Approximating p-centres in large δ-hyperbolic graphs

A quasi-linear time algorithm for distance-based clustering

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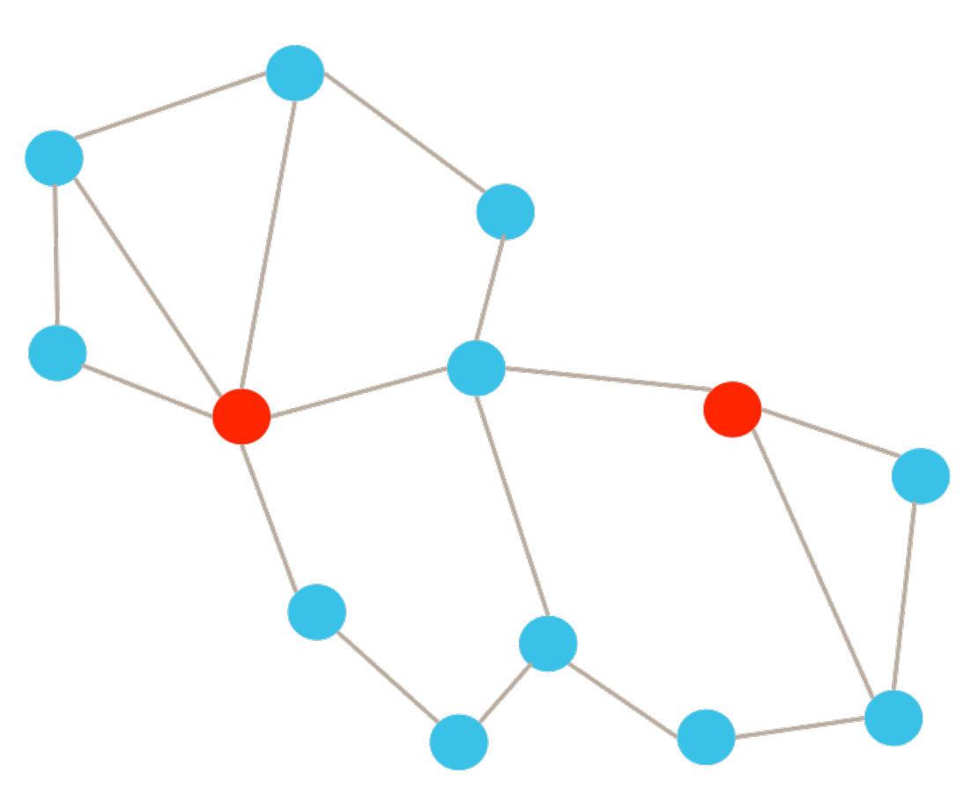
Clustering via p-centres

The p−center algorithm is a discrete variant of one of the most frequently used clustering algorithms, the k−means clustering. The goal of the p−center algorithm is to identify on a given graph a pre−specified number p of vertices or centers, such that the maximum distance of any graph vertex to its nearest center is minimized. For any given p, the algorithm naturally partitions a graph into p clusters induced by the position of its p centers. Unfortunately, as a clustering algorithm the complexity of the p−center algorithm is generally prohibitive, $O(n^p)$ for an n−vertex graph, making it inapplicable to even moderate size graphs.
In this work we show how to approximate p−centres in a class of graphs which are common in real data, the δ−hyperbolic graphs. We show that that by giving up to 3δ in the (additive) approximation, one can achieve a quasilinear time p−center approximation. As such, this scheme is the **first p−center approximation applicable to large graphs**, particularly when p is relatively small, for example in the range 10−10^4 and n is large, for example, 10^5−10^9 vertices.

The p-centre Problem



Given input graph G and integer p:

- Find p **centres** $c_1, \dots, c_p$ minimizing $r(c_1, \dots, c_p) = \max_{v \in V(G)} \min_i d(v,c_i)$
- Optimal value of $r(c_1, \dots, c_p)$ is called the ***p*−radius** r_p
- Centres may be vertices or lie on edges

Known Algorithms

In general graphs: **p−centres is NP−hard**. In fact it is NP−hard to approximate the p−radius to a factor smaller than 2. A (multiplicative) 2−approximation exists in time $O(m \log m)$.

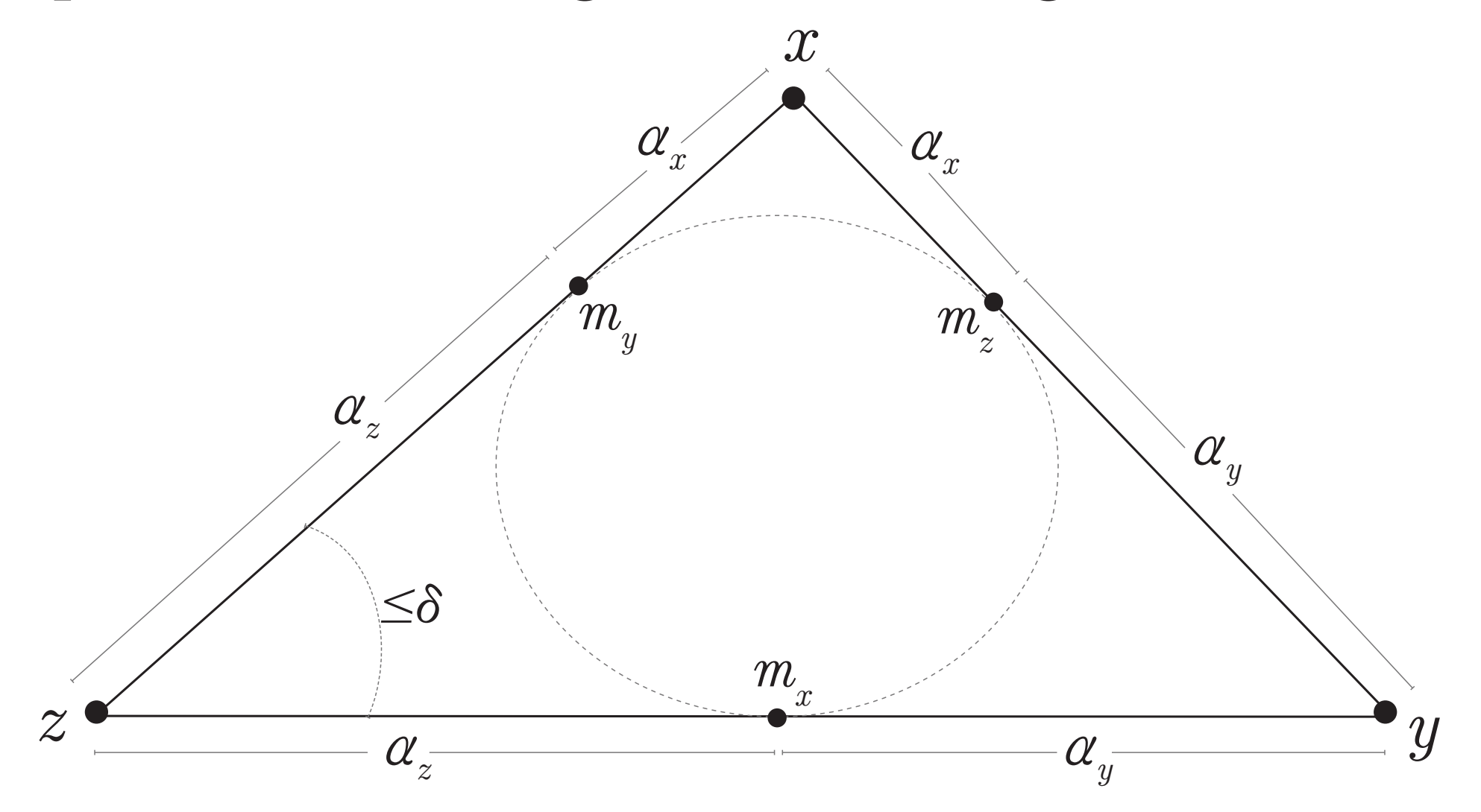
In trees: p−centers can be solved **exactly in linear time** $O(n)$.

In **δ−hyperbolic graphs:** previous result: p−centres can be solved in time $O(n^3)$ with an **additive error at most δ** on the p−radius [1].

δ-hyperbolicity

Hyperbolicity is an invariant of a graph which roughly measures how close its distance metric is to the distance metric of a tree.

Definition: Let x,y,z be any three vertices in G and let $[x,y]$, $[x,z]$, $[y,z]$ be three shortest paths. The union of the paths is called a **geodesic triangle**.



Let the perimeter $\pi = d(x,y) + d(y,z) + d(x,z)$ and define
$\alpha_x = ½\pi - d(y,z)$
$\alpha_y = ½\pi - d(x,z)$
$\alpha_z = ½\pi - d(x,y)$

The points m_x, m_y, m_z are located where the inscribed circle would meet the edges of a triangle with side lengths $d(y,z)$, $d(x,z)$ and $d(x,y)$.

The **insize** of a geodesic triangle is $\max_{v \in \{x,y,z\}} \max_{\theta \in [0,\alpha_v]} d(p,q)$ where p,q are points on the geodesic triangle that are both at distance α_v from v.

The **hyperbolicity $\boldsymbol{\delta}$** of G is the maximum insize of a geodesic triangle.

Fact: Trees are 0−hyperbolic.

Fact: Graphs arising from **real networks have small hyperbolicity**. Kennedy et al. [3] studied a large number of publicly available graphs arising from social media, collaboration and citation networks, IP−layer networks and web graphs. They found that as the size of these graphs grows very large, their hyperbolicities δ remain small (less than 10). This is in contrast with the random graph, which has logarithmic hyperbolicity.

So in this sense, real−world graphs are tree−like.

The objective

Real graphs are big: a quadratic−time algorithm is too slow in practice on a billion−vertex graph.
Real graphs are $\boldsymbol{\delta}$−hyperbolic: can get an additive constant error on the p−radius for graphs with fixed δ.
We want an approximation algorithm with a small additive approximation error that runs in nearly linear time.

Our results

In **$\boldsymbol{\delta}$−hyperbolic graphs:**

Theorem: If $p \in \{1,2\}$ and G is δ−hyperbolic: there is an approximation algorithm with
- additive error at most δ on the p−radius
- running time $O((2\ + 1)(m + n))$

Theorem: If $p \geq 3$ and G is δ−hyperbolic: there is an approximation algorithm with
- additive error at most 3δ on the p−radius
- running time $O(p(\delta + 1)(m + n) \log n)$

Proof Ideas

Hyperbolic graphs are similar to trees, a class of graphs where p−centres is easy. So we want to exploit this similarity.

The p−centre problem is a **covering** problem: we want to cover the vertices of G with p balls of a smallest possible radius. The natural dual is a **packing** problem, known as $(p+1)$−dispersion, which asks for $p+1$ disjoint balls each with a largest possible diameter.

The (p+1)-dispersion Problem

Given input graph G and integer $p+1$:

- Find p **vertices** $x_1, \dots, x_{p+1}$ maximizing $d(x_1, \dots, x_{p+1}) = \min_{i \neq j} d(x_i, x_j)$
- Optimal value of $r(c_1, \dots, c_p)$ is called the **$(p+1)$−diameter** d_{p+1}

In **general graphs:** $r_p \geq d_{p+1}$. This follows from the pigeonhole principle (but r_p and d_{p+1} may be arbitrarily far apart).

In **trees:** $r_p = d_{p+1}$

In **$\boldsymbol{\delta}$−hyperbolic graphs:** $r_p \leq d_{p+1} + \delta$

Previous work on δ−hyperbolic graphs solved the primal p−centre problem and its dual $(p+1)$−dispersion problem simultaneously, resulting in a solution to p−centres with an additive error of δ on the p−radius r_p. But this cubic time algorithm is prohibitively slow.

Locally Dispersed Sets

Our key observation is that starting from a solution to a **'local'** version of dispersion, we can easily obtain p centres with an additive error of at most 3δ on the p−radius. For small values of $p \in \{1,2\}$ our additive error is just δ.

In the local $(p+1)$−dispersion, rather than look for $p+1$ vertices which are pairwise as far apart as possible in G, we look for a set of $p+1$ vertices which **we can't improve by swapping a vertex in our set with a new vertex** (i.e. by making a local improvement).

We give an algorithm to find an **optimal locally dispersed set** by performing a **quasilinear number of vertex swaps**. We then show how to obtain the approximate p−centres from it in constant time.

References & Contact

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