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Towards a simple construction method for Hamiltonian decomposition of the hypercube

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Abstract. We consider the problem of Hamiltonian decomposition on the hypercube. It is known that there exist $\lfloor n/2 \rfloor$ edge-disjoint Hamiltonian cycles on a binary n-cube. However, there are still no simple algorithms to construct such cycles. We present some promising results that may lead to a very simple method to obtain the Hamiltonian decomposition. The binary n-cube is equivalent to the Cartesian product of cycles of length four ($C_4 \times C_4 \ldots \times C_4$). Case $n = 4$ is trivial. For the case $n = 6$, we first partition the set of edges of the $C_4 \times C_4 \times C_4$ into 12 disjoint cycles of length 16. We then present an operator to merge the cycles to produce the desired Hamiltonian cycles. In general the edge set of $n/2$ products $C_4 \times C_4 \ldots \times C_4$, can be partitioned into $n2^n/32$ disjoint cycles of length 16. It remains to formalize the merge operator in the general case.

1. Introduction

The problem of finding edge-disjoint cycles on a hypercube can be important in fault-tolerant distributed computing [**1**]. When the cube size is large, the probability of faulty communication links also increases. Thus for applications that require the usage of the nodes of the hypercube connected in a cycle, it is important to have knowledge of the existence of alternate cycles. Obviously, the mere knowledge of their existence is not sufficient. We should be able to construct the edge-disjoint cycles by using simple algorithms.

Several results concerning the existence of disjoint Hamiltonian cycles on graphs, in particular, hypercubes, are known. Alspach, Bermond and Sotteau [**2**] present a nice compilation of some of these results. The interested readers should

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also refer to [**3, 4, 5, 7**] for details. Several other properties of the hypercube are described in [**6, 8**].

It is known [**2**] that on a hypercube of dimension n, there exist $\lfloor n/2 \rfloor$ disjoint Hamiltonian cycles. Unfortunately the proof of this result, based mainly on [**4**], does not lead to any simple algorithm to construct the disjoint Hamiltonian cycles. The problem of finding such algorithms remains an open problem to this date.

In this paper we present an interesting approach and some preliminary results that may lead to a very simple method to obtain the Hamiltonian decomposition of the binary n-cube. Observe first that the binary n-cube is equivalent to a 4-ary $n/2$-cube, or the Cartesian product of cycles of length four: $C_4 \times C_4 \ldots \times C_4$. First we consider the case $n = 4$. M. F. Foregger presents a construction of the two Hamiltonian cycles on $C_4 \times C_4$. We then consider the case of $n = 6$. We show how to partition the set of edges of the $C_4 \times C_4 \times C_4$ into 12 disjoint cycles of length 16. We then present an operator that merges the 16 cycles to produce the desired Hamiltonian cycles. An amazing result is that, in general, the edge set of $n/2$ products $C_4 \times C_4 \ldots \times C_4$, can be partitioned into $n2^n/32$ disjoint cycles of length 16. It remains to formalize the merge operator in the general case. However, though we considered only a small example, the same ideas can be applied to hypercubes in general. The details for this are still being worked out and will be published later.

§2 discusses Hamiltonian decomposition of the hypercube and presents a two-step approach to construct the cycles. §3 shows how to partition the edge set of the hypercube into cycles of length 16. §4 shows a merge operation that joins cycles of length 16 into cycles of length 64, thus obtaining the desired Hamiltonian cycles. Finally §5 discusses the generalization of the merge operation.

2. Hamiltonian decomposition

The Hamiltonian decomposition of a graph is the partitioning of the set of edges of a graph into Hamiltonian cycles. Obviously, for such a partition to exist, the graph should be regular of even degree. In [**2**] a more general definition of Hamiltonian decomposition is presented, that includes the case when the degree is odd. We are interested in the Hamiltonian decomposition of the binary n-cube. For the sake of simplicity, we will consider n even and throughout the paper consider its equivalent 4-ary $n/2$-cube, i.e. the Cartesian product of $n/2$ cycles of length four: $C_4 \times C_4 \ldots \times C_4$. We state without proof the following result (see [**2**] for details).

THEOREM 2.1. *The binary n-cube, with n even, or equivalently the product of $n/2$ cycles, $C_4 \times C_4 \ldots \times C_4$, can be partitioned into $n/2$ Hamiltonian cycles.*

We propose an approach that may lead to a very simple algorithm to construct the edge-disjoint Hamiltonian cycles. The following two major steps are used.

(i) Partition the edge set into cycles of length 16.
(ii) Merge the resulting cycles into larger cycles to obtain the desired Hamiltonian cycles.

3. Construction of cycles of length 16

In what follows, we consider the binary n-cube, with n even. We use m to denote $n/2$.

Consider first $n = 4$, we want to obtain two edge-disjoint cycles of length 16 on the $C_4 \times C_4$. Foregger [**5**] gives a construction of the desired cycles, shown in Figures 1 and 2. (The $C_4 \times C_4$ graph is shown in dashed lines, without the toroidal edges, for simplicity. Cycles are shown in full thick lines.) To distinguish the two cycles, consider one cycle colored with color 0 and the other color 1.

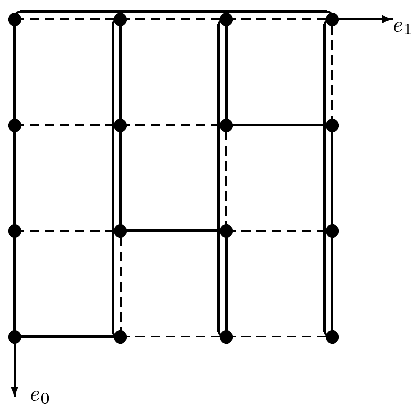

FIGURE 1. Cycle (color 0) of length 16 on $C_4 \times C_4$

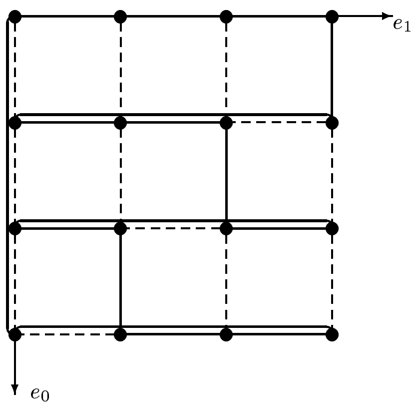

FIGURE 2. Cycles (color 1) of length 16 on $C_4 \times C_4$

In the figures, we denote the m directions by $e_0, e_1, \dots, e_{m-1}$ (in Figures 1 and 2 we have $m = 2$). Notice the symmetry of the two cycles. In Figure 1, starting from node $(0,0)$, the cycle of color 0 uses three edges along direction e_0, followed by one edge along direction e_1, and so on.

DEFINITION 3.1. *We say the a cycle is e_i-**dominant** if it alternately consists of three edges along direction e_i followed by one edge along direction $e_{(i+1 \mod m)}$.*

A e_i-dominant cycle is therefore entirely contained on a face determined by e_i and $e_{(i+1 \mod m)}$.

The cycle of color 0 of Figure 1 is e_0-dominant. Likewise the cycle of color 1 of Figure 2 is e_1-dominant. (Notice here $m = 2$.)

We now show the partitioning of the set of edges of the $C_4 \times C_4 \times C_4$ into 12 disjoint cycles of length 16. Here we divide the 12 cycles into

- 4 cycles of color 0,
- 4 cycles of color 1, and
- 4 cycles of color 2.

Cycles of length 16 of the same color will be merged in the next section to form a Hamiltonian cycle.

Figure 3 shows the 4 cycles of color 0. The cycles of color 0 are e_0-dominant. The same cycles are again shown in Figures 4, 5, 6 and 7, corresponding to the faces $e_2 = 0, 1, 2, 3$, respectively. Notice that when we move from one face to the next, the cycle is shifted to the left.

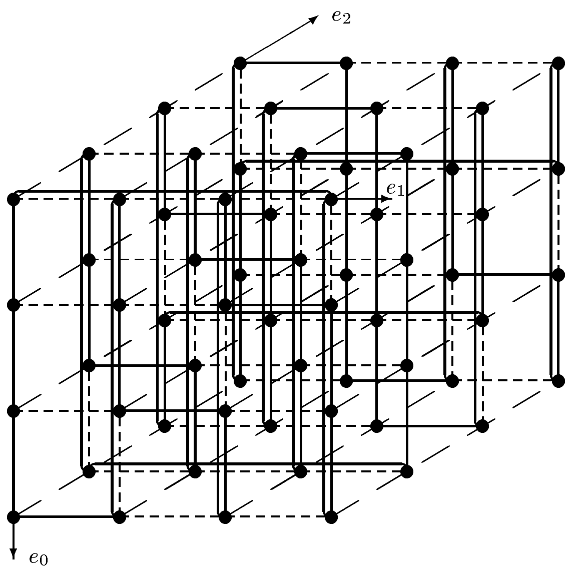

FIGURE 3. Cycles of color 0 (e_0-dominant)

Likewise, Figures 8 shows the 4 cycles of color 1. The cycles of color 1 are e_1-dominant. Notice again that when we move from one top face ($e_0 = 0, 1, 2, 3$) to the next, the cycle is shifted. Also notice that the cycles on such faces are the same, with the proper orientation, as those of Figures 4, 5, 6 and 7.

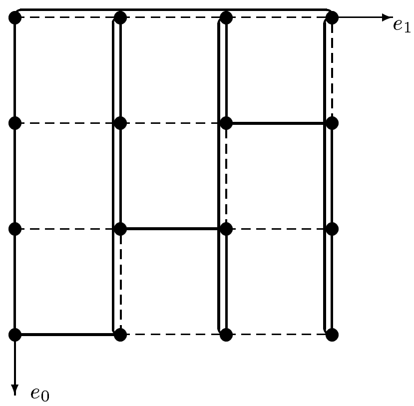

FIGURE 4. Cycle of color 0 on face $e_2 = 0$

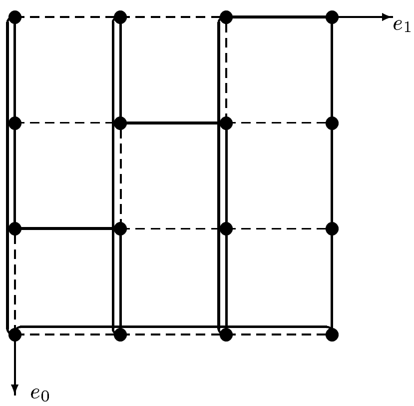

FIGURE 5. Cycle of color 0 on face $e_2 = 1$

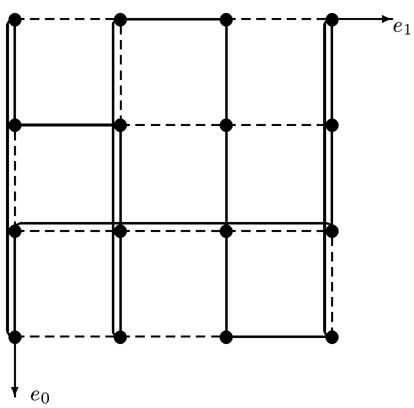

FIGURE 6. Cycle of color 0 on face $e_2 = 2$

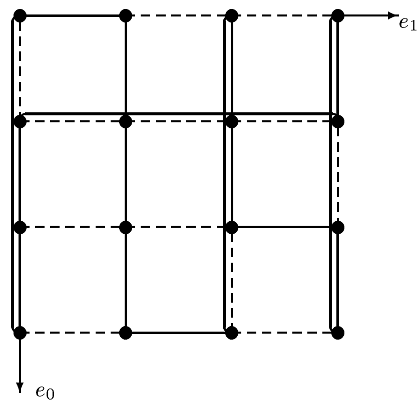

FIGURE 7. Cycle of color 0 on face $e_2 = 3$

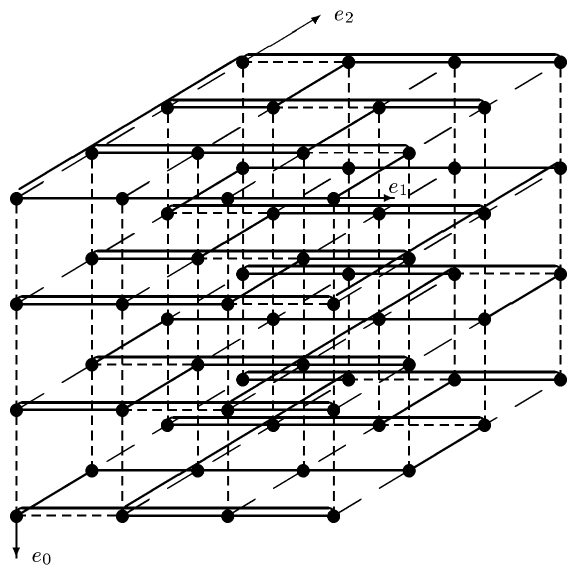

FIGURE 8. Cycles of color 1 (e_1-dominant)

Finally, Figures 9 show the 4 cycles of color 2. Again we notice that the cycles of color 2 are e_2-dominant and they are shifted when moving from one face to the next. Also, as before, these cycles are the same as those in Figures 4, 5, 6 and 7.

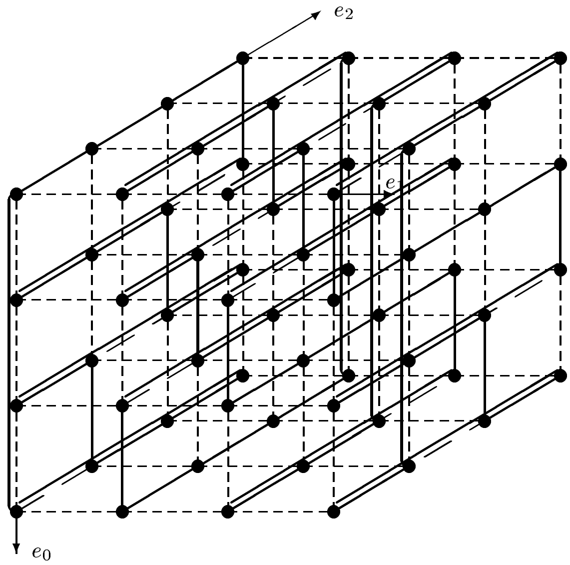

FIGURE 9. Cycles of color 2 (e_2-dominant)

In general, the edge set of $n/2$ products $C_4 \times C_4 \ldots \times C_4$, can be partitioned into $n2^n/32$ disjoint cycles of length 16. We use colors 0 to $m-1$ to color these cycles. There will be $2^n/16$ cycles of each color. Cycles of color i, $i = 0, \ldots, m-1$, are e_i-dominant.

4. Cycles merging

We show how the 12 cycles of length 16 of the previous section can be merged into the three desired Hamiltonian cycles. Cycles of the same color will be merged to form a Hamiltonian cycle.

Consider a node and edges incident with it. An edge-permutation operator is one that permutes the colors of these edges. (See Figure 10 for an example.)

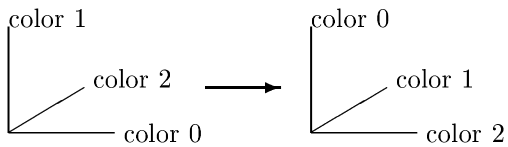

FIGURE 10. Example of an edge permutation operator

We use a set of edge-permutation operators to merge cycles of a given color to form larger cycles of the same color.

DEFINITION 4.1. *A set of edge-permutation operators constitutes a cycle merge operator (resp. cycle split operator) if it transforms a partition of the edge set of r cycles to a partition of s cycles, with $s < r$ (resp. $s > r$).*

Consider the partition of the edge set of $C_4 \times C_4 \times C_4$ into cycles of length 16 as seen in the previous section. Figure 11 shows a cycle merge operator that merges the cycles of a given color into a larger cycle of the same color. Figure 12 shows the locations (indicated by large circles) where reference point A of Figure 11 can be placed to apply the operator.

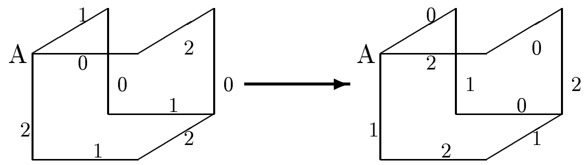

FIGURE 11. A merge operator for $C_4 \times C_4 \times C_4$

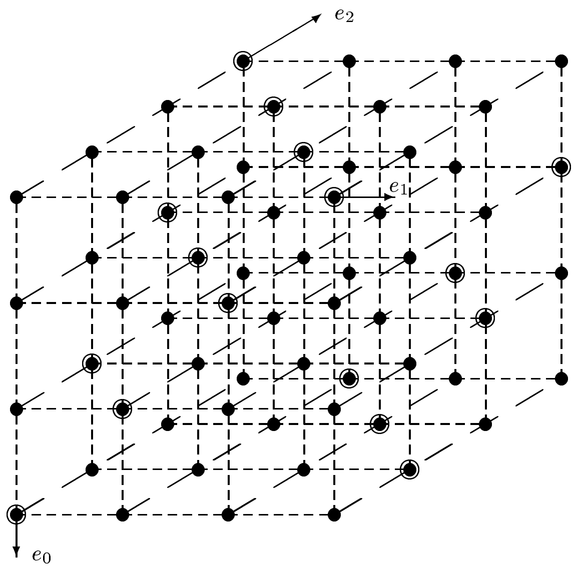

FIGURE 12. Possible locations (shown by large circles) to apply the merge operator

If the merge operator of Figure 11 is applied on one of the points of Figure 12, two cycles of each color (color 0, 1 and 2) will be merged to form a larger cycle of the same color. It will therefore transform six cycles to three.

The desired three cycles of the Hamiltonian decomposition of $C_4 \times C_4 \times C_4$ can be obtained by applying the merge operator to the points indicated in Figure 13.

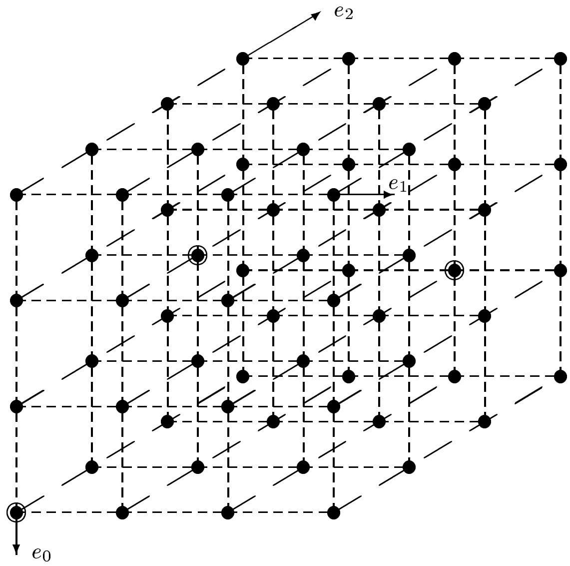

FIGURE 13. Locations to apply the merge operator for Hamiltonian decomposition

5. Generalization

To construct the Hamiltonian decomposition of $n/2$ products $C_4 \times C_4 \ldots \times C_4$, the approach is therefore as follows.

(i) Partition the edge set into cycles of length 16.
(ii) Merge the resulting cycles into larger cycles to obtain the desired Hamiltonian cycles.

The merge operator is contained in a binary $n/2$-cube. (For the case $n = 8$ the merge operator is inside a binary 4-cube, as shown in Figure 14.) It remains to formalize the merge operator in the general case.

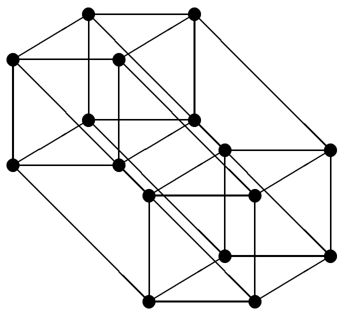

FIGURE 14. A merge operator for $C_4 \times C_4 \times C_4 \times C_4$

6. Conclusion

We have presented a simple approach to partition the edge set of a binary n-cube into Hamiltonian cycles. The starting point is the partitioning of the edge set into cycles of length 16. A merge operator constructs larger cycles until the desired cycles are obtained. Thus, this approach not only obtains the Hamiltonian decomposition, but also the partitioning into cycles of lengths 16, 32, $\ldots$, 2^n.

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