Hamilton Cycles in Hypergraphs below the Dirac Threshold

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Basic Definitions

A *k-uniform hypergraph* or *k-graph* $H = (V, E)$ is a finite vertex set V together with a set of edges $E \subseteq \binom{V}{k}$. This is a natural generalisation of a graph which coincides with the case $k = 2$.

Given any integer $1 \leq \ell < k$, we say that a k-graph C is an *ℓ-cycle* if the vertices of C may be cyclically ordered such that every edge of C consists of k consecutive vertices and each edge intersects the subsequent edge in precisely ℓ vertices.

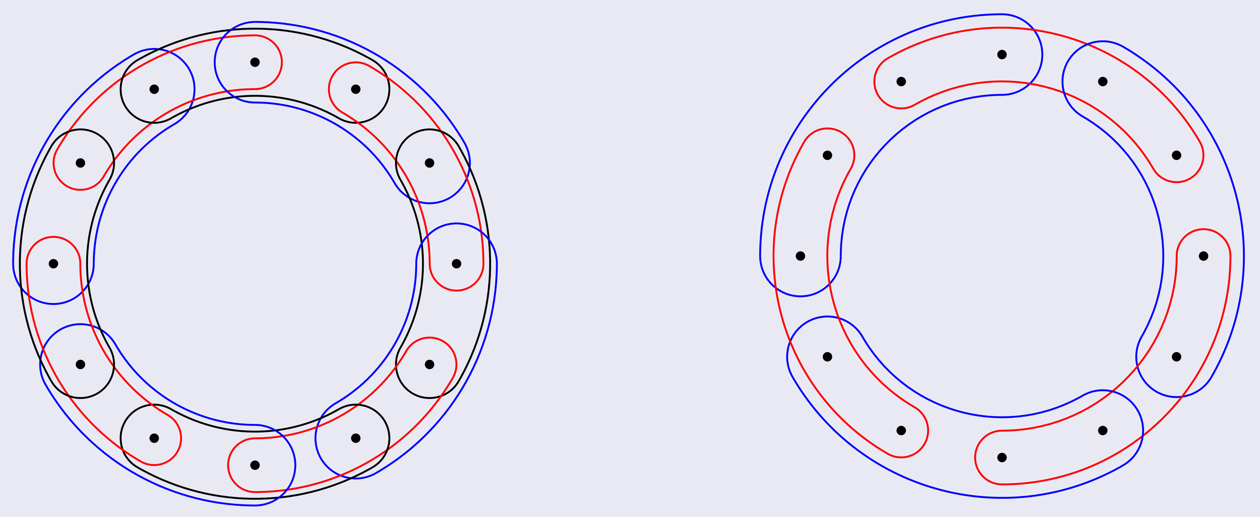

Figure: A 2-cycle 3-graph and a 2-cycle 4-graph.

We say that a k-graph H on n vertices contains a *Hamilton ℓ-cycle* if it contains an n-vertex ℓ-cycle as a subgraph.

The *degree* of a set $S \subseteq V$ is

$$d(S) = |\{e \in E \,:\, S \subseteq e\}| \,.$$

The *minimal codegree* of H is

$$\delta(H) = \min_{S \in \binom{V}{k-1}} \{d(S)\} \,.$$

Dirac-type Results

A classic result of graph theory is the theorem of Dirac.

Dirac's Theorem

If G is a graph on $n \geq 3$ vertices with minimum degree at least $n/2$, then G contains a Hamilton cycle.

A major focus in recent years has been to find hypergraph analogues of Dirac's theorem. The asymptotic Dirac threshold for any $1 \leq \ell < k$ can be collectively described by the following theorem which comprises results of independent collaboration among Hàn, Keevash, Kühn, Mycroft, Osthus, Rödl, Ruciński, Schacht and Szemerédi.

Theorem (Asymptotic Dirac for k-graphs)

For any $k \geq 3$, $1 \leq \ell < k$ and $\eta > 0$, there exists n_0 such that if $n \geq n_0$ is divisible by $k - \ell$ and H is a k-graph on n vertices with

$$\delta(H) \geq \begin{cases} \left(\frac{1}{2} + \eta\right) n & \text{if } k - \ell \text{ divides } k, \\ \left(\frac{1}{\lceil \frac{k}{k-\ell} \rceil (k-\ell)} + \eta\right) n & \text{otherwise,} \end{cases}$$

then H contains a Hamilton ℓ-cycle.

More recently the exact Dirac threshold has been identified in some cases, namely for $k = 3, \ell = 2$ by Rödl, Ruciński and Szemerédi, for $k = 3, \ell = 1$ by Czygrinow and Molla, and for any $k \geq 3$ and $\ell < k/2$ by Han and Zhao. We add the exact Dirac threshold for Hamilton 2-cycles in 4-graphs.

Theorem (G. and Mycroft, 2016)

There exists n_0 such that if $n \geq n_0$ is even and H is a 4-graph on n vertices with

$$\delta(H) \geq \begin{cases} \frac{n}{2} - 2 & \text{if } n \text{ is divisible by } 8, \\ \frac{n}{2} - 1 & \text{otherwise,} \end{cases}$$

then H contains a Hamilton 2-cycle. Moreover, this condition is best-possible for any even $n \geq n_0$.

Characterisation of 2-Hamiltonian 4-graphs

We now want to analyse dense 4-graphs below this Dirac threshold. The key insight is that each such 4-graph which is not 2-Hamiltonian is a subgraph of one of two extremal examples together with some very few additional edges.

Theorem (G. and Mycroft, 2016)

There exist $\varepsilon, n_0 > 0$ such that the following statement holds for any even $n \geq n_0$. Let H be a 4-graph on n vertices with $\delta(H) \geq n/2 - \varepsilon n$. Then H admits a Hamilton 2-cycle if and only if every bipartition of $V(H)$ is both even-good and odd-good.

Let $H = (V, E)$ be a 4-graph of order n, where n is even, and let $\{A, B\}$ be a bipartition of V. We say an edge $e \in E$ is *odd*, if $|e \cap A|$ is odd, otherwise we say that e is *even*. Furthermore we say that a pair $p \in \binom{V}{2}$ is a split pair, if $|p \cap A| = 1$.

Even-Good

We say that $\{A, B\}$ is *even-good* if at least one of the following statements holds.

- $|A|$ is even or $|A| = |B|$.
- E contains odd edges e and e' such that either $e \cap e' = \emptyset$ or $e \cap e'$ is a split pair.
- $|A| = |B| + 2$ and E contains odd edges e and e' with $e \cap e' \in \binom{A}{2}$.
- $|B| = |A| + 2$ and E contains odd edges e and e' with $e \cap e' \in \binom{B}{2}$.

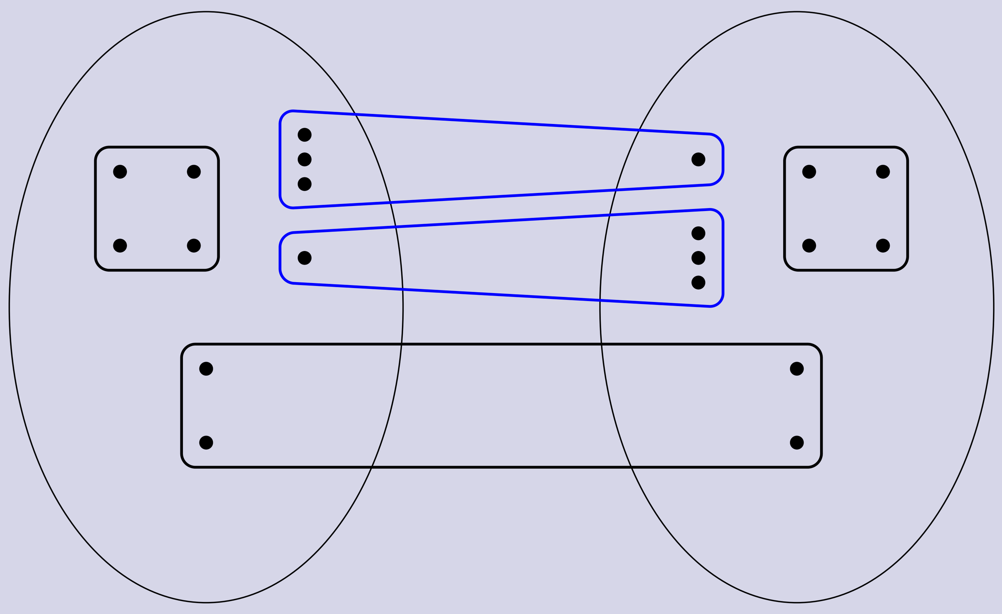

Figure: Note that with $|A| \approx \frac{n}{2}$, $|A| \neq |B|$, both odd, and without the blue edges this leads to an extremal example of a non 2-Hamiltonian 4-graph.

Let $m \in \{0, 2, 4, 6\}$ and $d \in \{0, 2\}$ be such that $m \equiv n \mod 8$ and $d \equiv |A| - |B| \mod 4$.

Odd-Good

We say that $\{A, B\}$ is *odd-good* if at least one of the following statements holds.

- $(m, d) \in \{(0, 0), (4, 2)\}$.
- $(m, d) \in \{(2, 2), (6, 0)\}$ and E contains an even edge.
- $(m, d) \in \{(4, 0), (0, 2)\}$ and E contains two even edges e, e' with $|e \cap e'| \in \{0, 2\}$.
- $(m, d) \in \{(6, 2), (2, 0)\}$ and either there is an even edge $e \in E$ with $|e \cap A| = 2$ or E contains 3 even edges which induce a disjoint union of 2-paths.

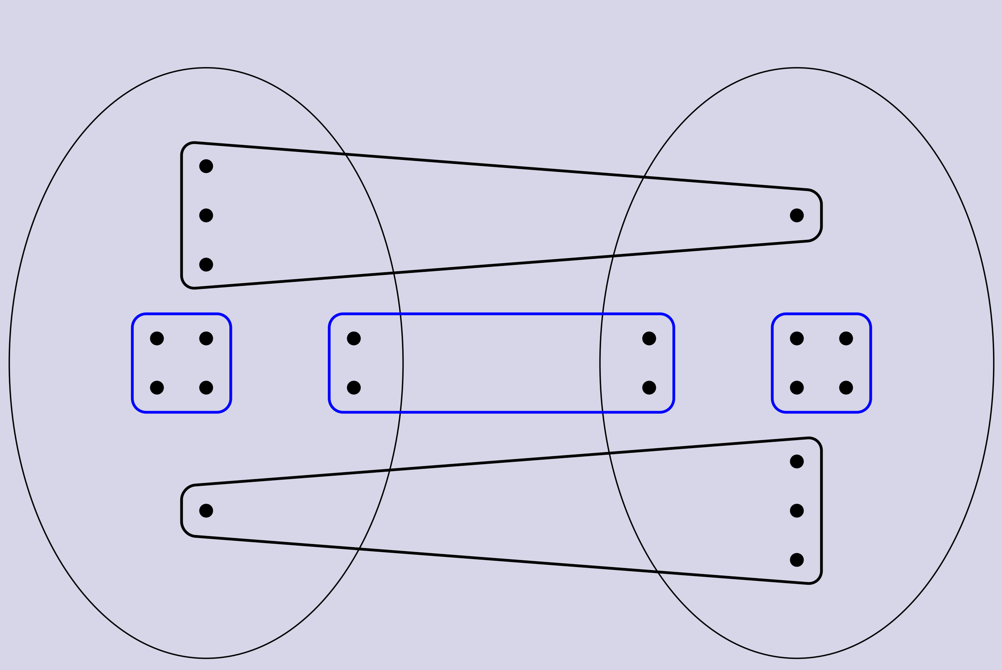

Figure: Note that with $|A| \approx \frac{n}{2}$, $n \equiv 2 \mod 4$ and without the blue edges this leads to an extremal example of a non 2-Hamiltonian 4-graph.

Consequences and further results

A consequence of this characterisation is that we can decide the Hamilton 2-cycle problem in 4-graphs with high minimal codegree in polynomial time.

Theorem (G. and Mycroft, 2016)

There exist a constant $\varepsilon > 0$ and an algorithm which, given a 4-graph H on n vertices with $\delta(H) \geq n/2 - \varepsilon n$, runs in time $O(n^{32})$ and returns either a Hamilton 2-cycle in H or a certificate that no such cycle exists.

We can also prove that the situation presents itself very differently in the case of tight cycles, i.e. $\ell = k - 1$. Indeed, if P≠NP, we should not expect a characterisation similar to the case of 2-cycles as the following theorem shows.

Theorem (G. and Mycroft, 2016)

For any $k \geq 3$ there exists $C \in \mathbb{N}$ such that it is NP-hard to determine whether a k-graph H with $\delta(H) \geq \frac{1}{2}|V(H)| - C$ admits a tight Hamilton cycle.