Hamiltonian cycles in the matroid basis graph

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Background

A *matroid* is an ordered pair $(E, \mathcal{C})$ consisting of a finite set E and a collection $\mathcal{C}$ of subsets of E satisfying the following three conditions:

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member $C_3 \subseteq (C_1 \cup C_2) - e$.

The members of $\mathcal{C}$ are the *circuits* of M, and E is the *ground set* of M.

FACT:	Let E be the set of edges of a graph G and $\mathcal{C}$ be the set of edge sets of cycles of G. Then $\mathcal{C}$ is the set of circuits of a matroid on E.

The matroid derived above from the graph G is called the *cycle matroid* of G. It is denoted by M_G.

The $\mathcal{I}$ be the collection of subsets of E that contain no member of $\mathcal{C}$. The members of $\mathcal{I}$ are the *independent sets* of M. We call a maximal independent set in M a *basis* of M.

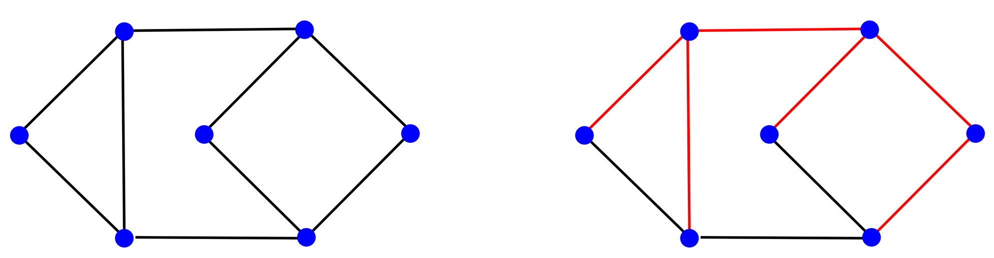
It is not hard to see that the collection $\mathcal{I}$ of independent sets of a matroid M satisfies the following properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 \setminus I_1$ such that $I_1 + e \in \mathcal{I}$.

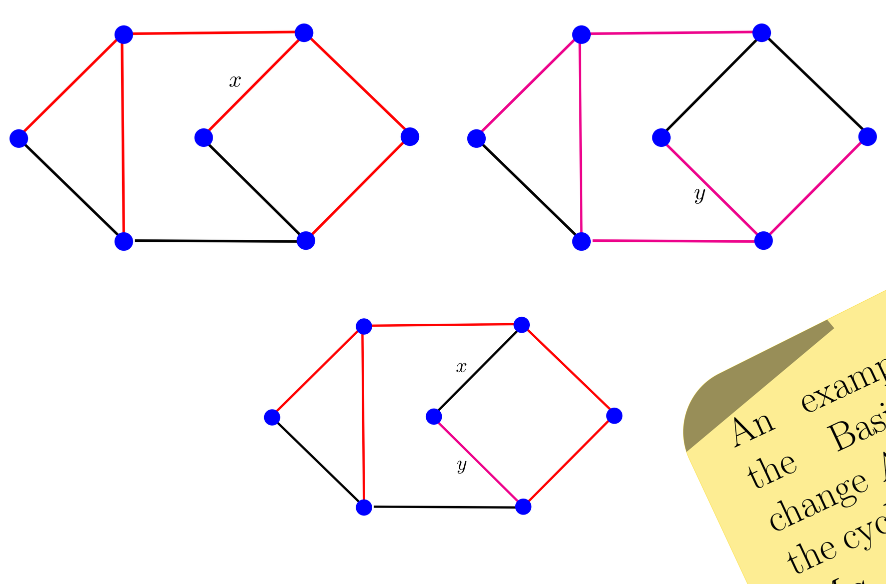
Let G be a graph. A subset X of $E(G)$ is independent in M_G exactly when $G[X]$, the subgraph of G induced by X, is a forest. Thus X is a basis of M_G when $G[X]$ is a spanning forest. It follows that, when G is connected, X is a basis of M_G if and only if $G[X]$ is a spanning tree.



A connected graph G. A basis of the cycle matroid M_G.

If M is a matroid and $\mathcal{B}$ is its collection of bases, then $\mathcal{B}$ satisfies the following *Basis Exchange Axiom*:

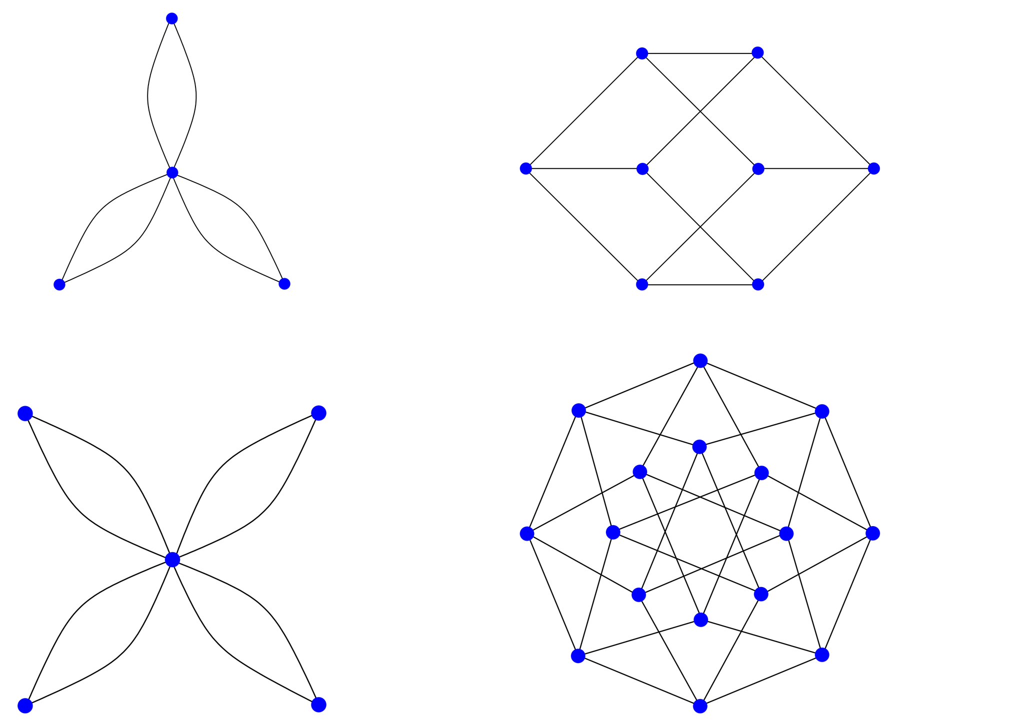
(BEA) If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) + y \in \mathcal{B}$.



An example of the Basis Exchange Axiom in the cycle matroid M_G

Matroid basis graph

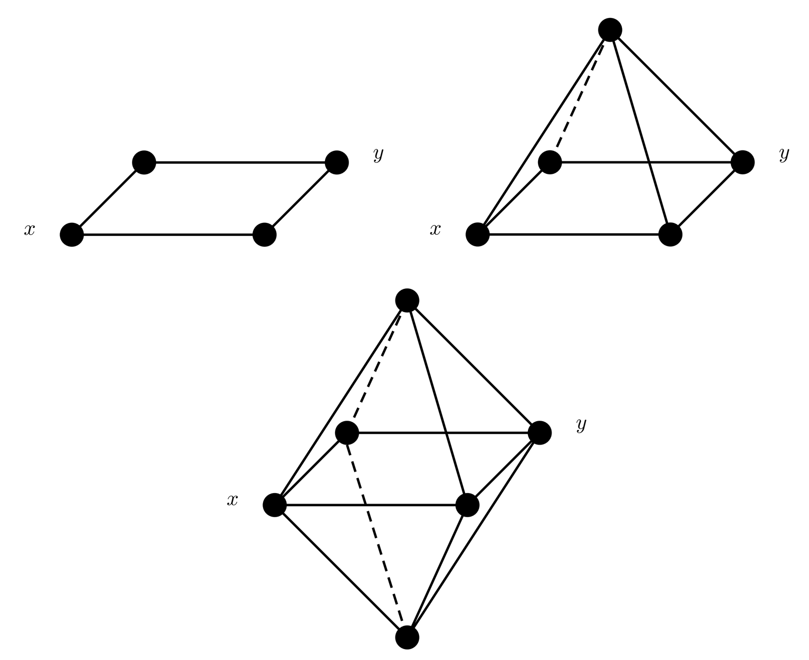
Let M be a matroid and $\mathcal{B}$ be its collection of bases. The *matroid basis graph* $\mathrm{BG}(M)$ of M is the graph with vertex set $\mathcal{B}$ in which two vertices/bases B_1 and B_2 are adjacent if and only if B_1 and B_2 differ by exactly one element; that is, if $|B_1 \triangle B_2| = 2$.



A graph G. The matroid basis graph $\mathrm{BG}(M_G)$.

Maurer characterized the matroid basis graph.

He proved that if the distance between the vertices x and y of a matroid basis graph is two, then the subgraph induced by their neighbors and themselves is either a square, a pyramid, or a octahedron.



The three types of subgraphs induced by the vertices x and y at distance two in a matroid basis graph.

Bondy and Ingleton proved that for every edge of the basis graph $\mathrm{BG}(M)$ of any matroid M, there exists a Hamiltonian cycle passing through it.

In the case of the basis graph of a cycle matroid, we proved that not only exists a Hamiltonian cycle passing through every edge but many of them. Formally, we proved the following.

Theorem[CGF, CH-V, JCP, JLRA]

Let G be a 3-edge-connected graph of order n, for $n \geq 3$. Then the number of Hamiltonian cycles passing through any edge of the basis graph $\mathrm{BG}(M_G)$ of the cycle matroid M_G is at least

$$(n-2)!2^{\binom{n-1}{2}}$$

.

Theorem[CGF, CH-V, JCP, JLRA]

Let G be a k-edge-connected graph of order 3, for $k \geq 3$. Then the number of Hamiltonian cycles passing through any edge of the basis graph $\mathrm{BG}(M_G)$ of the cycle matroid M_G is at least

$$\mathrm{sf}(k-1),$$

where $\mathrm{sf}(n) = n!(n-1)! \cdots 0!$ for a nonnegative integer n.

Theorem[CGF, CH-V, JCP, JLRA]

Let G be a k-edge-connected graph of order n, for $k, n \geq 4$. Then the number of Hamiltonian cycles passing through any edge of the basis graph $\mathrm{BG}(M_G)$ of the cycle matroid M_G is at least

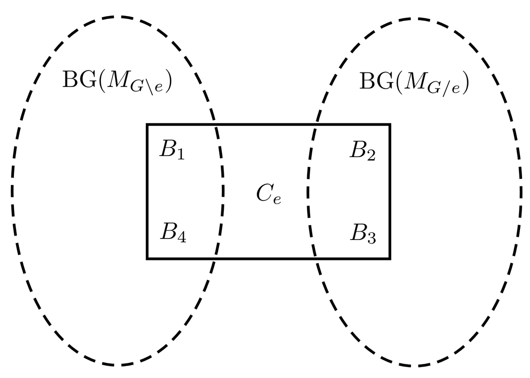
$$\frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1)k} \prod_{r=4}^{k} \left(r\,\mathrm{sf}(r-1)\right)^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}}.$$

Sketch of the proof

Let B_1B_2 be an edge of $\mathrm{BG}(M_G)$. We may assume that the element e belongs to B_1 and not to B_2. We partition the set of bases $\mathcal{B}$ into the set of bases $\mathcal{B}/e$ containing the element e and the set of bases $\mathcal{B}\backslash e$ that do not contain the element e.

We apply induction on $\mathcal{B}/e$ since there exists a $1{-}1$ correspondence with the bases of the cycle matroid derived from the graph G/e. Also, we apply induction on $\mathcal{B}\backslash e$ because the $1{-}1$ correspondence with the bases of the cycle matroid derived from the graph $G\backslash e$.

Finally, we joint both parts, $\mathcal{B}/e$ and $\mathcal{B}\backslash e$, with a "good" cycle.



Of course, we need the base cases given by the prior theorems and to prove the existence of such a good cycle.

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