Lecture notes on sparse color-critical graphs

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1 Introduction

This text together with the attached paper [8] surveys results on color-critical graphs, with emphasis on sparse ones. The first two sections discuss the important contributions by Dirac and Gallai and present proofs of some remarkable results of them. The next two sections discuss the later progress and a number of applications of the recent results. We also use [8] for description of some applications. In Section 6 we present a proof for 4-critical graphs of a conjecture of Gallai on sparsest color-critical graphs. In the last section, we briefly survey similar problems for hypergraphs and triangle-free graphs and mention some unsolved problems.

Recall that a *(proper) k-coloring* of a graph G is a mapping $g : V(G) \to \{1, \dots, k\}$ such that $g(v) \neq g(u)$ for each $vu \in E(G)$. The minimum k such that G has a k-coloring is the *chromatic number of* G, denoted by $\chi(G)$.

For a positive integer k, a graph G is *k-critical* if $\chi(G) = k$, but every proper subgraph of G is $(k-1)$-colorable.

It is easy to check that the complete k-vertex graph K_k is k-critical and that each odd cycle is 3-critical.

Exercise 1. *Let $k \geq 3$. Prove that there are no k-critical $(k+1)$-vertex graphs. Describe all k-critical $(k+2)$-vertex graphs.*

2 Dirac

Dirac [10, 11, 12, 20, 15, 22, 19] introduced the notion of k-critical graphs and started a systematic study of them.

Lemma 1 (Dirac [15])**.** *Let $k \geq 3$ and let G be a k-critical graph. Then G is $(k-1)$-edge-connected. In particular, $\delta(G) \geq k-1$.*

Proof *(Kopon).* Suppose that $V(G)$ has a partition $V(G) = V_1 \cup V_2$ into nonempty sets such that $|E_G(V_1, V_2)| = t \leq k-2$. Let $E_G(V_1, V_2) = \{x_1y_1, \dots, x_ty_t\}$, where $\{x_1, \dots, x_t\} \subseteq V_1$ and $\{y_1, \dots, y_t\} \subseteq V_2$ (the vertices $x_1, \dots, x_t$ (respectively, $y_1, \dots, y_t$) do not need to be distinct). For $i = 1, 2$, let $G_i = G[V_i]$. Since G_1 and G_2 are proper subgraphs of G, by the definition of k-critical graphs, for $i = 1, 2$, graph G_i has a proper $(k-1)$-coloring g_i with colors $1, \dots, k-1$.

There are $(k-1)!$ ways to rename the colors in g_2 with $1, \dots, k-1$. For every $1 \leq j \leq t$, the number of color permutations such that $g_2(y_j) = g_1(x_j)$ is $(k-2)!$. Hence there are at least $(k-1)! - t(k-2)! = (k-2)!(k-1-t) \geq (k-2)!$ permutations such that $g_2(y_j) \neq g_1(x_j)$ for all

$j = 1, \ldots, t$. Any such permutation yields a proper $(k-1)$-coloring g of G; a contradiction. $\square$

Exercise 2 (Toft)**.** *Let $k \geq 3$ and let G be a k-critical graph. Suppose $V(G)$ has a partition $V(G) = V_1 \cup V_2$ into nonempty sets such that $|E_G(V_1, V_2)| = k-1$. Let $E_G(V_1, V_2) = \{x_1y_1, \ldots, x_{k-1}y_{k-1}\}$, where $\{x_1, \ldots, x_{k-1}\} \subseteq V_1$ and $\{y_1, \ldots, y_{k-2}\} \subseteq V_2$. For $i = 1, 2$, let $G_i = G[V_i]$. Then one of the following holds:*
(1) for each $(k-1)$-coloring of G_1, the colors of all $x_1, \ldots, x_{k-1}$ are the same and for each $(k-1)$-coloring of G_2, the colors of all $y_1, \ldots, y_{k-1}$ are distinct;
(2) for each $(k-1)$-coloring of G_1, the colors of all $x_1, \ldots, x_{k-1}$ are distinct and for each $(k-1)$-coloring of G_2, the colors of all $y_1, \ldots, y_{k-1}$ are the same.

Already this simple lemma yields the Heawood Formula for the chromatic number of graphs embeddable into surfaces of a given genus.

Theorem 2 (Heawood, 1890)**.** *If G is graph embeddable into an orientable surface S_γ of genus $\gamma \geq 1$, then $\chi(G) \leq \left\lfloor \frac{7+\sqrt{1+48\gamma}}{2} \right\rfloor$.*

Proof. Let $c := c_\gamma := \frac{7+\sqrt{1+48\gamma}}{2}$. Suppose $\chi(G) > c$. Then G contains a $(\lfloor c \rfloor + 1)$-critical subgraph G'. Let $n = |V(G')|$, $e = |E(G')|$ and f be the number of faces in an embedding of G' into S_γ. Then $n > c$. From the Euler Formula $n - e + f = 2(1-\gamma)$ and the fact that $3f \leq 2e$, we obtain

$$\frac{2e}{n} \leq 6 + \frac{12(\gamma-1)}{n} \leq 6 + \frac{12(\gamma-1)}{c}. \tag{1}$$

Since c is a root of the equation $c^2 - 7c - 12(\gamma-1) = 0$, we have $6 + \frac{12(\gamma-1)}{c} = c - 1$, so (1) yields $\frac{2e}{n} \leq c - 1$. But by Lemma 1, $\frac{2e}{n} \geq \delta(G') \geq \lfloor c \rfloor$, a contradiction. $\square$

In a series of papers [13, 14, 16, 17, 18], Dirac sharpened Theorem 2 by showing that for $\gamma \geq 1$ every graph embeddable into S_γ and having chromatic number $\lfloor c_\gamma \rfloor$ contains the complete graph with $\lfloor c_\gamma \rfloor$ vertices. For this he used properties of critical graphs with few vertices, but a really short proof he obtained in [18] by using the following general lower bound on the number of edges in critical graphs.

Theorem 3 (Dirac [20])**.** *If $n > k \geq 4$ and G is an n-vertex k-critical graph, then*

$$2|E(G)| \geq (k-1)n + k - 3. \tag{2}$$

Proof *(Deuber, A.K., Sachs).* For a graph F, let $\epsilon(F) := 2|E(F)| - (k-1)|V(F)|$. Then the theorem is equivalent to the assertion that if $k \geq 4$, then

$$\epsilon(G) \geq k - 3 \quad \textit{for each } k\textit{-critical graph } G \not\cong K_k. \tag{3}$$

We will use induction on $|V(G)|$ for a fixed $k \geq 4$. So, let G be a smallest k-critical graph G distinct from K_k for which (3) does not hold.

If $y, z \in V(G)$ and $yz \notin E(G)$, then $H(G; y, z)$ is the graph obtained from G by gluing y and z into one vertex. Then $\chi(H(G; y, z)) \geq \chi(G) = k$. So, $H(G; y, z)$ contains a k-critical subgraph $G^* = G^*(y, z)$. Since G itself is k-critical,

$$y * z \in V(G^*). \tag{4}$$

Let $H = H(G; y, z)$ and $U = U(G^*) := V(G) - V(G^*) - y - z$. If $x \in V(G)$ with $d_G(x) = k - 1$ and $y, z \in N_G(x)$, then $d_H(x) = k - 2$ and hence by Lemma 1,

$$x \in U. \tag{5}$$

The main idea of the proof is the following relation:

$$2e(G) \geq 2e(G^*) + \sum_{u \in U} d_G(u) + e_G(U, V(G) - U) + 2|(N_G(y) \cap N_G(z)) - U|.$$

It implies that

$$\epsilon(G) \geq \epsilon(G^*) + \sum_{u \in U} (d_G(u) - k + 1) - (k - 1) + e_G(U, V(G) - U) + 2|(N_G(y) \cap N_G(z)) - U|. \tag{6}$$

We claim that

$$G^* \cong K_k \quad \textit{for any } x, y, z \in V(G) \textit{ with } d_G(x) = k - 1,\ xy, xz \in E(G) \textit{ and } yz \notin E(G). \tag{7}$$

Indeed, if $G^* \ncong K_k$, then by the minimality of G, $\epsilon(G^*) \geq k - 3$. By (5), $U \neq \emptyset$, and so by Lemma 1, $e_G(U, V(G) - U) \geq k - 1$. Then by (6),

$$\epsilon(G) \geq \epsilon(G^*) + \sum_{u \in U} (d_G(u) - k + 1) - (k - 1) + (k - 1) \geq \epsilon(G^*) \geq k - 3.$$

This proves (7).

Since $|V(G)| \geq k + 2$ and $\epsilon(G) \leq k - 4$, there is $v \in V(G)$ with $d_G(v) = k - 1$. Then by (7), there is $W \subset V(G)$ with $G[W] = K_{k-1}$. Again, since $\epsilon(G) \leq k - 4$, there are $x_1, x_2, x_3 \in W$ with $d_G(x_i) = k - 1$ for $1 \leq i \leq 3$. Let y_i be the neighbor of x_i in $V(G) - W$. Let $W_1 := W \cap N_G(_1)$ and $W_1' = W - W_1$. Choose W and x_1 to maximize $|W_1|$.

Let z_1 be a vertex in W_1' of minimum degree. Let $H := H(G; y_1, z_1)$, $G^* := G^*(y_1, z_1)$, $U = U(G^*)$ and $U_W := U \cap W$. By the symmetry between x_2 and x_3, we may assume $z_1 \neq x_2$. Since $d_{G-x_1}(x_2) = k - 2$, by (5), $\{x_1, x_2\} \subseteq U_W$. So, by (4),

$$2 \leq |U_W| \leq k - 2. \tag{8}$$

Case 1: $j := |U_W| = k-2$. Let $S := V(G^*) - y_1 * z_1$. By (7), $G[S] = K_{k-1}$. Let $S' := S \cap N(y_1)$, $s := |S'|$, and $S'' = S - S'$. Since $G^* = K_k$, $S'' \subset N_G(z_1)$ and so $d_G(z_1) \geq (|W| - 1) + |S''| = k - 2 + (k - 1 - s)$. Thus if $d_G(v) \geq k$ for each $v \in S'$, then $\epsilon(G) \geq (d_G(z_1) - k + 1) + s \geq k - 2$, a contradiction. Hence we may assume that S' contains a vertex x' with $d_G(x') = k - 1$ and hence by the choice of W and x_1,

$$s \leq |W_1|. \tag{9}$$

Also, since G does not contain K_k, $1 \le s \le k-2$ and $1 \le |W_1| \le k-2$. By the choice of z_1 and (9),

$$\epsilon(G) \ge (d_G(y_1) - k + 1) + |W_1'|(d_G(z_1) - k + 1)$$

$$\ge (|W_1| + s - k + 1) + (k - 1 - |W_1|)(k - 2 - s) = |W_1| - 1 + (k - 2 - |W_1|)(k - 2 - s)$$

$$\ge |W_1| - 1 + (k - 2 - |W_1|)^2 \ge |W_1| - 1 + (k - 2 - |W_1|) = k - 3;$$

a contradiction.

Case 2: $2 \le j \le k-3$. Each of the $k-2-j$ vertices in $W - U - z_1$ has $k-2$ neighbors in W and $j+1$ neighbors in $V(G^*) - W$. Thus

$$\epsilon(G) \ge \sum_{z \in W-U-z_1} (d_G(z) - k + 1) \ge (k-2-j)(k-2+j+1-k+1) = j(k-2-j) \ge k-3;$$

a contradiction. $\square$

Example 1 (Dirac)**.** *Let $k \ge 4$. Every graph G in the family $D(k)$ has $2k-1$ vertices partitioned into 3 sets: V_0, V_1 and V_2, where $|V_0| = 2$, $|V_1| = k-1$ and $|V_2| = k-2$. We have $G[V_1] = K_{k-1}$, $G[V_2] = K_{k-2}$, each $v \in V_2$ is adjacent to both vertices in V_0, and each vertex in V_1 is adjacent to exactly one vertex in V_0. Furthermore each of the two vertices in V_0 has a neighbor in V_1. There are no other edges.*

Exercise 3 (Dirac)**.** *Let $k \ge 5$. Prove that each graph $G \in D(k)$ is k-critical and has $0.5((k-1)|V(G)| + k - 3)$ edges, i.e., is a sharpness example for Theorem 3.*

Exercise 4 (Dirac)**.** *Let $k \ge 5$. Extending the ideas of a proof of Theorem 3, show that every k-critical graph G distinct from K_k and not belonging to $D(k)$ satisfies $\epsilon(G) \ge k-1$.*

Exercise 5. *Using Theorem 3, mimic the proof of Theorem 2 to prove the Dirac's result that for $\gamma \ge 1$, every graph embeddable into S_γ with chromatic number $\lfloor c_\gamma \rfloor$ contains the complete graph on $\lfloor c_\gamma \rfloor$ vertices.*

3 Gallai

In his fundamental papers [25] and [26], Gallai proved a series of important properties of color-critical graphs.

Theorem 4 (Gallai)**.** *If $k \ge 4$, $k+2 \le n \le 2k-2$ and G is an n-vertex k-critical graph, then the complement of G is disconnected.*

Theorem 5 (Gallai)**.** *Let $k \ge 4$ and G be a k-critical graph. Let $B = B(G)$ be the set of vertices of degree $k-1$ in G. Then each block of $G[B]$ is a complete graph or an odd cycle.*

Let $f(n,k)$ denote the minimum number of edges in an n-vertex k-critical graph. Then $f(k,k) = \binom{k}{2}$ and $f(k+1,k)$ is not well defined. Theorem 3 states that if $k \ge 4$ and $n \ge k+2$, then $f(n,k) \ge \frac{1}{2}((k-1)n + k - 3)$. Using Theorem 4, Gallai found exact values of $f(n,k)$ for small n.

Theorem 6 (Gallai)**.** *If $k \geq 4$ and $k+2 \leq n \leq 2k-1$, then*

$$f(n,k) = \frac{1}{2}\left((k-1)n + (n-k)(2k-n)\right) - 1.$$

Note that the function is quadratic in k.

Theorem 5 in turn implies the following lower bound on $f(n,k)$.

Theorem 7 (Gallai)**.** *If $k \geq 4$ and $k+2 \leq n$, then*

$$f(n,k) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n. \tag{10}$$

For large n, this bound is much stronger than the bound in Theorem 3.

3.1 Deriving Theorem 7 from Theorem 5

A *Gallai tree* is a graph in which every block is a complete graph or an odd cycle.

Lemma 8. *Let $k \geq 4$ and let T be an n-vertex Gallai tree with maximum degree $\Delta(T) \leq k-1$ not containing K_k. Then*

$$2|E(T)| \leq \left(k-2+\frac{2}{k-1}\right)n. \tag{11}$$

Proof. If T is a block, then, since $T \not\cong K_k$ and $k \geq 4$, $\Delta(T) \leq k-2$ which is stronger than (11).

Suppose (11) holds for all Gallai trees with at most s blocks and T is a Gallai tree with $s+1$ blocks. Let B be a leaf block in T and x be the cut vertex in $V(B)$. Let $D := \Delta(B)$.

Case 1: $D \leq k-3$. Let $T' := T - (V(B) - \{x\})$. Then T' is a Gallai tree with s blocks. So $2|E(T)| = 2|E(T')| + D|V(B)|$ and, by induction, $2|E(T')| \leq \left(k-2+\frac{2}{k-1}\right)(n-|V(B)|+1)$. If $B = K_r$, then $r = D+1 \leq k-2$. So in this case

$$\begin{aligned}
&2|E(T)| - \left(k-2+\frac{2}{k-1}\right)n \\
&\leq \left(k-2+\frac{2}{k-1}\right)(n-D) + D(D+1) - \left(k-2+\frac{2}{k-1}\right)n \\
&= D\left(-k+2-\frac{2}{k-1}+D+1\right) \leq -D\frac{2}{k-1} < 0,
\end{aligned}$$

as claimed. Similarly, if $B = C_t$, then, by the case, $k \geq 5$ and

$$\begin{aligned}
&2|E(T)| - \left(k-2+\frac{2}{k-1}\right)n \\
&\leq \left(k-2+\frac{2}{k-1}\right)(n-t+1) + 2t - n\left(k-2+\frac{2}{k-1}\right) \\
&= (t-1)\left(-k+2-\frac{2}{k-1}+2\right) + 2 < 2\left(-k+4\right) + 2 \leq 0.
\end{aligned}$$

Case 2: $D = k-2$. Since $\Delta(T) \leq k-1$, only one block B' apart from B may contain x and this B' must be K_2. Let $T'' = T - V(B)$. Then T'' is a Gallai tree with $s-1$ blocks. So

$2|E(T)| = 2|E(T'')| + D|V(B)| + 2$ and, by induction, $2|(T'')| \le \left(k - 2 + \frac{2}{k-1}\right)(n - |V(B)|)$. Hence in this case, since $|V(B)| \ge D + 1 = k - 1$,

$$\begin{aligned} &2|E(T)| - \left(k - 2 + \frac{2}{k-1}\right) n \\ &\le \left(k - 2 + \frac{2}{k-1}\right)(n - |V(B)|) + (k-2)|V(B)| + 2 - \left(k - 2 + \frac{2}{k-1}\right) n \\ &= |V(B)| \left(-k + 2 - \frac{2}{k-1} + k - 2\right) + 2 \le -\frac{2}{k-1}|V(B)| + 2 \le 0, \end{aligned}$$

again. $\square$

Proof of Theorem 7. We use discharging. Let G be an n-vertex k-critical graph distinct from K_k. By Lemma 1, the minimum degree of G is at least $k-1$. The initial charge of each vertex $v \in V(G)$ is $\mathrm{ch}(v) := d_G(v)$. The only discharging rule is this:

(R1) Each vertex $v \in V(G)$ with $d_G(v) \ge k$ sends to each neighbor the charge $\frac{k-1}{k^2-3}$.

Denote the new charge of each vertex v by $\mathrm{ch}^*(v)$. We will show that

$$\sum_{v \in V(G)} \mathrm{ch}^*(v) \ge \left(k - 1 + \frac{k-3}{k^2-3}\right) n. \tag{12}$$

Indeed, if $d_G(v) \ge k$, then

$$\mathrm{ch}^*(v) \ge d_G(v) - \frac{k-1}{k^2-3} \cdot d_G(v) \ge k \left(1 - \frac{k-1}{k^2-3}\right) = k - 1 + \frac{k-3}{k^2-3}. \tag{13}$$

Also, if T is a component of the subgraph G' of G induced by the vertices of degree $k-1$, then

$$\sum_{v \in V(T)} \mathrm{ch}^*(v) \ge (k-1)|V(T)| + \frac{k-1}{k^2-3} \left|E_G(V(T), V(G) - V(T)\right|.$$

Since T is a Gallai tree and does not contain K_k, by Lemma 8,

$$|E(V(T), V(G) - V(T)| \ge (k-1)|V(T)| - \left(k - 2 + \frac{2}{k-1}\right)|V(T)| = \frac{k-3}{k-1}|V(T)|.$$

Thus for every component T of G' we have

$$\sum_{v \in V(T)} \mathrm{ch}^*(v) \ge (k-1)|V(T)| + \frac{k-1}{k^2-3} \cdot \frac{k-3}{k-1} \cdot |V(T)| = \left(k - 1 + \frac{k-3}{k^2-3}\right)|V(T)|.$$

Together with (13), this implies (12). $\square$

3.2 List coloring and proving Theorem 5

The original proof of Theorem 5 was difficult, but the notion of list coloring as a biproduct yields a significantly simpler proof. This notion was introduced by Vizing [57] and independently by Erdős, Rubin and Taylor [23].

A *list* L for a graph G is a map $L : V(G) \to \mathrm{Pow}(\mathbb{Z}_{>0})$ that assigns to each vertex $v \in V(G)$ a set $L(v) \subseteq \mathbb{Z}_{>0}$. An *$L$-coloring* of G is a mapping $f : V(G) \to \mathbb{Z}_{>0}$ such that $f(v) \in L(v)$ for each $v \in V(G)$ and $f(v) \neq f(u)$ whenever $vu \in E(G)$. The *list chromatic number*, $\chi_\ell(G)$, is the minimum k such that G has an L-coloring for each L satisfying $|L(v)| = k$ for every $v \in V(G)$.

Since G is k-colorable if and only if it is L-colorable with the list $L : v \mapsto [k]$, we have $\chi_\ell(G) \geq \chi(G)$ for every G; however, the difference $\chi_\ell(G) - \chi(G)$ can be arbitrarily large. Moreover, graphs with chromatic number 2 may have arbitrarily high list chromatic number. While 2-colorable graphs may have arbitrarily high minimum degree, Alon [2] showed that $\chi_\ell(G) \geq (1/2 - o(1)) \log_2 \delta$ for each graph G with minimum degree δ. On the other hand, some well-known upper bounds on $\chi(G)$ in terms of vertex degrees hold for $\chi_\ell(G)$ as well. For example, Brooks' theorem [9] and the degeneracy upper bound hold for $\chi_\ell(G)$. The following simple fact also holds.

Lemma 9 (Vizing [57])**.** *Suppose that G is a connected graph and L is a list for G such that $|L(v)| \geq d_G(v)$ for every $v \in V(G)$, and there is $x \in V(G)$ with $|L(x)| > d_G(v)$. Then G is L-colorable.*

Proof. Suppose the lemma does not hold and choose a counter-example (G, L) with smallest $|V(G)|$. Consider $(G - x, L)$. Then each component C_i of $G - x$ has a vertex z_i adjacent to x and hence with $|L(z_i) > d_{G-x}(z_i)$. By induction, each of C_i and hence the whole $G - x$ has an L-coloring. We now can choose a color for x from $L(x)$ distinct from the colors of all $d_G(x)$ neighbors of x. □

Furthermore, Borodin [4, 5] and independently Erdős, Rubin, and Taylor [23] generalized Brooks' Theorem to degree lists. Recall that a list L for a graph G is a *degree list* if $|L(v)| = d_G(v)$ for every $v \in V(G)$.

Theorem 10 ([4, 5, 23]; a simple proof in [34])**.** *Suppose that G is a connected graph. Then G is not L-colorable for some degree list L if and only if each block of G is either a complete graph or an odd cycle.*

Proof. Suppose there exists a pair (G, L), where G is a connected graph that is not a Gallai tree and L is a list for G with $|L(v)| \geq d_G(v)$ for each $v \in V(G)$ such that G is not L-colorable. We may assume that (G, L) is such a pair with the smallest $|V(G)|$. If $|V(G)| = 1$, then $G = K_1$, i.e., is a Gallai tree. So $|V(G)| \geq 2$.

Given $y \in V(G)$ and $\alpha \in L(y)$, let (G'_y, L'_α) denote the pair such that $G'_y = G - y$ and L'_α be the list for $G'(y)$ where $L'_\alpha(v) = \begin{cases} L(v) & \text{if } yv \notin E(G); \\ L(v) - \alpha & \text{if } yv \in E(G). \end{cases}$

Case 1: G is a block. First, we show that

$$L(x) = L(y) \textit{ for all } x, y \in V(G), \textit{ and } G \textit{ is regular.} \tag{14}$$

If there are vertices in G with distinct lists, then there are such vertices that are adjacent to each other. Suppose that $xy \in E(G)$ and $\alpha \in L(y) - L(x)$. Consider (G'_y, L'_α). Since G is a block, G'_y is connected. By construction, $d_{G'_y}(v) \leq |L'_\alpha(v)|$ for each $v \in V(G'_y)$. Moreover, by the choice of α, $d_{G'_y}(x) < |L'_\alpha(x)|$. Thus, by Lemma 9, G'_y has an L'_α-coloring g. We extend g to an L-coloring of G by letting $g(y) := \alpha$. This proves the first part of (14). The second part follows from the first and the fact that vertices of distinct degrees have distinct lists (of the size of the degrees).

So by (14), we are seeking an ordinary d-coloring of a d-regular graph G (for some d). Then G is a complete graph or an odd cycle by Brooks' Theorem (also by Theorem 3).

Case 2: G has a cut vertex. Let B_1 and B_2 be distinct leaf blocks. For $i = 1, 2$, let b_i be the cut vertex, let a_i be a non-cut vertex in B_i, and let $\alpha_i \in L(a_i)$. Again for $i = 1, 2$, consider the pair $(G'_{a_i}, L'_{\alpha_i})$. Since a_i is a non-cut vertex, G'_{a_i} is connected. By definition, L'_{α_i} is a degree list for G'_{a_i}. Since G is not L-colorable, G'_{a_i} is not L'_{α_i}-colorable. So by the minimality of G, each block of G'_{a_i} is a complete graph or an odd cycle. In particular, this holds for each block of G distinct from B_i. This implies the theorem. $\square$

Deriving Theorem 5 from Theorem 10: Let B_1 be a component of $G[B]$. Since G is k-critical, there is a $(k-1)$-coloring g of $G - B_1$. For every $v \in B_1$, define $L(v) := \{1, \dots, k-1\} - \{g(u) : u \in N(v)\}$. Then L is a degree list for $G[B_1]$. So Theorem 10 yields the claim. $\square$

Remark 1. Similarly to k-critical graphs, one can define *list-k-critical graphs* as the graphs whose list chromatic number is k but the list chromatic number of any proper subgraph is less than k. And similarly to $f(n,k)$ one can define $f_\ell(n,k)$ - the minimum number of edges in an n-vertex list-k-critical graph. Then the proof in the previous paragraph shows that the claim of Theorem 5 holds also for list-critical graphs. This in turn implies that similarly to (10) we have

$$f_\ell(n,k) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-3)}n. \tag{15}$$

Remark 2. Bounds (10) and (15) imply that
for every fixed γ and any $k \geq 6$, there is a polynomial-time algorithm for checking any graph G embeddable into S_γ whether G is k-colorable and whether G is list-k-colorable.

Exercise 6. *Prove the claim in Remark 2.*

3.3 Critical graphs with one high vertex and a conjecture

Theorem 5 allowed Gallai to describe for $k \geq 4$ all k-critical graphs with exactly one vertex of degree $\geq k$. Indeed, if G is a k-critical graph and x is the only vertex of degree $\geq k$, then $B(G) = V(G) - x$. By Theorem 5, $G - x$ is a Gallai tree with maximum degree at most $k-1$ and minimum degree at least $k-2$. And for every such special Gallai tree T, the graph, obtained by adding an extra vertex x adjacent to all vertices of degree $k-2$ and only to them is k-critical. If $k \geq 5$, then the blocks of such special T are of only two types: K_{k-1}s and K_2s. In particular, every k-critical graph G with exactly one vertex of degree $\geq k$ has $1 \pmod{k-1}$ vertices and $\frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$ edges.

Gallai thought that for $n \geq k$ there are no k-critical n-vertex graphs with fewer edges and posed the following.

Conjecture 11 (Gallai [25]). *If $k \geq 4$ and $n \equiv 1 \pmod{k-1}$, then*

$$f(n,k) = \frac{(k+1)(k-2)n - k(k-3)}{2(k-1)}.$$

4 Ore and others

For a graph G and vertex $u \in V(G)$, a *split* of u is a construction of a new graph G' such that $V(G') = V(G) - u + \{u', u''\}$, where $G - u \cong G' - \{u', u''\}$, $N(u') \cup N(u'') = N(u)$, and $N(u') \cap N(u'') = \emptyset$. A *DHGO-composition* $O(G_1, G_2)$ of graphs G_1 and G_2 is a graph obtained as follows: Delete some edge yz from G_2, split some vertex x of G_1 into two vertices x_1 and x_2 of positive degree, and identify x_1 with y and x_2 with z. Note that DHGO-composition could be found in Dirac's paper [21] and has roots in [15]. It was also used by Gallai [25] and Hajós [28]. Ore [46] used it for a composition of complete graphs.

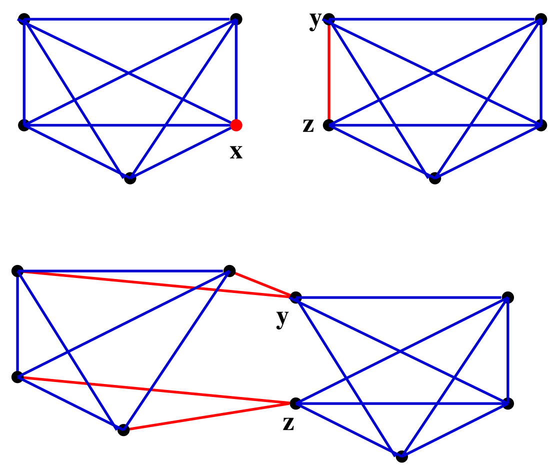

Figure 1: DHGO-composition $O(K_5, K_5)$.

The mentioned authors observed that if G_1 and G_2 are k-critical and G_1 is not k-critical after x has been split, then $O(G_1, G_2)$ also is k-critical. This observation implies

$$f(n+k-1,k) \leq f(n,k) + \frac{(k+1)(k-2)}{2} = f(n,k) + (k-1)\frac{(k+1)(k-2)}{2(k-1)}, \tag{16}$$

which yields that $\phi_k := \lim_{n\to\infty} \frac{f_k(n)}{n}$ exists and satisfies

$$\phi_k \leq \frac{k}{2} - \frac{1}{k-1}. \tag{17}$$

Gallai's bound gives $\phi_k \geq \frac{1}{2}\left(k - 1 + \frac{k-3}{k^2-3}\right)$. Ore believed that using this construction starting from an extremal graph on at most $2k$ vertices repeatedly with $G_2 = K_k$ at each iteration is best possible for constructing sparse critical graphs.

Conjecture 12 (Ore [46]). *If $k \geq 4$, $n \geq k$ and $n \neq k+1$, then*

$$f(n+k-1,k) = f(n,k) + (k-2)(k+1)/2.$$

Note that Conjecture 11 is equivalent to the case $n \equiv 1 \pmod{k-1}$ of Conjecture 12.

Krivelevich [42, 43] improved the bound of Theorem 7 to

$$f(n,k) \geq \frac{k-1}{2}n + \frac{k-3}{2(k^2-2k-1)}n \tag{18}$$

and demonstrated nice applications of his bound: he constructed graphs with high chromatic number and low independence number such that the chromatic numbers of all their small subgraphs are at most 3 or 4. We discuss a couple of his applications later. Then Kostochka and Stiebitz [36] proved that for $k \geq 6$ and $n \geq k+2$,

$$f(n,k) \geq \frac{k-1}{2}n + \frac{k-3}{k^2+6k-11-6/(k-2)}n. \tag{19}$$

Farzad and Molloy [24] proved the claim of Conjecture 11 in the case when $k=4$ and the subgraph of G induced by the vertices of degree 3 is connected.

Some time ago, Kostochka and Yancey [39] proved Conjecture 11 valid.

Theorem 13 ([39]). *If $k \geq 4$ and G is k-critical, then $|E(G)| \geq \left\lceil \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)} \right\rceil$. In other words, if $k \geq 4$ and $n \geq k$, $n \neq k+1$, then*

$$f(n,k) \geq F(n,k) := \left\lceil \frac{(k+1)(k-2)n-k(k-3)}{2(k-1)} \right\rceil.$$

The result also confirms Conjecture 12 in several cases.

Corollary 14 ([39]). *Conjecture 12 is true if (i) $k=4$, (ii) $k=5$ and $n \equiv 2 \pmod 4$, or (iii) $n \equiv 1 \pmod{k-1}$.*

Also, it determines ϕ_k:

Corollary 15. *For every $k \geq 4$ and $n \geq k+2$,*

$$0 \leq f(n,k) - F(n,k) \leq \frac{k(k-1)}{8} - 1.$$

In particular, $\phi_k = \frac{k}{2} - \frac{1}{k-1}$.

A simple but helpful tool was the following claim.

Corollary 16. *Let $k \geq 4$ and G be a k-critical graph. Let disjoint vertex subsets A and B be such that*
(a) either A or B is independent;
(b) $d(a) = k-1$ for every $a \in A$;
(c) $d(b) = k$ for every $b \in B$;
(d) $|A| + |B| \geq 3$.
Then (i) $e(G(A,B)) \leq 2(|A|+|B|) - 4$ and (ii) $e(G(A,B)) \leq |A| + 3|B| - 3$.

Call a graph G *k-extremal*, if G is k-critical and $|E(G)| = \frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$. By definition, if G is k-extremal, then $\frac{(k+1)(k-2)|V(G)|-k(k-3)}{2(k-1)}$ is an integer, and so $|V(G)| \equiv 1 \pmod{k-1}$. For example, K_k is k-extremal. Another example of a 5-extremal graph is on the bottom of Fig. 1.

Suppose that G_1 and G_2 are k-extremal and $G = O(G_1, G_2)$. Then

$$|E(G)| = |E(G_1)| + |E(G_2)| - 1 = \frac{(k+1)(k-2)(|V(G_1)| + |V(G_2)|) - 2k(k-3)}{2(k-1)} - 1$$

$$= \frac{(k+1)(k-2)|V(G)| - k(k-3)}{2(k-1)}.$$

After x is split, G_1 will still have $F(|V(G_1)|, k) < F(|V(G_1)| + 1, k)$ edges, and therefore will not be k-critical. Thus the DHGO-composition of any two k-extremal graphs is again k-extremal.

A graph is a *k-Ore graph* if it is obtained from a set of copies of K_k by a sequence of DHGO-compositions. By the above, every k-Ore graph is k-extremal. This yields an explicit construction of infinitely many k-extremal graphs. Kostochka and Yancey [41] proved that there are no other k-extremal graphs.

Theorem 17. *Let $k \geq 4$ and G be a k-critical graph. Then G is k-extremal if and only if it is a k-Ore graph. Moreover, if G is not a k-Ore graph, then $|E(G)| \geq \frac{(k+1)(k-2)|V(G)|-y_k}{2(k-1)}$, where $y_k = \max\{2k-6, k^2-5k+2\}$. Thus $y_4 = 2$, $y_5 = 4$, and $y_k = k^2 - 5k + 2$ for $k \geq 6$.*

The message of Theorem 17 is that although for every $k \geq 4$ there are infinitely many k-extremal graphs, they all have a simple structure. In particular, every k-extremal graph distinct from K_k has a separating set of size 2. The theorem gives a slightly better approximation for $f(n, k)$ and adds new cases for which we now know the exact values of $f(n, k)$:

Corollary 18. *Conjecture 12 holds and the value of $f(n, k)$ is known if (i) $k \in \{4, 5\}$, (ii) $k = 6$ and $n \equiv 0 \pmod 5$, (iii) $k = 6$ and $n \equiv 2 \pmod 5$, (iv) $k = 7$ and $n \equiv 2 \pmod 6$, or (v) $k \geq 4$ and $n \equiv 1 \pmod{k-1}$.*

This value of y_k in Theorem 17 is best possible in the sense that for every $k \geq 4$, there exist infinitely many 3-connected graphs G with $|E(G)| = \frac{(k+1)(k-2)|V(G)|-y_k}{2(k-1)}$. The idea of this construction and the examples for $k = 4, 5$ are due to Toft ([55], based on [54]). There are other examples for $k \geq 6$.

5 Some applications

5.1 Ore-degrees

The *Ore-degree*, $\Theta(G)$, of a graph G is the maximum of $d(x) + d(y)$ over all edges xy of G. Let $\mathcal{G}_t = \{G \,:\, \Theta(G) \leq t\}$.

Exercise 7. *Prove that $\chi(G) \leq 1 + \lfloor t/2 \rfloor$ for every $G \in \mathcal{G}_t$.*

Clearly $\Theta(K_{d+1}) = 2d$ and $\chi(K_{d+1}) = d+1$. The graph O_5 in Fig 2 is the only 9-vertex 5-critical graph with Θ at most 9. We have $\Theta(O_5) = 9$ and $\chi(O_5) = 5$.

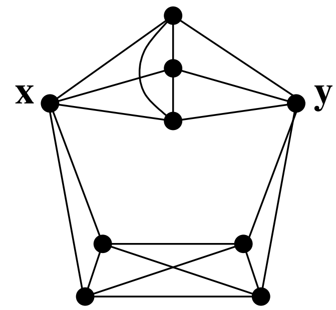

Figure 2: The graph O_5.

A natural question is to describe the graphs in $\mathcal{G}_{2d+1}$ with chromatic number $d+1$. Kierstead and Kostochka [30] proved that for $d \geq 6$ each such graph contains K_{d+1}. Then Rabern [50] extended the result to $d = 5$. Each $(d+1)$-chromatic graph G contains a $(d+1)$-critical subgraph G'. Since $\delta(G') \geq d$ and $\Theta(G') \leq \Theta(G) \leq 2d+1$,

$$\Delta(G') \leq d+1, \text{ and vertices of degree } d+1 \text{ form an independent set.} \tag{20}$$

Thus the results in [30] and [50] mentioned above could be stated in the following form.

Theorem 19 ([30, 50])**.** *Let $d \geq 5$. Then the only $(d+1)$-critical graph G' satisfying (20) is K_{d+1}.*

The case $d = 4$ was settled by Kostochka, Rabern, and Stiebitz [35]:

Theorem 20 ([35])**.** *Let $d = 4$. Then the only 5-critical graphs G' satisfying (20) are K_5 and O_5.*

Theorem 13 and Corollary 16 yield simpler proofs of Theorems 19 and 20. The key observation is the following.

Lemma 21. *Let $d \geq 4$ and let G' be a $(d+1)$-critical graph satisfying (20). If G' has n vertices of which $h > 0$ vertices have degree $d+1$, then*

$$h \geq \left\lceil \frac{(d-2)n - (d+1)(d-2)}{d} \right\rceil \tag{21}$$

and

$$h \leq \left\lfloor \frac{n-3}{d-1} \right\rfloor. \tag{22}$$

Proof. By definition, $2e(G') = dn + h$. So, by Theorem 13 with $k = d+1$,

$$dn + h \geq (d + 1 - \frac{2}{d})n - \frac{(d+1)(d-2)}{d},$$

which yields (21).

Let B be the set of vertices of degree $d+1$ in G' and $A = V(G') - B$. By (20), $e(G'(A,B)) = h(d+1)$. So, by Corollary 16(ii) with $k = d+1$,

$$h(d+1) \leq 3h + (n-h) - 3 = 2h + n - 3,$$

which yields (22). □

Another ingredient is Exercise 1: *Let $k \geq 3$. There are no k-critical graphs with $k+1$ vertices, and the only k-critical graph (call it D_k) with $k+2$ vertices is obtained from the 5-cycle by adding $k-3$ all-adjacent vertices.*

Suppose G' with n vertices of which h vertices have degree $d+1$ is a counter-example to Theorems 19 or 20. Since the graph D_{d+1} from Exercise 1 has a vertex of degree $d+2$, $n \geq d+4$. So since $d \geq 4$, by (21),

$$h \geq \left\lceil \frac{(d-2)(d+4)-(d+1)(d-2)}{d} \right\rceil = \left\lceil \frac{3(d-2)}{d} \right\rceil \geq 2.$$

On the other hand, if $n \leq 2d$, then by (22),

$$h \leq \left\lfloor \frac{2d-3}{d-1} \right\rfloor = 1.$$

Thus $n \geq 2d+1$.

Combining (21) and (22) together, we get

$$\frac{(d-2)n-(d+1)(d-2)}{d} \leq \frac{n-3}{d-1}.$$

Solving with respect to n, we obtain

$$n \leq \left\lfloor \frac{(d+1)(d-1)(d-2)-3d}{d^2-4d+2} \right\rfloor. \tag{23}$$

For $d \geq 5$, the RHS of (23) is less than $2d+1$, a contradiction to $n \geq 2d+1$. This proves Theorem 19.

Suppose $d = 4$. Then (23) yields $n \leq 9$. So, in this case, $n = 9$. By (21) and (22), we get $h = 2$. Let $B = \{b_1, b_2\}$ be the set of vertices of degree 5 in G'. By a theorem of Stiebitz [53], $G' - B$ has at least two components. Since $|B| = 2$ and $\delta(G') = 4$, each such component has at least 3 vertices. Since $|V(G') - B| = 7$, we may assume that $G' - B$ has exactly two components, C_1 and C_2, and that $|V(C_1)| = 3$. Again because $\delta(G') = 4$, $C_1 = K_3$ and all vertices of C_1 are adjacent to both vertices in B. So, if we color both b_1 and b_2 with the same color, this can extended to a 4-coloring of $G' - V(C_2)$. Thus to have G' 5-chromatic, we need $\chi(C_2) \geq 4$ which yields $C_2 = K_4$. Since $\delta(G') = 4$, $e(V(C_2), B) = 4$. So, since each of b_1 and b_2 has degree 5 and 3 neighbors in C_1, each of them has exactly two neighbors in C_2. This proves Theorem 20.

Remark. Recently Postle [47] and independently Kierstead and Rabern [31] have used Theorem 17 to describe the infinite family of 4-critical graphs G with the property that for each edge $xy \in E(G)$, $d(x)+d(y) \leq 7$. It turned out that such graphs form a subfamily of the family of 4-Ore graphs.

5.2 Local vs. global graph properties

Krivelevich [42] presented several nice applications of his lower bounds on $f(n,k)$ and related graph parameters to questions of existence of complicated graphs whose small subgraphs are simple. We indicate here how to improve two of his bounds using Theorem 13.

Let $f(\sqrt{n},3,n)$ denote the maximum chromatic number over n-vertex graphs in which every $\sqrt{n}$-vertex subgraph has chromatic number at most 3. Krivelevich proved that for every fixed $\epsilon > 0$ and sufficiently large n,

$$f(\sqrt{n},3,n) \geq n^{6/31-\epsilon}. \tag{24}$$

For this, he used his result that every 4-critical t-vertex graph with odd girth at least 7 has at least $31t/19$ edges. If instead of this result, we use our bound on $f(n,4)$, then repeating almost word by word Krivelevich's proof of (24) (Theorem 4 in[42]) and choosing $p = n^{-4/5-\epsilon'}$, we get that for every fixed ϵ and sufficiently large n,

$$f(\sqrt{n},3,n) \geq n^{1/5-\epsilon}. \tag{25}$$

Another result of Krivelevich is:

Theorem 22 ([42])**.** *There exists $C > 0$ such that for every $s \geq 5$ there exists a graph G_s with at least $C\left(\frac{s}{\ln s}\right)^{\frac{33}{14}}$ vertices and independence number less than s such that the independence number of each 20-vertex subgraph is at least 5.*

He used the fact that for every $m \leq 20$ and every m-vertex 5-critical graph H,

$$\frac{|E(H)|-1}{m-2} \geq \frac{\lceil 17m/8 \rceil - 1}{m-2} \geq \frac{33}{14}.$$

From Theorem 13 we instead get

$$\frac{|E(H)|-1}{m-2} \geq \frac{\left\lceil \frac{9m-5}{4} \right\rceil - 1}{m-2} \geq \frac{43}{18}.$$

Then repeating the argument in [42] we can replace $\frac{33}{14}$ in the statement of Theorem 22 with $\frac{43}{18}$.

5.3 Coloring planar graphs

In the attached paper [8], we use Theorem 13 to give simple proofs of some well-known results on 3-coloring of planar graphs, in particular of the Axenov-Grünbaum Theorem, and an one-paragraph proof of Grötzsch's Theorem [27]. Note that although the proof of the general case of Theorem 13 is somewhat long, the proof of the used case $k = 4$ is quite reasonable, and we present it in the next section.

In [7], Theorem 17 was used to describe the 4-critical planar graphs with exactly 4 triangles. This problem was studied by Axenov [1] in the seventies, and then mentioned by Steinberg [52] (quoting Erdős from 1990), and Borodin [6]. In particular, it was proved that the 4-critical planar graphs with exactly 4 triangles and no 4-faces are exactly the 4-Ore graphs with exactly 4 triangles.

6 Proof of Case $k = 4$ of Theorem 13

Theorem 13 for $k = 4$ reads:

$$f(n,4) = \left\lceil \frac{5n-2}{3} \right\rceil. \tag{26}$$

The proof in this section is from [40].

Definition 1. *For $R \subseteq V(G)$, define* the potential of R *to be* $\rho_G(R) = 5|R| - 3|E(G[R])|$. *When there is no chance for confusion, we will use $\rho(R)$. Let $P(G) = \min_{\emptyset \neq R \subseteq V(G)} \rho(R)$.*

Exercise 8. *Calculate that $\rho_{K_1}(V(K_1)) = 5$, $\rho_{K_2}(V(K_2)) = 7$, $\rho_{K_3}(V(K_3)) = 6$, $\rho_{K_4}(V(K_4)) = 2$.*

By definition, we have the following.

Exercise 9. *Let G be a graph and $A, B, C \subseteq V(G)$ be such that $A \supset B$ and $A \cap C = \emptyset$. Prove that $\rho_G(A - B) = \rho_G(A) - \rho_G(B) + 3|E_G(A - B, B)|$ (equivalently, $\rho_G(A \cup C) = \rho_G(A) + \rho_G(C) - 3|E_G(A, C)|$).*

Note that $|E(G)| < \frac{5|V(G)|-2}{3}$ is equivalent to $\rho(V(G)) > 2$. Let G be a vertex-minimal 4-critical graph with $\rho(V(G)) > 2$. This implies that

$$\text{if } |V(H)| < |V(G)| \text{ and } P(H) > 2, \text{ then } H \text{ is 3-colorable.} \tag{27}$$

Definition 2. *For a graph G, a set $R \subset V(G)$ and a 3-coloring ϕ of $G[R]$, the graph $Y(G, R, \phi)$ is constructed as follows. First, for $1 \leq i \leq 3$, let R'_i denote the set of vertices in $V(G) - R$ adjacent to at least one vertex $v \in R$ with $\phi(v) = i$. Second, let $X = \{x_1, x_2, x_3\}$ be a set of new vertices disjoint from $V(G)$. Now, let $Y = Y(G, R, \phi)$ be the graph with vertex set $V(G) - R + X$, such that $Y[V(G) - R] = G - R$ and $N(x_i) = R'_i \cup (X - x_i)$ for $1 \leq i \leq 3$.*

Claim 1. *Suppose $R \subset V(G)$, and ϕ is a 3-coloring of $G[R]$. Then $\chi(Y(G, R, \phi)) \geq 4$.*

Proof. Let $G' = Y(G, R, \phi)$. Suppose G' has a 3-coloring $\phi' : V(G') \to C = \{1, 2, 3\}$. By construction of G', the colors of all x_i in ϕ' are distinct. So we may assume that $\phi'(x_i) = i$ for $1 \leq i \leq 3$. By construction of G', for all vertices $u \in R'_i$, $\phi'(u) \neq i$. Therefore $\phi|_R \cup \phi'|_{V(G)-R}$ is a proper 3-coloring of G, a contradiction. □

Claim 2. *There is no $R \subsetneq V(G)$ with $|R| \geq 2$ and $\rho_G(R) \leq 5$.*

Proof. Let $2 \leq |R| < |V(G)|$ and $\rho(R) = m = \min\{\rho(W) : W \subsetneq V(G),\ |W| \geq 2\}$. Suppose $m \leq 5$. Then by Exercise 8, $|R| \geq 4$. Since G is 4-critical, $G[R]$ has a proper coloring $\phi : R \to C = \{1, 2, 3\}$. Let $G' = Y(G, R, \phi)$. By Claim 1, G' is not 3-colorable. Then it contains a 4-critical subgraph G''. Let $W = V(G'')$. Since $|R| \geq 4 > |X|$, $|V(G'')| < |V(G)|$. So, by the minimality of G, $\rho_{G'}(W) \leq 2$. Let $X' = W \cap X$. Since G is 4-critical by itself, every proper subgraph of G is 3-colorable and so $X' \neq \emptyset$. Since $0 < |X'| \leq 3$, by Exercise 8, $\rho_{G'}(X') \geq 5$. Since

$$|E_{G'}(W - X', X')| \leq |E_{G'}(W - X', X)| = |E_G(W - X', R)|,$$

by Exercise 9,

$$\begin{aligned} \rho_G((W - X') + R) &= \rho_G(W - X') + \rho_G(R) - 3|E_G(W - X', R)| \\ &= \rho_{G'}(W - X') + m - 3|E_{G'}(W - X', X)| \\ &\leq \rho_{G'}(W) - \rho_{G'}(X') + 3|E_{G'}(W - X', X')| + m - 3|E_{G'}(W - X', X)| \\ &\leq \rho_{G'}(W) - \rho_{G'}(X') + m \leq 2 - 5 + m. \end{aligned} \tag{28}$$

Since $W - X + R \supset R$, $|W - X + R| \geq 2$. Since $\rho_G(W - X + R) < \rho_G(R)$, by the choice of R, $W - X + R = V(G)$. But then $\rho_G(V(G)) \leq m - 3 \leq 2$, a contradiction. $\square$

Claim 3. *If $R \subsetneq V(G)$, $|R| \geq 2$ and $\rho_k(R) \leq 6$, then R is a K_3.*

Proof. Let R have the smallest $\rho(R)$ among $R \subsetneq V(G)$, $|R| \geq 2$. Suppose $m = \rho(R) \leq 6$ and $G[R] \neq K_3$. Then $|R| \geq 4$. By Claim 2, $m = 6$.

Let $R_* = \{u_1, \dots, u_s\}$ be the set of vertices in R that have neighbors outside of R. Because G is 2-connected, $s \geq 2$. Let $H = G[R] + u_1u_2$. Since $R \neq V(G)$, $|V(H)| < |V(G)|$. By the minimality of $\rho(R)$, for every $U \subseteq R$ with $|U| \geq 2$, $\rho_H(U) \geq \rho_G(U) - 3 \geq \rho_G(R) - 3 \geq 3$. Thus $P(H) \geq 3$, and by (27), H has a proper 3-coloring ϕ with colors in $C = \{1, 2, 3\}$. Let $G' = Y(G, R, \phi)$. Since $|R| \geq 4$, $|V(G')| < |V(G)|$. By Claim 1, G' is not 3-colorable. Thus G' contains a 4-critical subgraph G''. Let $W = V(G'')$. By the minimality of $|V(G)|$, $\rho_{G'}(W) \leq 2$. Since G is 4-critical by itself, $W \cap X \neq \emptyset$. Let $X' = W \cap X$. By Exercise 8, if $|X'| \geq 2$ then similarly to (28), $\rho_{k,G}(W - X' + R) \leq \rho_{G'}(W) - 6 + 6 \leq 2$, a contradiction again. So, we may assume that $X' = \{x_1\}$. Then again as in (28),

$$\rho_G(W - \{x_1\} + R) \leq (\rho_{G'}(W) - 5) + \rho_G(R) \leq \rho_G(R) - 3. \tag{29}$$

By the minimality of $\rho_G(R)$, $W - \{x_1\} + R = V(G)$. This implies that $W = V(G') - X + x_1$.

Let $R_1 = \{u \in R_* : \phi(u) = \phi(x_1)\}$. If $|R_1| = 1$, then $\rho_G(W - x_1 \cup R_1) = \rho_H(W) \leq 2$, a contradiction. Thus, $|R_1| \geq 2$. Since R_1 is an independent set in H and $u_1u_2 \in E(H)$, we may assume that $u_2 \notin R_1$. Then $E_{G'}(W - x_1, X - x_1) \neq \emptyset$. So, in this case repeating the argument of (28), instead of (29) we have

$$\rho_G(W - \{x_1\} + R) \leq \rho_{G'}(W) - 5 + \rho_G(R) - 3|E_{G'}(W - x_1, X - x_1)| \leq \rho_G(R) - 6 \leq 0. \quad \square$$

Claim 4. *G does not contain $K_4 - e$.*

Proof. If $G[R] = K_4 - e$, then $\rho_G(R) = 5(4) - 3(5) = 5$, a contradiction to Claim 3. $\square$

Claim 5. *Each triangle in G contains at most one vertex of degree 3.*

Proof. By contradiction, assume that $G[\{x_1, x_2, x_3\}] = K_3$ and $d(x_1) = d(x_2) = 3$. Let $N(x_1) = X - x_1 + a$ and $N(x_2) = X - x_2 + b$. By Claim 4, $a \neq b$. Define $G' = G - \{x_1, x_2\} + ab$. Because $\rho_G(W) \geq 6$ for all $W \subseteq G - \{x_1, x_2\}$ with $|W| \geq 2$, and adding an edge decreases the potential of a set by 3, $P(G') \geq \min\{5, 6 - 3\} = 3$. So, by (27), G' has a proper 3-coloring ϕ' with $\phi'(a) \neq \phi'(b)$. This easily extends to a proper 3-coloring of G. $\square$

Claim 6. *Let $xy \in E(G)$ and $d(x) = d(y) = 3$. Then both, x and y are in triangles.*

Proof. Assume that x is not in a K_3. Suppose $N(x) = \{y, u, v\}$. Then $uv \notin E(G)$. Let G' be obtained from $G - y - x$ by gluing u and v into a new vertex $u * v$. Then $|V(G')| < |V(G)|$. If G' has a 3-coloring $\phi' : V(G') \to C = \{1, 2, 3\}$, then we extend it to a proper 3-coloring ϕ

of G as follows: define $\phi|_{V(G)-x-y-u-v} = \phi'|_{V(G')-u*v}$, then let $\phi(u) = \phi(v) = \phi'(u*v)$, choose $\phi(y) \in C - (\phi'(N(y) - x))$, and $\phi(x) \in C - \{\phi(y), \phi(u)\}$.

So, $\chi(G') \geq 4$ and G' contains a 4-critical subgraph G''. Let $W = V(G'')$. Since G'' is smaller than G, $\rho_{G'}(W) \leq 2$. Since G'' is not a subgraph of G, $u*v \in W$. Let $W' = W - u*v + u + v + x$. Then $\rho_G(W') \leq 2 + 5(2) - 3(2) = 6$, since $G[W']$ has two extra vertices and at least two extra edges in comparison with G''. Because $y \notin W'$, we have $W' \neq V(G)$, and therefore by Claim 3, W induces a K_3 in G. This contradicts our assumption that x is not in a K_3. $\square$

By Claims 5 and 6, we have

$$\text{Each vertex of degree 3 has at most one neighbor of degree 3.} \tag{30}$$

We will now use discharging to show that $|E(G)| \geq \frac{5}{3}|V(G)|$, which will finish the proof to Case $k = 4$ of Theorem 13. Each vertex begins with charge equal to its degree. If $d(v) \geq 4$, then v gives charge $\frac{1}{6}$ to each neighbor with degree 3. Note that v will be left with charge at least $\frac{5}{6}d(v) \geq \frac{10}{3}$. By (30), each vertex of degree 3 will end with charge at least $3 + \frac{2}{6} = \frac{10}{3}$. $\square$

7 Triangle-free graphs, hypergraphs and unsolved problems

Kostochka and Stiebitz [37] proved that for large k and $n > k$, k-critical n-vertex triangle-free graphs must have almost $2f(n,k)$ edges. Asymptotically (in k) this is best possible: Some simple constructions of k-critical n-vertex graphs of arbitrary girth with average degree at most $2k - 1$ one can find in [33]. For small k, Postle [48, 49] recently obtained nice results. He proved that the asymptotical average degree for 4-critical graphs of girth 5 must be larger (not much, but larger) than the bound in Theorem 13. In [49] he proved a similar result for triangle-free 5-critical graphs. But these bounds most likely are not sharp, and finding exact bounds is a challenging problem.

The situation with hypergraphs with no graph edges is similar: it is proved in [37] that for large k, k-critical n-vertex hypergraphs must have almost $2f(n,k)$ edges, and constructions in [33] show that this bound for large k is asymptotically best. Again, exact bounds are not known, and values for small $k \geq 4$ are not known.

For list coloring, recently Kierstead and Rabern [32] and Rabern [51] using new ideas significantly improved the lower bounds on $f_\ell(n,k)$. Still, asymptotics of $f_\ell(n,k)$ is not known.

Another challenge is to prove Ore's Conjecture in full.

Many interesting unsolved problems on k-critical graphs are in [29]. In particular, there and in [56] the following problem by Dirac and Erdős is stated:

What is the **maximum** *number of edges $h(n,k)$ in a k-critical n-vertex graph, when k is fixed and n is large?*

Even for $k = 6$, $h(n,6)$ is quadratic in n: for $n = 4t + 2$, take two disjoint cycles C_1 and C_2 of length $2t + 1$ and join by an edge each vertex of C_1 with each vertex of C_2. It is not proved that this construction is best possible. Moreover, Toft [29][P. 97] conjectures that it is not best possible. He has a construction of **vertex**-6-critical n-vertex graphs with at least $3n^2/10$ edges.

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