The Asymptotic Behavior of the Correspondence Chromatic Number

Anton Bernshteyn

University of Illinois at Urbana–Champaign

bernsht2@illinois.edu

1. From list coloring to DP-coloring

LIST COLORING is an important generalization of ordinary graph coloring, introduced independently by Vizing [6] and Erdős, Rubin, and Taylor [4]. It is defined as follows. Let G be a graph and suppose that for each vertex $v \in V(G)$, a set of available colors $L(v)$, called the *list* of v, is specified. A proper coloring c of G is an *L-coloring* if $c(v) \in L(v)$ for all $v \in V(G)$. G is said to be *L-colorable* if it admits an L-coloring; G is *k-list-colorable* (or *k-choosable*) if it is L-colorable whenever $|L(v)| \geq k$ for all $v \in V(G)$. The least number k such that G is k-choosable is called the *list chromatic number* (or the *choosability*) of G and is denoted by $\chi_\ell(G)$ (or $\mathrm{ch}(G)$).

Consider a graph G and a list assignment L for G. For each $v \in V(G)$, let

$$\widetilde{L}(v) := \{(v, c) \,:\, c \in L(v)\}.$$

(Note that the sets $\widetilde{L}(v)$ are pairwise disjoint.) Let H be the graph with vertex set

$$V(H) := \bigcup_{v \in V(G)} \widetilde{L}(v)$$

and edge set given by

$$(v_1, c_1)(v_2, c_2) \in E(H) :\Longleftrightarrow (v_1 = v_2 \text{ and } c_1 \neq c_2) \text{ or } (v_1v_2 \in E(G) \text{ and } c_1 = c_2).$$

In other words, the graph H is obtained from G by replacing each vertex $v \in V(G)$ by a clique of size $|L(v)|$ and each edge $v_1v_2 \in E(G)$ by a matching that pairs the vertices corresponding to coinciding colors in $L(v_1)$ and $L(v_2)$.

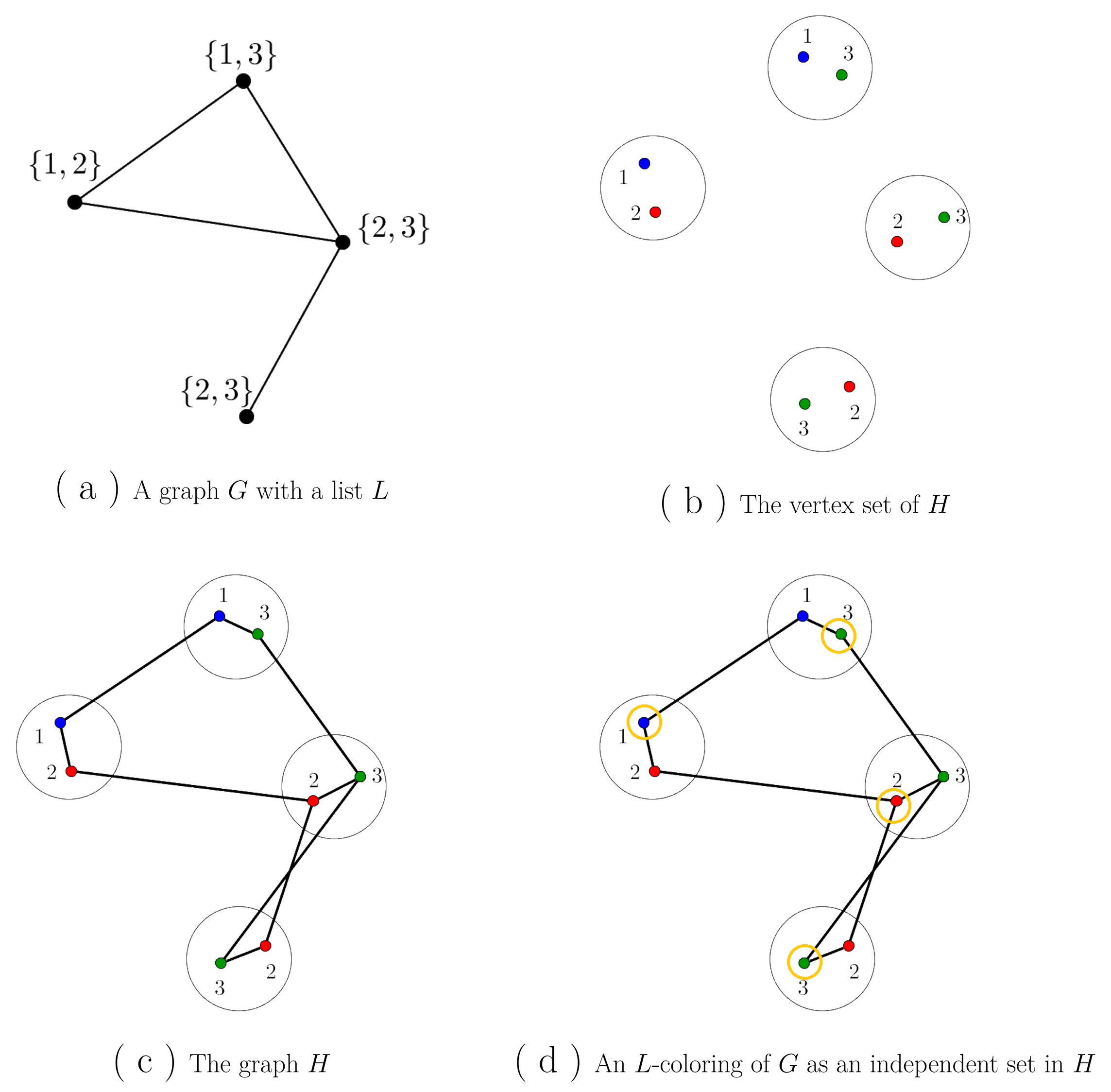
Given an L-coloring c of G, we define the set $I_c \subseteq V(H)$ as follows:

$$I_c := \{(v, c(v)) \,:\, v \in V(G)\}.$$

Observe that I_c is an independent set in H and for each vertex $v \in V(G)$, $|I_c \cap \widetilde{L}(v)| = 1$. Conversely, if $I \subseteq V(H)$ is an independent set such that $|I \cap \widetilde{L}(v)| = 1$ for all $v \in V(G)$, then, setting $c_I(v)$ to be the single color such that $(v, c_I(v)) \in I$, we obtain a proper L-coloring c_I of G.

So, we have that

$$G \text{ is } L\text{-colorable} \iff H \text{ contains an independent set of size } |V(G)|.$$



(a) A graph G with a list L (b) The vertex set of H (c) The graph H (d) An L-coloring of G as an independent set in H

Figure 1: *From list coloring to DP-coloring*

2. DP-colorings: the definition

The following definitions are essentially due to Dvořák and Postle [3] (they used the term "correspondence coloring" instead of "DP-coloring").

Definition 2.1: Covers

Let G be a graph. A *cover* of G is a pair (L, H), where L is an assignment of pairwise disjoint sets to the vertices of G and H is a graph with vertex set $\bigcup_{v \in V(G)} L(v)$, satisfying the following conditions.

1. For each $v \in V(G)$, $H[L(v)]$ is a complete graph.
2. For each $uv \in E(G)$, the edges of H between $L(u)$ and $L(v)$ form a matching (possibly empty).
3. For each distinct $u, v \in V(G)$ with $uv \notin E(G)$, no edges of H connect $L(u)$ and $L(v)$.

Definition 2.2: DP-colorings

Suppose G is a graph and (L, H) is a cover of G. An *(L, H)-coloring* of G is an independent set $I \subseteq V(H)$ of size $|V(G)|$.

Definition 2.3: DP-chromatic numbers

The *DP-chromatic number* of G (notation: $\chi_{DP}(G)$) is the minimum k such that G is (L, H)-colorable for each choice of (L, H) with $|L(v)| \geq k$ for all $v \in V(G)$.

3. First properties

Proposition 3.1: Dvořák–Postle [3]

- $\chi_{DP}(G) \leq \Delta(G) + 1$;
- $\chi_{DP}(G) \leq 5$ for planar G;
- $\chi_{DP}(G) \leq 3$ for planar G with girth ≥ 5.

Proposition 3.2

$\chi_{DP}(C_n) = 3$ for all $n \geq 3$.

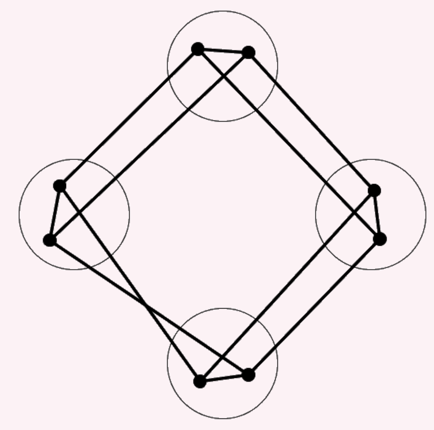


Figure 2: *A cover of C_4 showing that $\chi_{DP}(C_4) > 2$*

4. Alon's theorem for DP-colorings

A fundamental result of Alon asserts that the list chromatic number of a graph is bounded below by an increasing function of its average degree. More precisely:

Theorem 4.1: Alon [1]

Let G be a graph with average degree d. Then

$$\chi_\ell(G) \geq (1/2 - o(1)) \log_2 d.$$

I turns out that the dependence of the DP-chromatic number of a graph on its average degree is much stronger:

Theorem 4.2: Version of Alon's theorem for DP-colorings; A.B. [2]

Let G be a graph with average degree $d \geq 2e$. Then

$$\chi_{DP}(G) \geq \frac{d/2}{\ln(d/2)}.$$

Proof. Suppose that

$$k \leq \frac{d/2}{\ln(d/2)}.$$

Let $\{L(v)\}_{v \in V(G)}$ be a collection of pairwise disjoint sets, each of size k. Randomly construct a graph H with vertex set $\bigcup_{v \in V(G)} L(v)$ as follows: For each $u \in V(G)$, make $H[L(u)]$ a clique, and for each $uv \in E(G)$, connect $L(u)$ and $L(v)$ by a perfect matching chosen independently and uniformly at random. By construction, (L, H) is a cover of G.

Consider any set $I \subseteq \bigcup_{v \in V(G)} L(v)$ such that $|I \cap L(v)| = 1$ for all $v \in V(G)$. If $uv \in E(G)$, then the probability that the only vertex in $I \cap L(v)$ and the only vertex in $I \cap L(u)$ are nonadjacent in H is exactly $1 - 1/k$. Therefore, the probability that I is a proper (L, H)-coloring of G is exactly $(1 - 1/k)^{|E(G)|} \leq e^{-|E(G)|/k}$. Thus, the probability that there exists at least one (L, H)-coloring is at most

$$k^{|V(G)|} \cdot e^{-|E(G)|/k} = e^{|V(G)| \ln k - |E(G)|/k}.$$

We claim that it is less than 1. Indeed, it is enough to show that

$$k \ln k < \frac{|E(G)|}{|V(G)|} = d/2.$$

But

$$k \ln k < \frac{d/2}{\ln(d/2)} \cdot \ln(d/2) = d/2,$$

as desired. □

5. Johansson's theorem for DP-colorings

Theorem 5.1: Johansson [1]

Let G be a triangle-free graph with maximum degree Δ. Then

$$\chi_\ell(G) = O\left(\frac{\Delta}{\ln \Delta}\right).$$

It turns out that Johansson's bound holds for DP-chromatic number as well:

Theorem 5.2: Johansson's theorem for DP-colorings; A.B. [2]

Let G be a triangle-free graph with maximum degree Δ. Then

$$\chi_{DP}(G) = O\left(\frac{\Delta}{\ln \Delta}\right).$$

As a surprising corollary of Theorems 4.2 and 5.2, we see that the DP-chromatic number of a regular triangle-free graph is determined, up to a constant factor, by its degree.

Corollary 5.3: DP-chromatic number of regular triangle-free graphs

Let G be a d-regular triangle-free graph. Then

$$\chi_{DP}(G) = \Theta\left(\frac{d}{\ln d}\right).$$

References

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