Chromatic index, treewidth and maximum degree

Revisiting an old result of Vizing

Henning Bruhn[1], **Laura Gellert**[1] **and Richard Lang**[2]

[1]Universität Ulm, Ulm (Germany), [2]Universidad de Chile, Santiago (Chile)

Abstract

We conjecture that any graph G with treewidth k and maximum degree $\Delta(G) \geq k+\sqrt{k}$ satisfies $\chi'(G) = \Delta(G)$. In support of the conjecture we prove its fractional version.

Background

An *edge colouring* is an assignment of colours to the edges of a graph G such that no two adjacent edges receive the same colour. The least number of colours that admits an edge colouring is called the *chromatic index* and denoted by $\chi'(G)$. A classic theorem of Vizing states that a graph G has either chromatic index $\Delta(G)$ or $\Delta(G)+1$. But to decide whether Δ or $\Delta+1$ colours are needed is a difficult algorithmic problem.

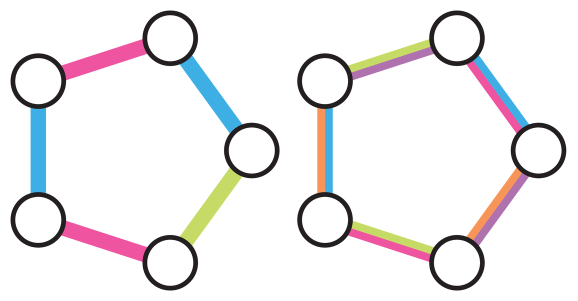


Figure 1: C_5 has chromatic index 3, while its fractional chromatic index is $5/2$.

Often, graphs with a relatively simple structure can be edge-coloured with only $\Delta(G)$ colours. This is the case for bipartite graphs (Kőnig's theorem) and for cubic Hamiltonian graphs. Arguably, one measure of simplicity is *treewidth*, how closely a graph resembles a tree. Vizing [4] (see also Zhou et al. [6]) observed a consequence of his adjacency lemma:

Theorem (Vizing, 1965):
Any graph of treewidth k and maximum degree $\Delta \geq 2k$ has chromatic index Δ.

Is this tight? No, it turns out. Using two recent adjacency lemmas ([2, 5]), we obtain:

Proposition 1 (Bruhn, Gellert, L., 2016):
For any graph G of treewidth $k \geq 4$ and maximum degree $\Delta \geq 2k-1$ it holds that $\chi'(G) = \Delta$.

This immediately suggests the question: how much further can the maximum degree be lowered? We conjecture:

Conjecture (Bruhn, Gellert, L., 2016):
Any graph of treewidth k and maximum degree $\Delta \geq k+\sqrt{k}$ has chromatic index Δ.

The bound is close to best possible: we construct for infinitely many k, graphs with treewidth k, maximum degree $\Delta = k+\lfloor\sqrt{k}\rfloor < k+\sqrt{k}$, and chromatic index $\Delta+1$; see Figure 2.

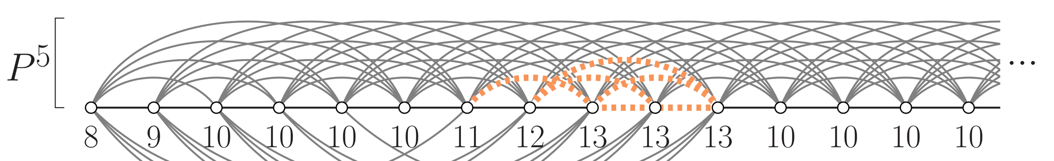


Figure 2: Removing the dashed edges from a P^5 gives a graph of treewidth $k=8$ and maximum degree $\Delta = 10$. Due to overfulness, its chromatic index is $\Delta+1$.

The problem of edge colouring can be relaxed as follows. A *fractional edge colouring* is an assignment of weights λ_M to each matching M in G such that $\sum_M \lambda_M \mathbf{1}_M(e) = 1$ for every $e \in E(G)$. The *fractional chromatic index*, $\chi'_f(G)$, is defined as the minimum of $\sum_M \lambda_M$ over all fractional edge colourings λ. Since an edge colouring can be interpreted as 0-1-valued fractional edge colouring, it follows that $\chi'_f(G) \leq \chi'(G)$. The converse, however, does not always hold as illustrated in Figure 1. Moreover, in contrast to the chromatic index, the fractional chromatic index can be computed efficiently.

In support of the conjecture we prove its fractional version.

Theorem 1 (Bruhn, Gellert, L., 2016):
Any graph of treewidth k and maximum degree $\Delta \geq k+\sqrt{k}$ has fractional chromatic index Δ.

The theorem follows from a new upper bound on the number of edges for these graphs, whose proof is quite intriguing.

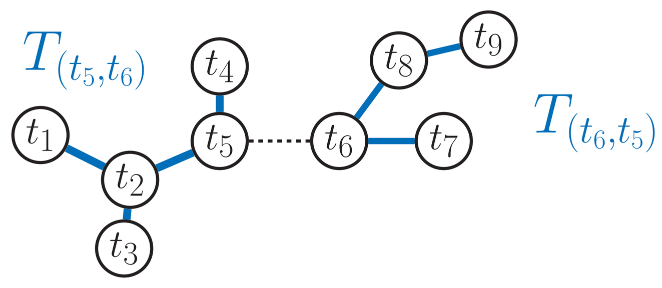
Proposition 1:
A graph G of treewidth k and maximum degree Δ satisfies

$$2|E(G)| \leq \Delta|V(G)| - (\Delta-k)(\Delta-k+1).$$

A graph G is *overfull* if it has an odd number n of vertices and strictly more than $\Delta(G)(n-1)/2$ edges; a subgraph H of G is an *overfull subgraph* if it is overfull and satisfies $\Delta(H) = \Delta(G)$. Proposition 2 implies quite directly that no graph with treewidth k and maximum degree $\Delta \geq k+\sqrt{k}$ can be overfull. It follows from Edmonds' matching polytope theorem that $\chi'_f(G) = \Delta$, if the graph G does not contain any overfull subgraph of maximum degree Δ; see [3, Ch. 28.5]. As the treewidth of a subgraph is never larger than the treewidth of the original graph, Theorem 1 is a consequence of Proposition 2.

Two strange tree inequalities

We use the following two lemmas about trees in the proof of Proposition 2. We denote the number of vertices in a tree T by $|T|$. For an edge $st \in E(T)$ let $T_{(s,t)}$ be the component of $T-st$ containing s:

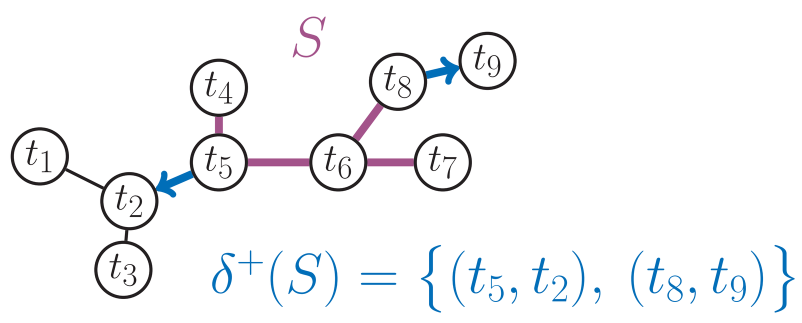


Lemma 1:
For a tree T and a positive integer $d \leq |T|$ it holds that

$$\sum_{(s,t):st\in E(T)} \max\left(d - |T_{(s,t)}|, 0\right) \geq d(d-1).$$

Remark that Lemma 1 is wrong if we omit $\max\left(\,\cdot\,, 0\right)$.

For a subtree $S \subset T$ let $\delta^+(S)$ be the set of oriented edges leaving S:



Lemma 2:
Let T be a tree and let $d \leq |T|$ be a positive integer. Then for any subtree $S \subset T$ it holds that

$$\sum_{(s,t)\in\delta^+(S)} \max\left(d - |T_{(s,t)}|, 0\right) \leq \max\left(d - |S|, 0\right).$$

Treewidth

For a graph G a *tree-decomposition* $(T, \mathcal{B})$ consists of a tree T and a collection $\mathcal{B} = \{B_t : t \in V(T)\}$ of *bags* $B_t \subset V(G)$ such that

(a) each vertex of G is in some bag,

(b) the vertices of each edge of G are in some common bag and

(c) if $v \in B_s \cap B_t$, then $v \in B_r$ for each vertex r on the path connecting s and t in T.

A tree-decomposition $(T,\mathcal{B})$ has *width* k if each bag has a size of at most $k+1$. The *treewidth* of G is the smallest integer k for which there is a width k tree-decomposition of G. A tree-decomposition $(T,\mathcal{B})$ of width k is *smooth* if

(d) each bag has size exactly $k+1$ and

(e) $|B_s \cap B_t| = k$ for all $st \in E(T)$.

A graph of treewidth at most k always admits a smooth tree-decomposition of width k; see [1].

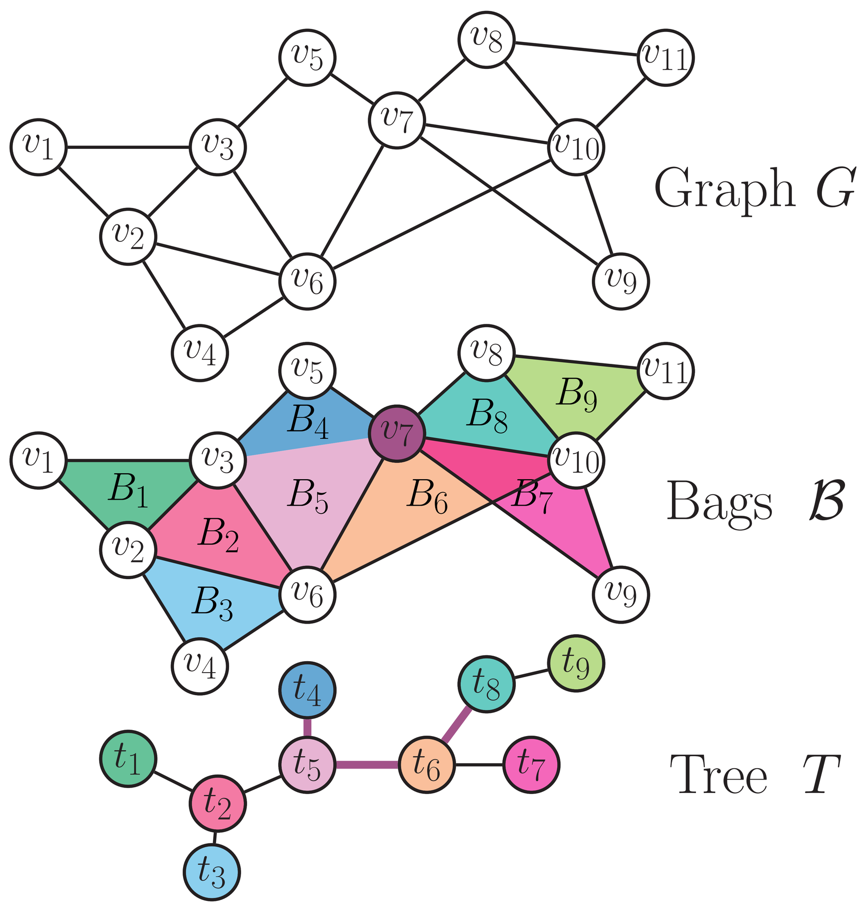


Figure 3: The graph G has a smooth tree-decomposition $(T,\mathcal{B})$ of width 2. Since $v_7 \in B_4 \cap B_8$ it follows that $v_7 \in B_5$ and $v_7 \in B_6$ as well.

.

Proving the edge bound

As explained in the last paragraph of the introduction, Theorem 1 follows from Proposition 2.

Proof of Proposition 2. Let $(T,\mathcal{B})$ be a smooth tree-decomposition of G of width k. Fix a vertex $v \in V(G)$ and denote by $T(v)$ the subtree of T that consists of those bags that contain v; see Figure 4.

The idea is to estimate $\deg(v)$ through the size of $T(v)$: since the tree-decomposition is smooth, there is exactly one new vertex in the union of all bags containing v is at most $|T(v)|+k$ and thus

$$\deg(v) \leq \max\left(|T(v)|+k-1, \Delta\right).$$

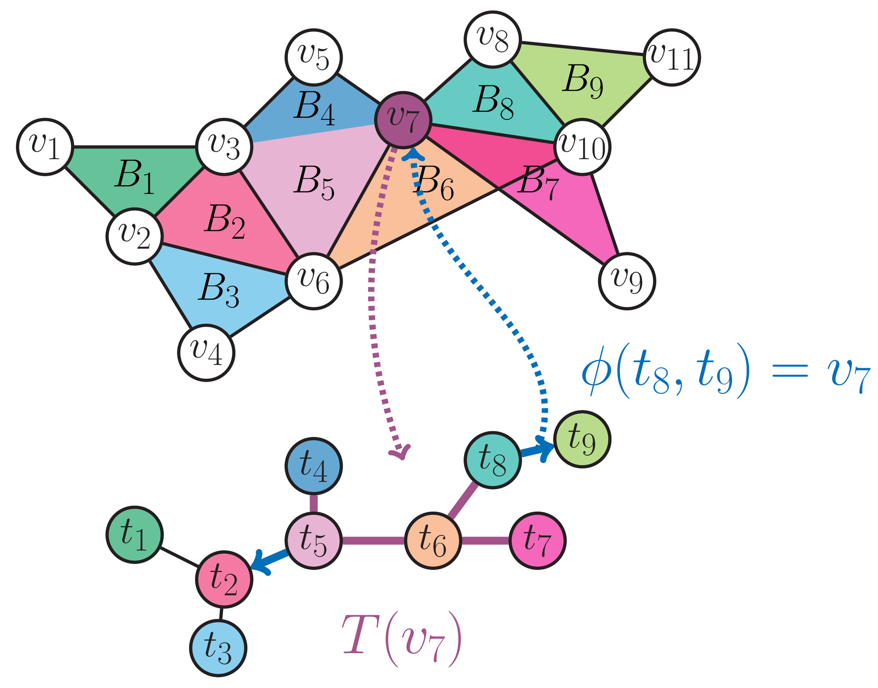


Figure 4: The vertex v_7 induces the subtree $T(v_7) \subset T$.

Set $d := \Delta - k + 1 \geq 1$, and observe that $d \leq |V(G)| - k = |T(v)|$ as the tree-decomposition is smooth. Using the above we calculate

$$\begin{aligned}\Delta - \deg(v) &\geq \max\left(\Delta - k + 1 - |T(v)|, 0\right)\\ &= \max\left(d - |T(v)|, 0\right)\\ &\geq \sum_{(s,t)\in\delta^+(T(v))} \max\left(d - |T_{(s,t)}|, 0\right),\end{aligned}$$

where the last inequality follows from Lemma 2.

Consider an edge $st \in E(T)$. Since the tree-decomposition is smooth there is exactly one vertex $v \in V(G)$ with $v \in B_s$ and $v \notin B_t$. Setting $\phi(s,t) = v$ then defines a function from the set of oriented edges of T into $V(G)$; see Figure 4.

Note that $\phi(s,t) = v$ if and only if $(s,t) \in \delta^+(T(v))$ and so

$$\{(s,t) \in \phi^{-1}(v) \ : \ v \in V(G)\} \longleftrightarrow \{(s,t) \ : \ st \in E(T)\}.$$

Summing the previous inequality over all vertices, we get

$$\begin{aligned}\Delta|V(G)| - 2|E(G)| &= \sum_{v\in V(G)} \left(\Delta - \deg(v)\right)\\ &\geq \sum_{v\in V(G)} \sum_{(s,t)\in\phi^{-1}(v)} \max\left(d - |T_{(s,t)}|, 0\right)\\ &= \sum_{(s,t):st\in E(T)} \max\left(d - |T_{(s,t)}|, 0\right)\\ &\geq d(d-1),\end{aligned}$$

where the last inequality is due to Lemma 1. By choice of d, this gives the desired inequality. □

Degenerate graphs

A graph G is *k-degenerate* if there is an elimination order $v_n, \ldots, v_1$ of the vertices such that v_{i+1} has degree at most k in $G - \{v_n, \ldots, v_i\}$ for every i. Graphs of treewidth k are in particular k-degenerate. Indeed, Vizing [4] originally showed that k-degenerate graphs, rather than treewidth k graphs, of maximum degree $\Delta \geq 2k$ have an edge-colouring with Δ colours.

By simple induction following the elimination order, we obtain for k-degenerate graphs a bound with half the degree loss of the bound of Proposition 2:

$$2|E(G)| \leq \Delta|V(G)| - \frac{1}{2}(\Delta-k)(\Delta-k+1).$$

(This bound turns out to be tight for some Δ, k as the construction in Figure 5 shows.) Consequently, Proposition 2 can easily be transferred: Any k-degenerate graph of maximum degree $\Delta \geq k+1/2+\sqrt{2k+1/4}$ is not overfull and therefore has fractional chromatic index $\chi_f(G) = \Delta$.

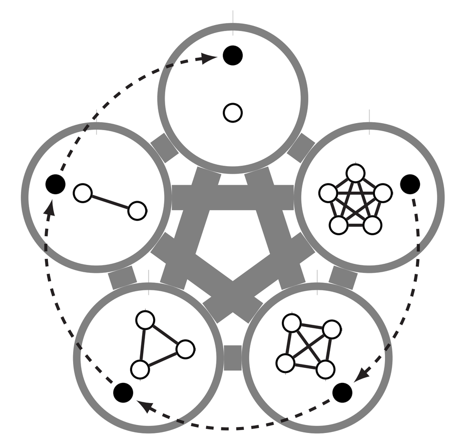


Figure 5: A 14-degenerate graph of maximum degree 19. Thick gray edges indicate that two vertex sets are complete to each other. The elimination order of the v_i is shown in dashed lines

References

[1] H. L. Bodlaender, *A partial k-arboretum of graphs with bounded treewidth*, Theoretical Computer Science **209** (1998), no. 1–2, 1 – 45.

[2] D. P. Sanders and Y. Zhao, *Planar graphs of maximum degree seven are class I*, Journal of Combinatorial Theory, Series B **83** (2001), no. 2, 201 – 212.

[3] A. Schrijver, *Combinatorial optimization: Polyhedra and efficiency*, Algorithms and Combinatorics, Springer, 2003.

[4] V. G. Vizing, *Critical graphs with given chromatic class [in Russian]*, Metody Diskretnogo Analiza **5** (1965), 9 – 17, English translation in *Graph Edge Coloring* by Favrholdt et al.

[5] L. Zhang, *Every planar graph with maximum degree 7 is of class 1*, Graphs and Combinatorics **16** (2000), no. 4, 467 – 495.

[6] X. Zhou, S.-I. Nakano, and T. Nishizeki, *Edge-coloring partial k-trees*, Journal of Algorithms **21** (1996), no. 3, 598 – 617.