Applying Spectral Graph Theory to Classical Extremal Problems

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Introduction

Certain extremal problems are classified as Turán type problems. Here we shall consider the Zarankiewicz problem, which aims to maximize the number of edges $e(G)$ under the condition that the graph G does not contain any copy of a complete bipartite graph $K_{s,t}$. We say that G is $K_{s,t}$-free.

In addition to the classical version of the Zarankiewicz problem, we shall discuss it under a spectral viewpoint, since it is possible to derive bounds for the classical version from inequalities of Spectral Graph Theory.

In this work, we consider two matrix representations of graphs:

Adjacency Matrix: Given a graph $G = (V, E)$ with vertex set $\{v_1, \dots, v_n\}$ the adjacency matrix $A(G) = (a_{ij})$ is given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise} \end{cases}$$

Signless Laplacian Matrix: $Q(G) = A(G) + D(G)$, where $D(G)$ is the diagonal matrix whose i-th entry is given by the degree of vertex v_i.

The largest eigenvalues associated with the matrices $A(G)$ and $Q(G)$ are called the **spectral radius** $\lambda(G)$ and the **q-index** $q(G)$, respectively.

Bounds for the Zarankiewicz problem

Reference	Bound
Kövári, Sós, and Turán [5]	$\frac{2e(G)}{n} \leq (s-1)^{1/t} n^{1-1/t} + t - 1$
Füredi [7]	$\frac{2e(G)}{n} \leq (s-t+1)^{1/t} n^{1-1/t} + t n^{1-2/t} + t$

Table. 1: Bounds on the average degree of a $K_{s,t}$-free graph G with n vertices, where $2 \leq t \leq s$.

Bounds for the spectral radius

The spectral radius λ satisfies the following inequality

$$\frac{2e(G)}{n} \leq \lambda.$$

Therefore, bounds on the spectral radius lead to bounds on the edges number.

Hence, we may state a spectral question associated with the Zarankiewicz problem: *What is the maximum value of $\lambda(G)$ if G is a $K_{s,t}$-free graph of order n?*

Reference		Bound
Babai and Guiduli [1]		$\lambda \leq \left((s-1)^{1/t} + o(1)\right) n^{1-1/t}$
Nikiforov [6]	$t = 2$	$\lambda \leq 1/2 + \sqrt{(s-1)(n-1) + 1/4}$
	$t \geq 3$	$\lambda \leq (s-t+1)^{1/t} n^{1-1/t} + (t-1) n^{1-2/t} + t - 2$

Table. 2: Bounds on the spectral radius of a graph G as in the above problem. Note that in the first inequality the asymptotics is for t and s fixed and $n \to \infty$, while the latter is an improvement on the results in [7].

Tightness of upper bounds on the spectral radius

$t = 2$	$s = 2$	$\lambda \leq 1/2 + \sqrt{n - 3/4}$
		Equality for friendship graphs.
	$s > 2$	$\lambda \geq \sqrt{sn} + O\left(n^{1/3}\right)$ given by [8]
		Equality for strongly regular graph in which every two vertices have exactly $s-1$ common neighbors.
$t = 3$	$s = 3$	$\lambda \leq n^{2/3} + 2n^{1/3} + 1$
		$\lambda \geq n^{2/3} + 2/3\, n^{1/3} + C$ with $C > 0$ by Alon, Rònyai and Szabò, in [2]
$t \geq 2$ and $s \geq (t-1)! + 1$		$\lambda \geq n^{1-1/t} + O\left(n^{1-1/t-C}\right)$ with $C > 0$ as shown in [2]

Table. 3: Upper and lower bounds on λ that allow us to assess the quality of the bounds in [6] for some values of s and t.

Bounds for the q-index

Since the spectral radius and the q-index are related as follows

$$2\lambda \leq q$$

bounds on the q-index imply bounds on the spectral radius, and thus on the number of edges. This leads to the following spectral question: *What is the maximum q of a graph G is a $K_{s+1,t}$-free graph of order n?*

Reference	Bound
de Freitas, Nikiforov and Patuzzi, [4]	$q \leq \frac{n+2s}{2} + \frac{1}{2}\sqrt{(n-2s)^2 + 8s}$
	$q < n$
	$q \leq \frac{n}{2} + s + t - 2 + \frac{1}{2}\sqrt{(n-2+2s)^2 - 8s(n-2) + 4(t-1)(n-t+1)}$

Table. 4: Bounds on the q-index of a graph G as in the above problem. Note that the first inequality is the resultant of this problem on the following conditions: $n \geq s^2 + 6s + 6$ and $t = 2$. The second inequality holds under the restriction $\triangle(G) < n - 1$. The last inequality is a conjecture proposed in [4].

Tightness of upper bounds on the q-index

$t = 2$	$s = 1$	$q \leq n/2 + 1 + 1/2\sqrt{(n-2)^2 + 8}$ in [3]
	$s > 1$	The equality is valid if and only if $G = K_1 \vee H$ where H is an s-regular graph of order $n-1$, as shown in [4]

Table. 5: Values for s and t presented by de Freitas, Nikiforov and Patuzzi, [4], such that the bounds are tight.

Future Work

- Verify whether the above conditions may be relaxed in some instances.
- Look for better upper and lower bounds on the q-index.
- Consider other spectral parameters related with the Zarankiewicz problem.
- Consider other classical problems and their relation with spectral parameters.

References

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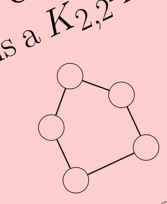
Consider the C_5 graph, it has order 5 and is a $K_{2,2}$-free graph.



Its spectral radius is $\lambda = 2$ so, by the first inequality presented in *Tightness of upper bounds on the spectral radius*, is tight, as we can see below.

$$\lambda \leq 1/2 + \sqrt{5 - 3/4} \cong 2.5615528.$$

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