Monochromatic tree covers and Ramsey *numbers for set-coloured graphs*

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Abstract

We extend results on monochromatic tree covers and from classical Ramsey theory to a generalised setting, where each of the edges of an underlying host graph (here, either a complete graph or a complete bipartite graph), is coloured with a *set* of colours.

Our results for tree covers in this setting have an application to Ryser's Conjecture. Every r-partite r-uniform hypergraph whose edges pairwise intersect in at least $k \geq (r-2)/2$ vertices, has a transversal of size at most $r-k$. In particular, for these hypergraphs, Ryser's conjecture holds.

Introduction

We consider complete (and complete bipartite) graphs G whose edges are each coloured with a set of k colours, chosen among r colours in total. That is, we consider functions $\varphi : E(G) \to \binom{[r]}{k}$. We call any such φ an (r,k)-colouring (so, the usually considered r-colourings for Ramsey problems are $(r,1)$-colourings).

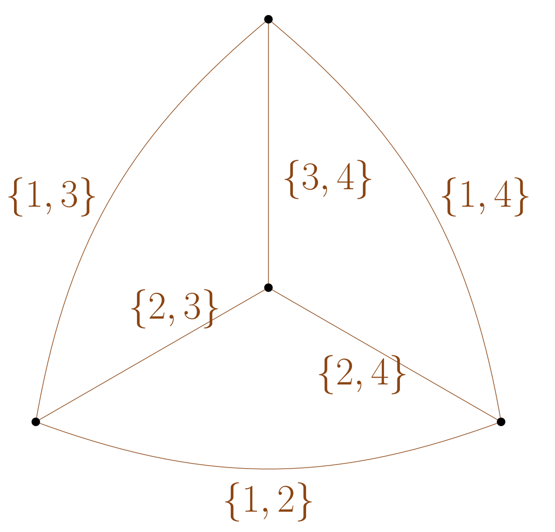

Figure 1: $(4,2)$-colouring of K_4

The first problem we consider is the tree covering problem. In the traditional setting, one is interested in the minimum number $tc_r(K_n)$ such that each r-colouring of $E(K_n)$ admits a cover with $tc_r(K_n)$ monochromatic trees (not necessarily of the same colour). The following conjecture has been put forward by Gyárfás:

Conjecture 1 (Gyárfás [5]). *For all $n \geq 1$, we have $tc_r(K_n) \leq r-1$.*

Note that this conjecture becomes trivial if we replace $r-1$ with r, as for any colouring, all monochromatic stars centered at any fixed vertex cover K_n. Also, the conjecture is tight when $r-1$ is a prime power, and holds for $r \leq 5$. This is due to results from [1, 4, 5].

In our setting, for a given graph G we define $tc_{r,k}(G)$ as the minimum number m such that each (r,k)-colouring of $E(G)$ admits a cover with m monochromatic trees. In this context, a monochromatic tree in G is a tree $T \subseteq G$ such that there is a colour i which, for each $e \in E(T)$, belongs to the set of colours assigned to e.

Tree coverings have also been studied for complete bipartite graphs $K_{n,m}$. Chen, Fujita, Gyárfás, Lehel and Tóth [2] proposed the following conjecture.

Conjecture 2 (Chen et al. [2]). *If $r > 1$ then $tc_{r,1}(K_{n,m}) \leq 2r-2$, for all $n, m \geq 1$.*

It is shown in [2] that this conjecture is tight; that it is true for $r \leq 5$; and that $tc_{r,1}(K_{n,m}) \leq 2r-1$ for all $r, n, m \geq 1$.

Also classical Ramsey problems extend to (r,k)-colourings. Define the *set-Ramsey number* $r_{r,k}(H)$ of a graph H as the smallest n such that every (r,k)-colouring of K_n contains a monochromatic copy of H. (As above, a monochromatic subgraph H of G is a subgraph $H \subseteq G$ such that there is a colour i that appears on each $e \in E(H)$.) So the usual r-colour Ramsey number of H equals $r_{r,1}(H)$. Note that $r_{r,k}(H)$ is increasing in r, if H and k are fixed, and decreasing in k, if H and r are fixed.

Results

Let φ be an (r,k)-colouring of a graph G. Note that deleting $k-1$ fixed colours from all edges, and, if necessary, deleting some more colours from some of the edges, we can produce an $(r-k+1)$-colouring from φ. So, Conjecture 1, if true, implies that $tc_{r,k}(K_n) \leq r-k$. We confirm this bound, in the case that r is not much larger than $2k$.

Theorem 1. *If $r \leq 2k+2$ then $tc_{r,k}(K_n) \leq r-k$ for all $n \geq 1$.*

Clearly, the bound from Theorem 1 is tight for $k = r-1$, and it is also tight for $k = r-2$, as Figure 1 shows, but in general, the bound is not tight. The smallest example (in terms of r and k) corresponds to $r = 5$ and $k = 2$.

Theorem 2. *For all $n \geq 4$, we have $tc_{5,2}(K_n) = 2$.*

There is an interesting connection between Theorem 1 and Ryser's Conjecture. The latter conjecture states that $\tau(\mathcal{H}) \leq (r-1)\nu(\mathcal{H})$ for each r-partite r-uniform hypergraph with $r > 1$, where $\tau(H)$ is the size of a smallest transversal of $\mathcal{H}$, and $\nu(\mathcal{H})$ is the size of a largest matching in $\mathcal{H}$. Is not hard to see (Figure 2) that there is a correspondence between r-partite r-uniform hypergraphs and r-colourings of graphs in such a way that, for intersecting hypergraphs, transversals in the hypergraph become monochromatic tree covers in a suitable r-colouring.

Ryser's conjecture for k-*intersecting* hypergraphs (where every two hyperedges intersect in at least k vertices) is equivalent to the statement $tc_{r,k}(K_n) \leq r-1$. But as Theorem 1 gives the stronger bound $tc_{r,k}(K_n) \leq r-k$, we obtain a stronger version of Ryser's conjecture for these hypergraphs:

Corollary 1. *We have $\tau(\mathcal{H}) \leq r-k$ for all r-partite r-uniform k-intersecting hypergraphs $\mathcal{H}$ with $r \leq 2k+2$.*

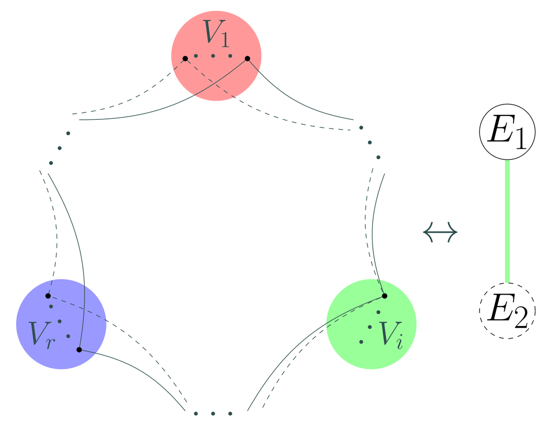

Figure 2: Correspondence between r-partite r-uniform hypergraphs and r-colourings

In the case of G being a complete bipartite graph, we can use the argument from above, deleting $k-1$ fixed colours, to see that $tc_{r,k}(K_{n,m}) \leq 2r-2k+1$. However, it is possible to improve the upper bound as the following theorem shows.

Theorem 3. *For all r, k, n, m,*

$$tc_{r,k}(K_{n,m}) \leq \begin{cases} r-k+1, & \text{if } r \leq 2k \\ 2r-3k+1, & \text{if } 2k < r \leq 5k/2 \\ 2r-3k+2, & \text{otherwise.} \end{cases}$$

For the case $r \leq 2k$, our bound is sharp for large graphs.

Theorem 4. *For each r, k with $r > k$ there is m_0 such that if $n \geq m \geq m_0$ then $tc_{r,k}(K_{n,m}) \geq \max\{r-k+1, r-k+\lfloor \frac{r}{k} \rfloor - 1\}$.*

Considering set-Ramsey numbers, we can bound $r_{r,k}(H)$ with the help of the usual r-colour Ramsey number $r_r(H)$. In fact, in the same way as we obtained our trivial bounds on $tc_{r,k}$, one can prove (see also [6]) that for every graph H and integers $r > k > 0$,

$$r_{r-k+1}(H) \geq r_{r,k}(H) \geq r_{\lfloor \frac{r}{k} \rfloor}(H). \tag{1}$$

Both bounds are not best possible as already the example of $r = 3$, $k = 2$ and $H = K_3$, or $H = K_4$, shows. Namely, it is not difficult to show that $r_{3,2}(K_3) = 5$, and the value $r_{3,2}(K_4) = 10$ follows from results of [3].

If $\frac{k}{r}$ surpasses $\frac{t-2}{t-1}$, we can estimate $r_{r,k}(K_t)$ using Turán's Theorem.

Theorem 5. *Let $\varepsilon \in (0,1)$, let $t \geq 2$ and let $r > k > 0$. If $\frac{t-2}{t-1} = (1-\varepsilon)\frac{k}{r}$, then $r_{r,k}(K_t) \leq \frac{1}{\varepsilon} + 1$. This bound is sharp if $k = r-1 = t-1$ is a prime power, in which case $r_{r,k}(K_t) = k^2 + 1$.*

We also establish a lower bound for cycles under this setting.

Theorem 6. *If ℓ is odd and $k \geq 2$, then $r_{r,k}(C_\ell) > \max\{2^{\frac{r-1}{k-1}}, 2^{\lfloor \frac{r}{k} \rfloor}(\ell-1)\}$.*

It is possible to show, by using Theorem 6 and some basic combinatorial arguments, that $r_{4,2}(K_3) = 9$, being this number another example in which Equation (1) is not sharp.

Open Questions

Related to Theorem 1, the best lower bound we know is $tc_{r,k}(K_n) \geq \lfloor \frac{r}{k} \rfloor - 1$, for $n \geq (r-1)^2$. We do not know where in the interval $[\lfloor \frac{r}{k} \rfloor - 1, r-k]$ the true value of $tc_{r,k}(K_n)$ lies.

Problem 1. *Determine $tc_{r,k}(K_n)$ for all r, k, n.*

For the case of complete bipartite graphs and for $r > 2k$, we do not know the true value of $tc_{r,k}(K_{n,m})$.

Problem 2. *Determine $tc_{r,k}(K_{n,m})$ for all $r, k, n, m \geq 1$.*

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