Strong Ramsey Games: drawing on an infinite board

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Strong Ramsey Games

For integers $3 \leq q \leq n$ consider the strong Ramsey game $\mathcal{R}(K_q, n)$, defined as follows:

- In each round of this game, the first player (FP) claims a free edge of K_n and then the second player (SP) claims a free edge of K_n
- The first player to build a copy of K_q] wins.
- If, once every edge of K_n is claimed, neither player built a copy of K_q, the game is declared a draw.

In general, the *board* of the game is some (possibly infinite) set X and the *winning sets* are the elements of a family $\mathcal{F}$ of subsets of X. Note that the game is a perfect information game.

The Strategy Stealing Argument

A simple yet elegant game-theoretic argument, called the *strategy stealing argument*, shows that FP can always guarantee at least a draw, for every n and q.

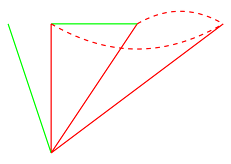
Unfortunately, it just provides the existence of a winning strategy but it does not provide an explicit strategy for FP. This is in fact true for any strong game.

Theorem In a strong game played on a hypergraph $(X, \mathcal{F})$, FP can always achieve at least a draw.

Proof (sketch) Assume to the contrary that SP has a winning strategy $\mathcal{S}$. Now FP starts with an arbitrary move and then pretends to play as the second player. Furthermore, he 'steals' $\mathcal{S}$ and plays according to it. Additional moves are only beneficial for FP, and so it is not hard to see that he can in fact play according to $\mathcal{S}$, so both players play according to a winning strategy - a contradiction.

Playing on the edges of K_n

Strong games are notoriously hard to analyse. Besides the strategy stealing argument, the essentially the only other tool is the existence of *Ramsey numbers* $r(q)$. In fact, once $n \geq r(q)$, Ramsey's theorem implies that in the game $R(K_q, n)$ there is no final drawing position and hence combining this fact with strategy stealing, one sees that FP has a winning strategy.

- $R(K_3, n)$ - an easy win for FP :  A double threat of FP after three moves.
- $R(K_4, n)$: Once $n \geq r(4) = 18$, we know that FP has a winning strategy. However an explicit winning strategy is unknown.

Of course the same is true for bigger values of q: For ***any*** $n \geq r(q)$, no matter how large, FP has a winning strategy in the game $R(K_q, n)$. So, of course, it should be true on the infinite complete graph...?

Playing on the edges of $K_{\mathbb{N}}$

Consider now the strong game $\mathcal{R}(K_q, \aleph_0)$. Its board is the edge set of the countably infinite complete graph $K_{\mathbb{N}}$ and its winning sets are the copies of K_q in $K_{\mathbb{N}}$. If no player has a strategy that ensures his win after finitely many rounds, the game is declared a draw.
Even though the board of this game is infinite, strategy stealing still applies, i.e., FP always has a winning strategy in $\mathcal{R}(K_q, \aleph_0)$. Clearly, Ramsey's Theorem applies as well, i.e. any red/blue colouring of the edges of $K_{\mathbb{N}}$ yields a monochromatic copy of K_q. Hence, as in the finite version of the game, one could expect to combine these two arguments to deduce that FP has a winning strategy in $\mathcal{R}(K_q, \aleph_0)$.

- $R(K_3, \aleph_0)$: This is again an easy win for FP - exactly as above.
- For any $q \geq 4$ the question whether it is a FP win or a draw is wide open. In fact, Beck considers it as one of his '7 most humiliating open problems', and he considers the case $q = 5$ to be hopeless.

We don't have to restrict ourselves to cliques

Playing Ramsey games, we do not have to restrict our attention to cliques, or even to graphs for that matter. For every integer $k \geq 2$ and every k-uniform hypergraph $\mathcal{H}$, we can study the finite strong Ramsey game $\mathcal{R}^{(k)}(\mathcal{H}, n)$ and the infinite strong Ramsey game $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$, where both players try to be the first to build a copy of $\mathcal{H}$.
As in the graph case, strategy stealing and Hypergraph Ramsey Theory shows that FP has a winning strategy in finite game. On the infinite board, both arguments individually work again. But does the combination work too?

Our result

Intuitively one might feel that if FP wins on every large enough *finite* board, then he should also win on the infinite board. Our main result shows that this is not true in general. We construct the following 5-uniform hypergraph $\mathcal{H}$, for which $R^{(5)}(\mathcal{H}, \aleph_0)$ is a draw.

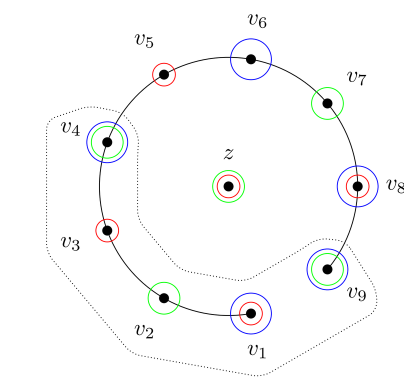


Fig. 1: The 5-uniform hypergraph $\mathcal{H}$. The black line from v_1 to v_9 represents a tight path.

Theorem (Hefetz, K., Narins, Pokrovskiy, Requilé, Sarid)

$\mathcal{R}^{(5)}(\mathcal{H}, \infty)$ is a draw.

Apart from being very surprising, our result indicates that strong Ramsey games are even more complicated than we originally suspected.

Sufficient conditions

Theorem (Hefetz, K., Narins, Pokrovskiy, Requilé, Sarid)

Let $\mathcal{H}$ be a k-graph which satisfies all of the following properties:

(i) $\mathcal{H}$ has a degree 2 vertex z, and two edges r and g going through it in $\mathcal{H}$.

(ii) $\delta(\mathcal{H} \setminus \{z\}) \geq 3$ and $d_{\mathcal{H}}(u) \geq 4$ for every $u \in V(\mathcal{H}) \setminus \{z\}$;

(iii) $\mathcal{H} \setminus \{z\}$ has a fast winning strategy;

(iv) For every two edges $e, e' \in \mathcal{H}$, if $\phi : V(\mathcal{H} \setminus \{e, e'\}) \longrightarrow V(\mathcal{H})$ is a monomorphism, then ϕ is the identity;

(v) $e \cap r \neq \emptyset$ and $e \cap g \neq \emptyset$ holds for every edge $e \in \mathcal{H}$.

(vi) $|V(\mathcal{H}) \setminus (r \cup g)| < k - 1$.

Then $\mathcal{R}^{(k)}(\mathcal{H}, \infty)$ is a draw.

Idea: The strategy is divided into three stages. In the first stage SP quickly builds a copy of $\mathcal{H} \setminus \{z\}$, in the second stage SP defends against FP's threats, and in the third stage (which we might never reach) SP makes his own threats.

The three stages of SP's strategy

- Let $\mathcal{H}$ be a k-graph which satisfies the conditions of the theorem and let $m = |E(\mathcal{H})|$.
- At any point during the game, let $\mathcal{G}_1$ denote FP's current graph and let $\mathcal{G}_2$ denote SP's current graph.

Stage I: Let e_1 denote the edge claimed by FP in his first move. In his first $m - 2$ moves, SP builds a copy of $\mathcal{H} \setminus \{z\}$ which is vertex-disjoint from e_1. SP then proceeds to Stage II.

Stage II: Immediately before each of SP's moves in this stage, he checks whether there are a subgraph $\mathcal{F}_1$ of $\mathcal{G}_1$ and a free edge $e' \in K_{\mathbb{N}}^k$ such that $\mathcal{F}_1 \cup \{e'\} \cong \mathcal{H}$. If such $\mathcal{F}_1$ and e' exist, then SP claims e' (we will show later that, if such $\mathcal{F}_1$ and e' exist, then they are unique). Otherwise, SP proceeds to Stage III.

Stage III: Let $\mathcal{F}_2$ be a copy of $\mathcal{H} \setminus \{z\}$ in $\mathcal{G}_2$ and let z' be an arbitrary vertex of $K_{\mathbb{N}}^k \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$. Let $r', g' \in K_{\mathbb{N}}^k$ be free edges such that $z' \in r' \cap g'$ and $\mathcal{F}_2 \cup \{r', g'\} \cong \mathcal{H}$. If, once SP claims r', FP cannot make a threat by claiming g', then SP claims r'. Otherwise he claims g'.

Some open problems

- Given an integer $d \geq 3$, is there a k-graph $\mathcal{H}$ such that $\delta(\mathcal{H}) \geq d$ and $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw?
- Is there a k-graph $\mathcal{H}$ with minimum degree at least 3 such that $\mathcal{R}^{(k)}(\mathcal{H}, \aleph_0)$ is a draw and, for every positive integer n, FP cannot win $\mathcal{R}^{(k)}(\mathcal{H}, n)$ in less than, say, $1000|V(\mathcal{H})|$ moves?
- A 2-uniform example?
- What about K_5?