Local Symmetry in Random Graphs

Jefferson Elbert Simões, Daniel R. Figueiredo, Valmir C. Barbosa
Systems Engineering and Computer Science Program (PESC)
Federal University of Rio de Janeiro (UFRJ) — Rio de Janeiro, Brazil
E-mails: {elbert, daniel}@land.ufrj.br, valmir@cos.ufrj.br

Motivation

- Symmetry: structural invariance w.r.t. certain transformations
- Symmetry in networks (graphs) relates structure of edges and identity of vertices
  - e.g.: using network structure for matching vertices of related networks, breaching users' privacy
- Global symmetry: traditional graph-theoretic approach
  - isomorphisms: adjacency-preserving vertex mappings
  - automorphisms: isomorphisms to the graph itself
  - group of automorphisms induces equivalence classes of vertices: "equivalent position" in the network
- In many cases, an asymmetric network contains vertices that are intuitively equivalent from a local perspective
  - Topologically similar areas of the network

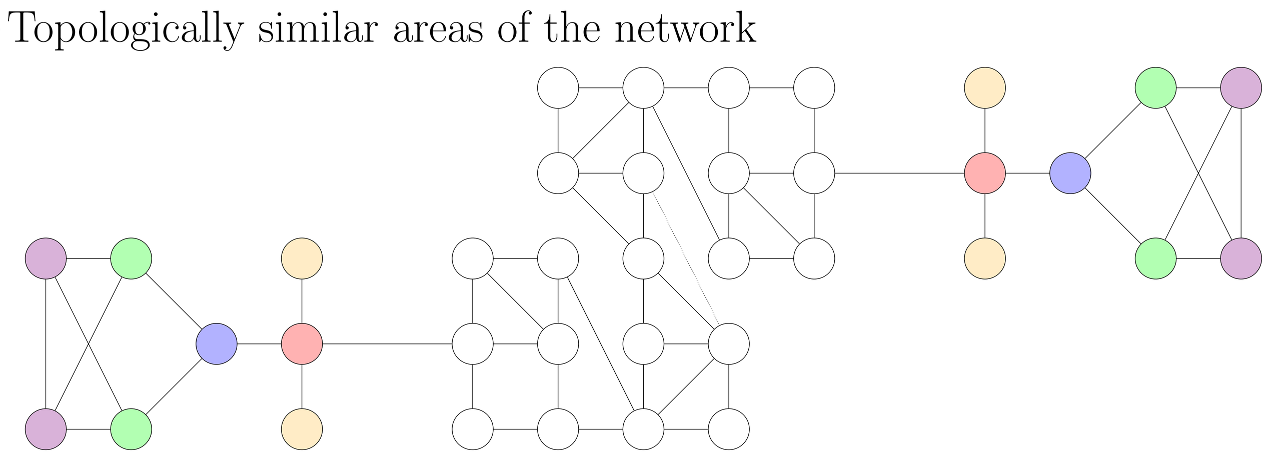

Fig. 1: Equivalence of local structures, highlighted by similarly colored subgraphs. Due to the presence of the dashed edge, no automorphism matches these subgraphs, so global symmetry between them does not exist.

- Can we capture this equivalence with the concept of symmetry?

Local symmetry

- Given a graph $G=(V,E)$ and node $v \in V$, let $\mathcal{N}^k[u]$ be the subgraph induced by vertices with distance at most k from u
  - $\mathcal{N}^0[v] = v$ (trivial graph)
  - $\mathcal{N}^1[v] = \mathcal{N}[v] = v$ and its neighbors with incident edges
  - Growing neighborhoods: proxy for locality

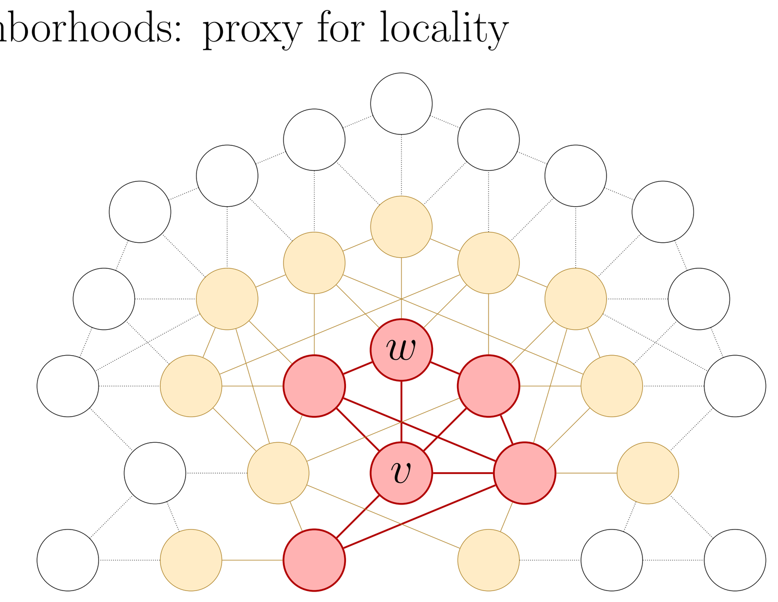

Fig. 2: Neighborhoods around v. $\mathcal{N}[v]$ is traditional closed neighborhood; $\mathcal{N}^2[v] = \mathcal{N}[\mathcal{N}[v]]$ includes both red and orange vertices; $\mathcal{N}^3[v] = G$

- Global symmetry (traditional):
  - **u and v are globally symmetric** $\iff$ there is an automorphism of G mapping u to v
  - **G is globally symmetric** $\iff$ there are $u, v \in V$ distinct and globally symmetric — i.e. G has a non-trivial automorphism
- Local symmetry (proposed here):
  - **u and v are k-locally symmetric** $\iff$ there is an isomorphism between $\mathcal{N}^k[u]$ and $\mathcal{N}^k[v]$ mapping u to v
  - **G is locally symmetric** $\iff$ there are $u, v \in V$ distinct and k-locally symmetric
  - e.g.: red nodes in Figure 1 are 4-local symmetric
- Intuition: u and v locally symmetric must
  - have topologically equivalent neighborhoods; AND
  - be equivalently located in these neighborhoods
- Size of neighborhood determines level of locality
  - We'll use "local symmetry" = 1-local symmetry by default
- Hierarchy of symmetries
  - $(k+1)$-local symmetry implies k-local symmetry
  - If $k \geq \operatorname{diam}(G)$, k-local symmetry = global symmetry

Known results

(see [Bollobás '81], [KSV '02])

- A $G(n,p)$ random graph is:
  - globally asymmetric a.a.s. if $p \in [\log n/n, 1 - \log n/n]$
  - globally symmetric if $p \ll \log n/n$ or $1 - p \ll \log n/n$
- A $G(n,m)$ random graph is globally asymmetric a.a.s iff $2m/\binom{n}{2} \geq \log n + \omega(1)$ and $n - 1 - 2m/\binom{n}{2} \geq \log n + \omega(1)$

Regime of local symmetry

Theorem: A $G(n,p)$ random graph, with $p = o(n^{-2/3})$, is locally symmetric a.a.s.

- Local symmetry persists to much higher average degree than global symmetry

Proof sketch:

- A $G(n,p)$ has few triangles ($\sim (np)^3 = o(n)$)
  - therefore most neighborhoods ($n - o(n)$) are stars
- Degrees in $G(n,p)$ are concentrated around np
  - range of length $\sim np$: $o(n)$ degree values used
- Same degree + star neighborhoods $\implies$ local symmetry

Regimes of local asymmetry

Theorem: A $G(n,p)$ random graph, with $\omega(n^{-1/2+\delta_1}) \leq p \leq o(n^{-3/7-\delta_2})$ for constant $\delta_1, \delta_2 > 0$, is locally asymmetric a.a.s.

- Local asymmetry eventually emerges, but much later

Proof sketch:

- Union bounds reduces problem to local asymmetry of two vertices in $G(n,p)$
- Isomorphism relates to distance metric Δ over degree sequences of neighborhoods
  - Local symmetry between u and v only if $\Delta(\mathcal{N}[u], \mathcal{N}[v]) = 0$
- Intersection between neighborhoods is small
  - Removing it affects Δ by $O((np^2)^2)$
- Neighborhood remainders are two independent $G(n,p)$ random graphs
  - Random size: $\sim np$ vertices
  - Degrees can be considered independent
  - Power-law decaying probabilities are preserved by approximation
- Degrees in each graph are grouped into "buckets"
  - Joint distributions of degrees in each bucket are multinomial
  - $\Delta \geq L_1$-distance between two independent multinomials
  - Larger than $(np)^{1/2-\varepsilon}$ with probability $1 - o(n^{-a})$, for any $\varepsilon, a > 0$

Additional details:

- *Degree sequence edit distance:* $\Delta(G, G') = \sum_k |\phi_G(k) - \phi_{G'}(k)|$, where $\phi_G(k)$ counts vertices in G with degree k
  - $\Delta(G, G')$ is the minimum mismatch when aligning vertices of G and G' by degree
  - Isomorphic graphs must have the same degree sequence: $\Delta = 0$
  - Useful fact: if $G = (V, E)$ and $S \subseteq G$, then $\Delta(G, G[S]) \leq |V \setminus S| + |C(S)|$, where $C(S)$ counts edges between S and $V \setminus S$
- Degrees in a $G(n,p)$ random graph are almost like independent random variables
  - McKay and Wormald '97: approximation framework, with asymptotic error guarantees for event probabilities
  - Three intermediate models to perform transition between $\mathcal{B}_{n,p}$ (*binomial model*, independent sequence) and $\mathcal{D}_{n,p}$ (*degree sequence model*, extracted from $G(n,p)$)

  Theorem: If p satisfies $\omega(\log n/n) \leq p \leq o(n^{-1/2})$, then for any event sequence A_n and any fixed $a > 0$, $\mathbb{P}_{\mathcal{B}_{n,p}}(A_n) = o(n^{-a})$ implies $\mathbb{P}_{\mathcal{D}_{n,p}}(A_n) = o(n^{-a})$.
  - Our extension: simultaneous approximation for two independent random graphs

Preprints

[arXiv 1601.02478] Power-law decay of the degree-sequence probabilities of two random graphs with application to graph isomorphism

[arXiv 1605.01758] Local symmetry in random graphs