Maximum number of triangles in wheel-free graphs and its relation with Turán number of expansion of wheels in 3-graphs

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Basic Definitions

Definition 1 (r-hypergraphs)**.** *Let V be a set of n elements. F is called a r-hypergraph or r-graph with vertex set V if the edge set $E(F)$ of F is a subset of $\binom{[V]}{r}$, where $\binom{[V]}{r}$ is the collection of all r-sets of V.*

Definition 2 (Turán Number)**.** *Let $ex_r(n, F)$ denote the maximum number of edges in an r-hypergraph with n vertices not containing any copy of the r-hypergraph F.*

Definition 3 (r-expansion)**.** *The r-expansion G^+ of a graph G is the r-hypergraph obtained from a graph G by enlarging each edge of G with a vertex subset of size $r-2$ disjoint from $V(G)$ such that distinct edges are enlarged by disjoint subsets.*

Definition 4 (k-wheels)**.** *W_k is denoted for $k-$wheels, determined by connecting a center K_1 to each vertex of a $k-$cycle C_k by an edge.*

Problem

Determine the order of magnitude of $ex_3(n, G^+)$ when G is an even wheel, that is $G = W_{2k}$. Specifically, determine whether $ex_3(n, G^+) = \Theta(n^2)$ for every even wheel G.

Partial Results

$ex_3(n, W_{2k}^+) = \Omega(n^2)$.

Ideas for Remaining Parts

Definition. *For a hypergraph H, the codegree of a set $S = \{x_1, x_2 ..., x_s\}$ of vertices of H is $d_H(S) = |\{e \in H : S \subset e\}|$. Let ∂H denote the $(r-1)$-graph of sets contained in some edge of H - this is the shadow of H. The edges of ∂H will be called the sub-edges of H. An r-graph H is d-full if every sub-edge of H has codegree at least d.*

The following is a useful lemma for the problem.

Lemma. *For $r \geq 2, d \geq 1$, every n-vertex r-graph H has a $(d+1)$-full subgraph F with $|F| \geq |H| - d|\partial H|$.*

Proof. Immediately by greedy algorithm. □

The following ideas are mainly focused on finding the Turán number of 3-expansion of W_4.

1. Consider an extramal 3-graph F for W_4. Find a vertex with large degree in ∂F and look for $C_4 = K_{2,2}$ in its neighbors.
2. View W_4^+ as a similar thing to tight 5-cycle or tight 5-cycle minus an edge.
3. Flag Algebra.
4. Consider Turán density.
5. $\cdots$

Reference

[1] Z. Füredi, M. Goemans, D. Kleitman: *On the Maximum Number of Triangles in Wheel-Free Graphs,* Combinatorics, Probability & Computing, 1994, 3(1): 63-75

Proofs

Lemma 1. *[1] Let WFG_n be the set of all graphs on n vertices with no wheels. Let $t(n)$ be the maximum number of triangles over WFG_n. Then there exists a wheel-free graph on n vertices with $\frac{n^2}{7.5} + \frac{n}{15}$ triangles whenever n is a multiple of 15, i.e., $t(n) \geq \frac{n^2}{7.5} + \frac{n}{15}$.*

Proof. Define the graph G_n on n vertices, where n is a multiple of 15 (see figure 1) as follows. Its vertex set $V(G_n)$ consists of a_i, b_i, c_i, d_i for $i = 1, ..., n/5$, and e_k, f_k, g_k for $k = 1, ..., n/15$. Its edge set $E(G_n)$ consists of
- two matchings of size $n/5$: (a_i, b_i) and (c_i, d_i),
- three matchings of size $n/15$: $(e_k, f_k), (f_k, g_k) and (e_k, g_k)$, and
- all the edges of types: $(a_i, c_j), (a_i, d_j), (a_i, g_k), (b_i, d_j), (b_i, f_k), (b_i, g_k), (c_j, e_k), (c_j, f_k), (d_j, e_k)$. (Here $1 \leq i, j \leq n/5$ and $1 \leq k \leq n/15$.)

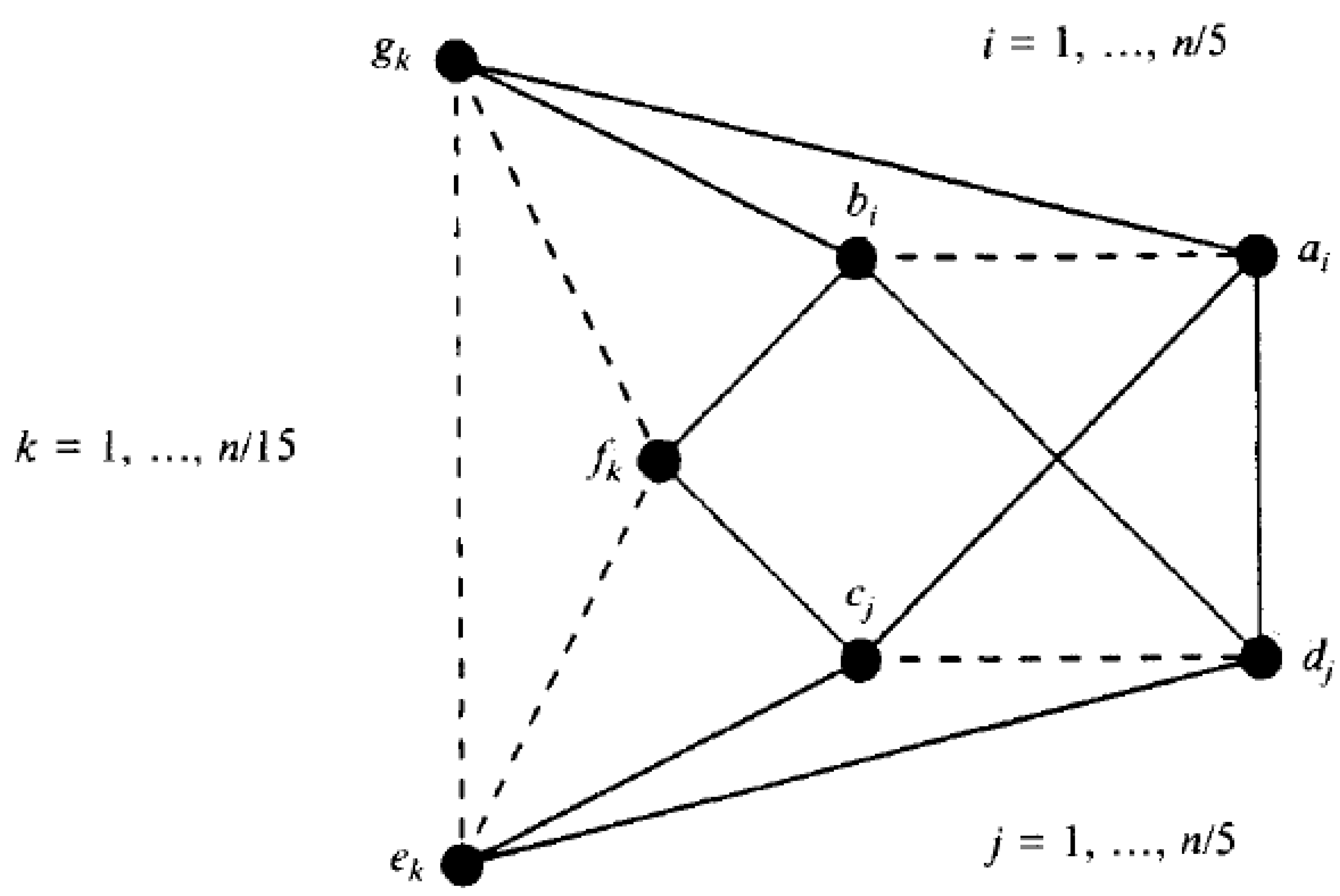

Figure 1 A wheel-free graph having $n^2/7.5 + n/15$ triangles.

It is easy to verify that this graph belongs to WFG_n. For example, the neighborhood of the vertex a_i, consists of the matching $\{(c_j, d_j) : 1 \leq j \leq n/5\}$ as well as a star rooted at b_i with edges $\{(b_i, g_k) : 1 \leq k \leq n/15\}$ and $\{(b_i, d_j) : 1 \leq j \leq n/5\}$.
Each triangle in G_n contains an edge from the matchings, and its vertices are in three different classes. An easy calculation shows that the number of triangles in G_n is $t(G_n) = n^2/7.5 + n/15$. □

Lemma 2. *Let G be a graph and G^+ be the 3-expansion of G. Let $g(n)$ be the maximum number of triangles over all $G-$free graph on n vertices. Then $g(n) \leq ex_3(n, G^+)$.*

Proof. Let F be a 3-graph. Let ∂F denote the 2-graph of sets contained in some edge of F, which is called *shadow* of F. Let H_2 be a $G-$free graphs with $g(n)$ triangles. View each triangle of H_2 as a 3-edge, we get a 3-graph H. Then H has $g(n)$ edges and is G^+-free. Otherwise, since $G^+ \subseteq H$, then $G \subseteq \partial G^+ \subseteq \partial H \subseteq H_2$, contradictory to H_2 is $G-$free. □

Theorem 1. $ex_3(n, W_{2k}^+) = \Omega(n^2)$.

Proof. Since wheel free implies some even wheel free. By Lemma 1 and 2, we get the result immediately. □