, H.$$

A graph sequence $\{G_n\}_{n=1}^{\infty}$ is p-quasi-random if $G_n \to W \equiv p$, i.e., $t_H(G_n) \to t_H(W) = p^{e(H)}$ holds for all H.

Intuitively, a graph sequence (G_n) is p-quasi-random if it has similar statistics to Erdös-Rényi random graphs $G(|V(G_n)|, p)$.



[Chung-Graham-Wilson] For p-quasi-randomness, the two graphs $H = K_2, C_4$ suffice,
$$t_{K_2}(W) = p, t_{C_4}(W) = p^4 \Rightarrow W \equiv p.$$
However, no single graph H suffices.

Quasi-random graph Problem [Yuster-Shapira'10, Janson-Sós'14]

[Y-S'10, J-S'14] Let $m = v(H)$ and $\sum_i \alpha_i \le 1$. Say $P(H, \alpha_1, \dots, \alpha_m)$ is a quasirandom property, if
$$\frac{1}{\alpha_1 \dots \alpha_m} \int_{A_1 \times \ldots \times A_m} \prod_{ij \in E(H)} W(x_i, x_j) = p^{e(H)}$$
for all disjoint $A_1, \dots, A_m$ of respective measures $\alpha_1, \dots, \alpha_m$ implies $W = p$ a.e..

Which $P(H, \alpha_1, \dots, \alpha_m)$ is a quasirandom property? Let
$$f(x_1, \dots, x_m) := \prod_{ij \in E(H)} W(x_i, x_j) - p^{e(H)}.$$
Then this graph theoretical problem reduces to the previous analytic problem, asking when $f = 0$?

Partial Answer: Twin vertices

[Janson-Sós'14] conjectured $P(P_3, \alpha_1, \alpha_2, \alpha_3)$ is a quasi-random property, where P_3 is the path of length 2. We proved the following more general result.

Theorem [HHL'15]

If H contains twin vertices(pair of vertices having same neighbour), then $P(H, \alpha_1, \dots, \alpha_m)$ is a quasi-random property.

Proof of P_3

In this case $f = w(x_1, x_2) w(x_1, x_3) - p^2$. By main Theorem,
$$\begin{aligned} w(x_1, x_2) w(x_1, x_3) - p^2 = F_{\{1\}} + F_{\{2\}} + F_{\{3\}} \\ + F_{\{1,2\}} + F_{\{1,3\}}. \end{aligned}$$
$F_{\{1,2,3\}} = 0$ and $F_{\{2,3\}} = 0$ as they are symmetric and anti-symmetric with respect to x_2, x_3.

Integrating with respect to x_2, x_3 gives
$$F_{\{1\}}(x_1) = p^2 + \left(\int w(x_1, y) dy\right)^2.$$
Setting $x_2 = x_3 = y$ and integrating gives
$$F_{\{1\}}(x_1) = p^2 + \int w(x_1, y)^2 dy.$$
Hence
$$\left(\int w(x_1, y) dy\right)^2 = \int w(x_1, y)^2 dy,$$
implies $w \equiv p$.

Conjecture [Janson-Sós'14]

Except $P(K_2, \frac{1}{2}, \frac{1}{2})$, all $P(H, \alpha_1, \dots, \alpha_m)$ are quasi-random property?

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