Polyhedral studies on vertex coloring problems

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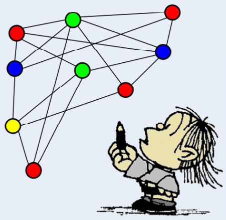
What is graph coloring?

It consists in assigning a color to each vertex of the graph such that adjacent vertices receive different colors.

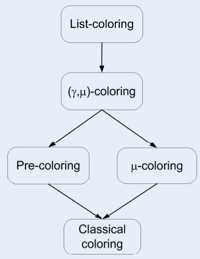
The classical problem asks for a coloring of the graph which uses the minimum number of colors, but **there are many variants of this problem.**



What is a variant of this problem?

There can be, for instance, forbidden colors for some vertices (*List-coloring*) or some vertices with pre-assigned colors (*Pre-coloring extension*), etc.

We can define, for some of these problems, a hierarchy in which a problem generalizes other problems, thus being "harder" to solve in practice.



Are these problems hard?

The classic coloring problem is a very hard problem to solve in general (it belongs to the NP-C class).

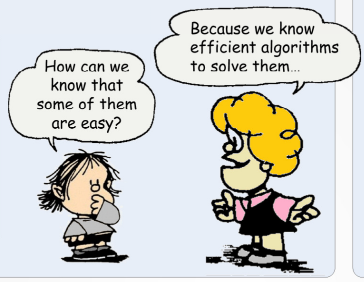
Therefore, its generalizations in the above hierarchy are also very difficult problems to solve (at least NP-C)

Are they difficult to solve for any graph?

Luckily, some of these problems are polynomial for some classes of graphs [1].

Class	Coloring	Pre-col	μ-col	(γ,μ)-col	List-col
Complete bipartite	P	P	P	P	NP-C
Bipartite	P	NP-C	NP-C	NP-C	NP-C
Cographs	P	P	P	**?**	NP-C
Distance-Hereditary	P	NP-C	NP-C	NP-C	NP-C
Interval	P	NP-C	NP-C	NP-C	NP-C
Unit interval	P	NP-C	NP-C	NP-C	NP-C
Complete split	P	P	P	P	NP-C
Split	P	P	NP-C	NP-C	NP-C
Line of $K_{n,n}$	P	NP-C	NP-C	NP-C	NP-C
Line of K_n	P	NP-C	NP-C	NP-C	NP-C
Complements of bipartites	P	P	**?**	**?**	NP-C
Block y cacti	P	P	P	P	P

"P": Polinomial problem "NP-C": NP-Complete problem "?": Open problem



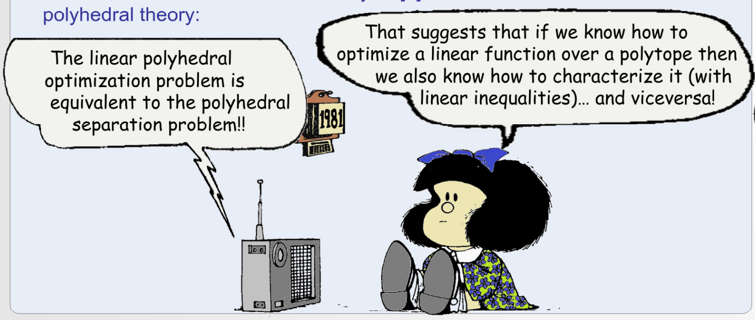
Linear programming [4]

One way to solve problems in polynomial time is by finding an appropriate **linear programming model** for it.

An LP model describes an **n-dimensional polytope** and by optimizing a certain (linear) function over this polytope, we can find the solution to the problem it models.

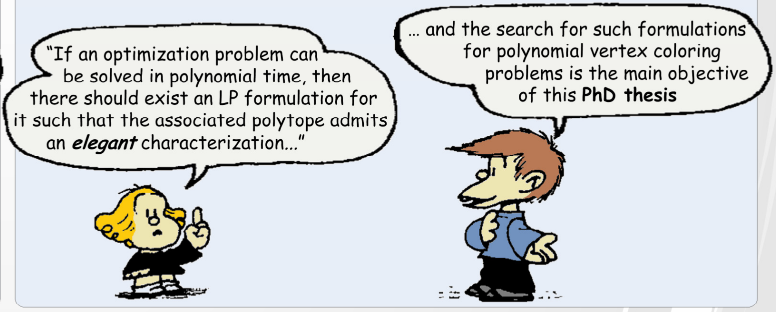
Geometric algorithms and combinatorial optimization

In 1981, Gröschel, Lovász and Schrijver [6] found a fundamental result for polyhedral theory:

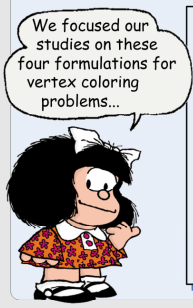


Conjecture and thesis proposal

From these results, a **generalized conjecture** arises...



A brief summary of the results from this thesis



Independent sets formulation [7]

Since this formulation yields models with exponentially-many variables, we just studied the polytopes associated with some simple classes.

We found complete descriptions for:

- Complete bipartites graphs
- Split graphs
- Co-interval graphs

Standard model [5]

- Found that the associated polytope **is a face of** the STAB polytope of another graph.
- Found complete descriptions for trees and block graphs.
- Gave a new family of valid inequalities generalizing others.
- Conjectured that the above family suffices to describe the polytopes when the graph is a cycle or a cactus.

Representatives formulation [3]

- Found that the associated polytope **is** the STAB polytope of another graph.
- Based on this result we give complete descriptions for the complements of the following families:
  - Graphs without
  - Graphs without
  - Graphs without

Orientation model [2]

- Presented new facet-generating procedures (i.e., the ***path lifting procedures***) wich take two general valid inequalities and a path, and generate infinitely-many valid inequalities (some of them inducing facets).
- Presented new families of valid inequalities by using the procedures, which we conjecture suffice to describe the polytopes when the graph is a path.

What's next?

We left many open questions about several topics:

- Conjectures about complete descriptions (cacti graphs and paths).
- "Translation" of valid inequalities from STAB to vertex coloring.
- Model adaptations for different variants of coloring.
- Perfectness results.

... and many more!

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[3] M. Campêlo, V. Campos y R. Corrêa, *On the asymmetric representatives formulation for the vertex coloring problem*, Disc. Applied Math. 156-7 (2008), 1097—1111.
[4] V. Chvátal, "*Linear Programming*", W. H. Freeman, 1983.
[5] P. Coll, J. Marenco, I. Méndez-Díaz y P. Zabala, *Facets of the Graph Coloring Polytope*, Annals of Operations Research 116-12 (2002) 79--90.
[6] M. Gröschel, L. Lovász y A. Schrijver, "*Geometric Algorithms and Combinatorial Optimization*", Springer-Verlag, 1988.
[7] A. Mehrotra y M. Trick, *A column generation approach for graph coloring.* INFORMS Journal On Computing 8, 4 (1996), 344–354.