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On properties of the vertical rotation interval for twist mappings

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Abstract. In this paper we consider twist mappings of the torus, $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, and their vertical rotation intervals $\rho_V(T) = [\rho_V^-, \rho_V^+]$, which are closed intervals such that for any $\omega \in]\rho_V^-, \rho_V^+[$ there exists a compact $\overline{T}$-invariant set $\overline{Q}_\omega$ with $\rho_V(\overline{x}) = \omega$ for any $\overline{x} \in \overline{Q}_\omega$, where $\rho_V(\overline{x})$ is the vertical rotation number of $\overline{x}$. In the case when ω is a rational number, $\overline{Q}_\omega$ is a periodic orbit. Here we analyze how ρ_V^- and ρ_V^+ behave as we perturb $\overline{T}$ and which dynamical properties for $\overline{T}$ can be obtained from their values.

1. *Introduction and main results*

The dynamics of a degree-one endomorphism of the circle has a very important invariant, the so-called rotation interval, which is a generalization of the concept of rotation number originally introduced by Poincaré in the study of circle homeomorphisms. Unprecisely, the rotation number of a point is the average speed at which the orbit of the point rotates around the circle, when this average speed exists. The definition of rotation interval had to be introduced in this context (see [**12, 21**]), because for degree-one mappings of the circle which are not one-to-one, different points may have different rotation numbers, something that does not happen for one-to-one mappings. To be more precise, given a degree-one endomorphism of the circle, denoted by $f : S^1 \to S^1$, there exists a closed interval $\rho(f) = [\rho^-, \rho^+]$ with the following properties:

- given $\omega \in \rho(f)$, there exists a compact f-invariant set Q_ω, such that the rotation number of every $x \in Q_\omega$ is equal to ω. And if $\omega = p/q$, then Q_ω is a q-periodic orbit for f;
- if $\omega \notin \rho(f)$, then the rotation number of every point in the circle (when it exists) is different from ω.

In [**7**], Boyland proved several results about the way ρ^- and ρ^+ behave as f is perturbed and gave some dynamical consequences of the rationality or not of their values.

The aim of the present paper is to study a similar problem in the context of twist mappings of the torus. First, we remember that twist mappings of the torus have an invariant called the *vertical rotation interval* (see [**1–3, 5**]), which has some similarities to the rotation interval of an $f : S^1 \to S^1$ as above. In [**2**] we proved that given an exact 'area'-preserving twist mapping $\widehat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ which induces a mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$ homotopic to the Dehn twist $(\phi, I) \to (\phi + I \bmod 1, I \bmod 1)$, then there exists a closed interval $\rho_V = [\rho_V^-, \rho_V^+]$ with the following properties.

For $\omega \in \mathrm{int}(\rho_V)$, there are two different situations:

(1) $\omega = p/q$ is a rational number. In this case there is a q-periodic point $\overline{x}$ for $\overline{T}$ (in fact, there are at least two such points) that lifts to a point $\widehat{x} \in S^1 \times \mathbb{R}$ such that $\widehat{T}^q(\widehat{x}) = \widehat{x} + (0, p)$;

(2) ω is an irrational number. Then there is a compact, $\overline{T}$-invariant set $\overline{Q} \subset \mathrm{T}^2$ that lifts to a set $\widehat{Q} \subset S^1 \times \mathbb{R}$ such that for any $\widehat{x} \in \widehat{Q}$ we have:

$$\lim_{n\to\infty} \frac{p_2 \circ \widehat{T}^n(\widehat{x}) - p_2(\widehat{x})}{n} = \omega,$$

where $p_2 : S^1 \times \mathbb{R} \to \mathbb{R}$ is the projection in the vertical coordinate.

As will be explained in definition (7) below, the vertical rotation interval is, in fact, the whole vertical rotation set as defined by Misiurewicz and Ziemian in [**19**]. Among other things, this means that if $\omega \notin \rho_V$, then for any sequences $x_i \in \mathbb{R}^2$ and $n_i \to \infty$,

$$\lim_{i\to\infty} \frac{p_2 \circ T^{n_i}(x_i) - p_2(x_i)}{n_i} \neq \omega,$$

which is much stronger than just saying that no $\overline{T}$-invariant set realizes this vertical rotation number.

In [**2**], we also proved that $\mathrm{int}(\rho_V) \neq \emptyset$ if and only if $\widehat{T}$ does not have rotational invariant curves. Note that in this case $0 \in \rho_V$.

This class of mappings is a very interesting one, as it contains, for instance, the well-known standard mapping

$$\widehat{S}_M : \begin{cases} \widehat{\phi}' = \widehat{\phi} + \widehat{I} + (k/2\pi)\sin(2\pi\widehat{\phi}) \quad \bmod 1, \\ \widehat{I}' = \widehat{I} + (k/2\pi)\sin(2\pi\widehat{\phi}), \end{cases} \tag{1}$$

and a surprising mapping analyzed in [**1, 3, 4**], related to the dynamics near a homoclinic orbit to a saddle-center equilibrium of a Hamiltonian system with 2 degrees of freedom. In appropriate coordinates this mapping may be written as

$$\widehat{F} : \begin{cases} \widehat{\phi}' = \mu(\widehat{\phi}) + \widehat{I} + \gamma\log(J(\widehat{\phi})) \quad \bmod \pi, \\ \widehat{I}' = \widehat{I} + \gamma\log(J(\widehat{\phi})), \end{cases} \tag{2}$$

where $J(\widehat{\phi}) = \alpha^2\cos^2(\widehat{\phi}) + \alpha^{-2}\sin^2(\widehat{\phi})$, $\mu(\widehat{\phi}) = \arctan(\tan(\widehat{\phi})/\alpha^2)$, $\mu(0) = 0$, and $\alpha, \gamma \in \mathbb{R}$ are parameters.

On the other hand, the proof presented in [**3**, Appendix] implies that even without the 'area' preservation and exactness hypothesis, there is still a closed interval (the vertical rotation interval) $\rho_V = [\rho_V^-, \rho_V^+]$ associated to $\overline{T}$ with the same properties (1) and (2) above. But, in this case it may happen that $0 \notin \rho_V$ and we cannot give a simple criterion to guarantee the non-degeneracy of ρ_V.

As already indicated, the main goal of this paper is to study how ρ_V^- and ρ_V^+ behave as $\overline{T}$ is perturbed and which dynamical implications can be obtained from their values and properties. One simple remark is that everything proved for ρ_V^+ is also true for ρ_V^-, so in the following we consider only ρ_V^+.

In [**5**] we proved that ρ_V^+ is a continuous function of $\overline{T}$ in the C^1-topology, so basically there are two situations to be analyzed:

(i) ρ_V^+ is a rational number;

(ii) ρ_V^+ is an irrational number.

The precise statements of the results we obtain and some of their consequences will be presented after the necessary definitions are introduced. In the next section we present an exposition of the results used in this paper. In the third section we prove our main results. In the fourth section we present other consequences of our results and in the last section we present an open question.

Notations and definitions.

(0) Let $(\widehat{\phi}, \widehat{I})$ denote the coordinates for the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where $\widehat{\phi}$ is defined modulo 1. Let (ϕ, I) denote the coordinates for the universal cover $\mathbb{R}^2$ of the cylinder. For all mappings $\widehat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ we define $(\widehat{\phi}', \widehat{I}') = \widehat{T}(\widehat{\phi}, \widehat{I})$ and $(\phi', I') = T(\phi, I)$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $\widehat{T}$.

(1) $\mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2) = \{T : \mathbb{R}^2 \to \mathbb{R}^2 / T$ is a C^1-diffeomorphism of the plane, $I'(\phi, I) \overset{I\to\pm\infty}{\to} \pm\infty$, $\partial_I \phi' > 0$ (twist to the right), $\phi'(\phi, I) \overset{I\to\pm\infty}{\to} \pm\infty$ and T is the lift of a C^1-diffeomorphism $\widehat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\}$.

(2) $\mathrm{Diff}^1_{\mathrm{r}}(S^1 \times \mathbb{R}) = \{\widehat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} / \widehat{T}$ is induced by an element of $\mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2)\}$.

(3) Let $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ be the standard projections, respectively in the ϕ and I coordinates ($p_1(\phi, I) = \phi$ and $p_2(\phi, I) = I$). We also use p_1 and p_2 for the standard projections of the cylinder.

(4) Let $TQ \subset \mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2)$ be the set of mappings T such that

$$T : \begin{cases} \phi' = T_\phi(\phi, I) & \text{with } \partial_I \phi' = \partial_I T_\phi(\phi, I) \geq K > 0 \quad \text{and} \\ I' = T_I(\phi, I) \end{cases}$$

$$\begin{cases} T_I(\phi + 1, I) = T_I(\phi, I), \\ T_I(\phi, I + 1) = T_I(\phi, I) + 1, \\ T_\phi(\phi + 1, I) = T_\phi(\phi, I) + 1, \\ T_\phi(\phi, I + 1) = T_\phi(\phi, I) + 1. \end{cases} \tag{3}$$

Every $T \in TQ$ induces a mapping $\widehat{T} \in \mathrm{Diff}^1_{\mathrm{r}}(S^1 \times \mathbb{R})$ and a mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, where $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the 2-torus. Coordinates in the torus are denoted by $(\overline{\phi}, \overline{I})$ and $p : \mathbb{R}^2 \to \mathrm{T}^2$ is the associated covering mapping.

(5) Let $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ be the following covering mapping:

$$\pi(\phi, I) = (\phi \bmod 1, I). \tag{4}$$

(6) Given a point $\overline{x} \in \mathrm{T}^2$, we define its vertical rotation number as (when the limit exists)

$$\rho_V(\overline{x}) = \lim_{n\to\infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n}, \quad \text{for any } x \in p^{-1}(\overline{x}). \tag{5}$$

(7) Given a mapping $T \in TQ$, following Misiurewicz and Ziemian [**19**], we define the vertical rotation set of T as follows:

$$\rho_V(T) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{n\geq i} \left\{ \frac{p_2 \circ T^n(x) - p_2(x)}{n} : x \in \mathbb{R}^2 \right\}}, \tag{6}$$

that is, $\sigma \in \rho_V(T)$, if and only if there are sequences $x_i \in \mathbb{R}^2$ and $n_i \to \infty$, such that

$$\lim_{i\to\infty} \frac{p_2 \circ T^{n_i}(x_i) - p_2(x_i)}{n_i} = \sigma.$$

If we denote $\omega^- = \inf \rho_V(T)$ and $\omega^+ = \sup \rho_V(T)$, [**19**, Theorem 2.4] gives two ergodic $\overline{T}$-invariant measures μ_- and μ_+ with vertical rotation numbers ω^- and ω^+, respectively. This means that

$$\int_{\mathrm{T}^2} \phi(\overline{x})\, d\mu_{-(+)} = \omega^{-(+)},$$

where $\phi : \mathrm{T}^2 \to \mathbb{R}$ is given by:

$$\phi(\overline{x}) = p_2 \circ T(x) - p_2(x), \quad \text{for any } x \in p^{-1}(\overline{x}).$$

Therefore, from the Birkhoff ergodic theorem, there are points $\overline{z}^+$ and $\overline{z}^-$ with $\rho_V(\overline{z}^+) = \omega^+$ and $\rho_V(\overline{z}^-) = \omega^-$. Finally, applying [**3**, Theorem 6 of the Appendix] and [**2**, Theorem 5], we get that for all $\alpha \in]\omega^-, \omega^+[$ there is a compact $\overline{T}$-invariant set $\overline{Q_\alpha}$, which is a periodic orbit if α is rational, such that $\rho_V(\overline{x}) = \alpha$, for all $\overline{x} \in \overline{Q_\alpha}$. So, $\rho_V(T) = [\omega^-, \omega^+]$. This justifies the title of the paper, because in this setting the vertical rotation set is an interval.

Now we are ready to state our main results.

THEOREM 1. *Let $T \in TQ$ be such that $\rho_V(T) = [\rho_V^-, p/q]$, with $p/q \in \mathbb{Q}$ and $(p, q) = 1$. Then, there exists a compact set $\widehat{A} \subset \mathrm{S}^1 \times \mathbb{R}$ such that $\widehat{G}(\widehat{A}) = \widehat{A}$, where $\widehat{G}(\bullet) \stackrel{\text{def}}{=} \widehat{T}^q(\bullet) - (0, p)$. Of course, we can choose $\widehat{A}$ as a minimal set, but it is not true that $\widehat{G}$ always has periodic points.*

Note that, in general, $\widehat{G}$ does not have a twist property, but in the literature there is a class of mappings called *tilt* that contains $\widehat{G}$. Tilt mappings share many properties with the twist mappings and many results that are true for twist mappings are also true for the tilt ones. See e.g. [**18, 8, 11**], for more information on this subject. The set $\widehat{A}$ constructed in the theorem is an invariant set for $\widehat{G}$, possibly an Aubry–Mather set (by 'Aubry–Mather set' we mean a compact, $\widehat{G}$-invariant minimal set that is contained in a graph over S^1). It is not hard to obtain a $\widehat{G}$ as in the above theorem without periodic points. For instance, it is easy to construct a $T \in TQ$ such that $\rho_V(T) = \{0\}$ and $\widehat{T}$ has no periodic points. For this, consider the Reeb homeomorphism F of $\mathbb{R} \times [0, 1]$ which has the following properties:

(i) $F(\mathbb{R} \times \{i\}) = \mathbb{R} \times \{i\}$, for $i = 0, 1$;

(ii) for all $x \in \mathbb{R} \times [0, 1]$, $F(x + (1, 0)) = F(x) + (1, 0)$;

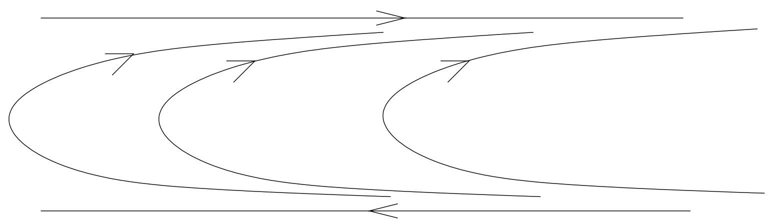

FIGURE 1. Diagram showing the dynamics of F.

(iii) for all $\phi \in \mathbb{R}$, $F(\phi, 1) = F(\phi, 0) + (1, 1)$;
(iv) the homeomorphism $\widehat{F} : S^1 \times [0, 1] \to S^1 \times [0, 1]$ induced by F satisfies

$$\rho(\widehat{F})|_{S^1\times\{0\}} = \sqrt{2} - 2 < 0 \quad \text{and} \quad \rho(\widehat{F})|_{S^1\times\{1\}} = \sqrt{2} - 1 > 0.$$

Now, let us define a $T \in TQ$ from F in the following way. Given $x \in \mathbb{R}^2$, define $T(x) = F(x - (0, m)) + (m, m)$, where $m \in \mathbb{Z}$ is chosen in such a way that $p_2(x) - m \in [0, 1[$. As F has the twist property, we have that $T \in TQ$. By construction, $\widehat{T}$ has no periodic points. See Figure 1 for a picture of the dynamics of F.

THEOREM 2. *Let $T \in TQ$ be such that $\rho_V(T) = [\rho_V^-, p/q]$, with $\rho_V^- < p/q$, $p/q \in \mathbb{Q}$ and $(p, q) = 1$. Then, given any $\eta > 0$ there exists $\alpha \in [0, \eta]$ such that $T_\alpha(\phi, I) = T(\phi, I) + (0, \alpha)$ induces a mapping $\overline{T}_\alpha$ on the torus with an nq-periodic orbit $\overline{Q}_{p/q}$ $(n \geq 1)$ that has $\rho_V(\overline{Q}_{p/q}) = np/nq = p/q$.*

COROLLARY 1. *Let $T \in TQ$ be such that $\rho_V(T) = [\rho_V^-, p/q]$, with $\rho_V^- < p/q$, $p/q \in \mathbb{Q}$ and $(p, q) = 1$. Suppose also that there is no periodic orbit for $\overline{T}$ with vertical rotation number $\rho_V = p/q$. Then, for all $\epsilon > 0$, we get that $\rho_V^+(T_{-\epsilon}) < p/q$.*

COROLLARY 2. *Let $T \in TQ$ be such that $\rho_V(T) = [\rho_V^-, p/q]$, with $\rho_V^- < p/q$, $p/q \in \mathbb{Q}$ and $(p, q) = 1$. Suppose also that there is a neighborhood $\mathcal{U}$ of T in TQ such that for any $T^* \in \mathcal{U}$, $\rho_V^+(T^*) \geq p/q$. Then, $\overline{T}$ has some (at least two) periodic orbits with vertical rotation number $\rho_V = p/q$ that cannot be destroyed by arbitrarily small perturbations.*

LEMMA 1. *The set $(\rho_V^+)^{-1}(\omega) = \{T \in TQ : \rho_V^+(T) = \omega\}$ is a path-connected subset of TQ for any $\omega \in \mathbb{R}$.*

Theorem 2 and Corollary 1 mean that given $T^* \in TQ$ with $\rho_V^+(T^*) = p/q$, if $\overline{T}^*$ does not have periodic points with vertical rotation number $\rho_V = p/q$, then $T^* \in \partial(\rho_V^+)^{-1}(p/q)$.

THEOREM 3. *Let $T \in TQ$ be such that $\rho_V(T) = [\rho_V^-, \omega]$, with $\omega \notin \mathbb{Q}$. Then, for any $\epsilon > 0$ we get that $\rho_V^+(T_\epsilon) > \omega$.*

So, given $T \in TQ$ as in Theorem 3, $\rho_V^+(T_\epsilon) \neq \omega$ for all $\epsilon \neq 0$. In other words, for any irrational number ω, $(\rho_V^+)^{-1}(\omega)$ has empty interior.

To conclude we present a corollary of the main result of [**16**], which says that we do not need to think of general perturbations in this setting. Vertical translations are enough for all applications, as the previous results have indicated.

COROLLARY 3. *Let $T \in TQ$ be such that $\rho_V(T) = [\omega^-, \omega^+]$. Suppose that by an arbitrarily C^1-small perturbation applied to T, we can change ω^+, that is, there exists T^* arbitrarily C^1-close to T, such that $\rho_V^+(T^*) \neq \omega^+$. Then, for any given $\epsilon > 0$, at least one of the following inequalities must hold:*

(1) $\rho_V^+(T_\epsilon) \neq \omega^+$, *or*

(2) $\rho_V^+(T_{-\epsilon}) \neq \omega^+$.

Moreover, given $T \in TQ$ with $\rho_V(T) = [\omega^-, \omega^+]$, there exists a neighborhood $T \in \mathcal{U} \subset TQ$ such that for any $T^* \in \mathcal{U}$, $\rho_V^+(T^*) = \omega^+$, if and only if, for some $\epsilon > 0$, $\rho_V^+(T_\alpha) = \omega^+$ for all $\alpha \in [-\epsilon, \epsilon]$.

2. *Basic tools*

2.1. *Some results for twist mappings.* First, we recall some topological results for twist mappings essentially due to Le Calvez (see [**14, 15**]). Let $\widehat{T} \in \mathrm{Diff}_{\mathrm{r}}^1(S^1 \times \mathbb{R})$, and $T \in \mathrm{D}_{\mathrm{r}}^1(\mathbb{R}^2)$ be one of its lifts. For every pair (s, q), $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we define the following sets:

$$K_{\mathrm{lift}}(s,q) = \{(\phi, I) \in \mathbb{R}^2 : p_1 \circ T^q(\phi, I) = \phi + s\}$$
$$\text{and} \quad K(s,q) = \pi \circ K_{\mathrm{lift}}(s,q). \tag{7}$$

Then we have the following.

LEMMA 2. *For every $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(s,q) \supset C(s,q)$, a connected compact set that separates the cylinder.*

Now we need a few definitions.

For every $q \geq 1$ and $\phi' \in \mathbb{R}$, let

$$\mu_q(t) = T^q(\phi', t), \quad \text{for } t \in \mathbb{R}. \tag{8}$$

We say that the first encounter between μ_q and the vertical line through some $\phi_0 \in \mathbb{R}$ is for

$$t_F \in \mathbb{R} \quad \text{such that} \quad t_F = \min\{t \in \mathbb{R} : p_1 \circ \mu_q(t) = \phi_0\}.$$

The last encounter is defined in the same way:

$$t_L \in \mathbb{R} \quad \text{such that} \quad t_L = \max\{t \in \mathbb{R} : p_1 \circ \mu_q(t) = \phi_0\}.$$

Of course, we have $-\infty < t_F \leq t_L < +\infty$.

LEMMA 3. *For all $\phi_0, \phi' \in \mathbb{R}$, let $\mu_q(t) = T^q(\phi', t)$, as in (8). So we have the following inequalities: $p_2 \circ \mu_q(t_L) \leq p_2 \circ \mu_q(t') \leq p_2 \circ \mu_q(t_F)$, for all $t' \in \mathbb{R}$ such that $p_1 \circ \mu_q(t') = \phi_0$.*

For all $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we can define the following functions on S^1:

$$\mu^-(\widehat{\phi}) = \min\{p_2(\widehat{z}) : \widehat{z} \in K(s,q) \text{ and } p_1(\widehat{z}) = \widehat{\phi}\},$$
$$\mu^+(\widehat{\phi}) = \max\{p_2(\widehat{z}) : \widehat{z} \in K(s,q) \text{ and } p_1(\widehat{z}) = \widehat{\phi}\}.$$

And we can define similar functions for $\widehat{T}^q(K(s,q))$:

$$\nu^-(\widehat{\phi}) = \min\{p_2(\widehat{z}) : \widehat{z} \in \widehat{T}^q \circ K(s,q) \text{ and } p_1(\widehat{z}) = \widehat{\phi}\},$$
$$\nu^+(\widehat{\phi}) = \max\{p_2(\widehat{z}) : \widehat{z} \in \widehat{T}^q \circ K(s,q) \text{ and } p_1(\widehat{z}) = \widehat{\phi}\}.$$

LEMMA 4. *Defining* $\mathrm{Graph}\{\mu^{\pm}\} = \{(\widehat{\phi}, \mu^{\pm}(\widehat{\phi})) : \widehat{\phi} \in S^1\}$ *we have*

$$\mathrm{Graph}\{\mu^-\} \cup \mathrm{Graph}\{\mu^+\} \subset C(s, q).$$

So, for all $\widehat{\phi} \in S^1$, *we have* $(\widehat{\phi}, \mu^{\pm}(\widehat{\phi})) \in C(s, q)$.

And we have the following simple corollary to Lemma 3.

COROLLARY 4. $\widehat{T}^q(\widehat{\phi}, \mu^-(\widehat{\phi})) = (\widehat{\phi}, \nu^+(\widehat{\phi}))$ *and* $\widehat{T}^q(\widehat{\phi}, \mu^+(\widehat{\phi})) = (\widehat{\phi}, \nu^-(\widehat{\phi}))$.

For proofs of all the previous results see Le Calvez [**14, 15**].

Now, we remember ideas and results from [**16**]. In the following, $T \in TQ$.

Give a triplet $(s, p, q) \in \mathbb{Z} \times \mathbb{N}^* \times Z$, if there is no point $(\phi, I) \in \mathbb{R}^2$ such that $T^q(\phi, I) = (\phi + s, I + p)$, it can be proved that the sets $\widehat{T}^q \circ K(s, q)$ and $K(s, q) + (0, p)$ can be separated by the graph of a continuous function from S^1 to $\mathbb{R}$, essentially because, from all the previous results, either one of the following inequalities must hold:

$$\nu^-(\widehat{\phi}) - \mu^+(\widehat{\phi}) > p, \tag{9}$$

$$\nu^+(\widehat{\phi}) - \mu^-(\widehat{\phi}) < p, \tag{10}$$

for all $\widehat{\phi} \in S^1$, where $\nu^+, \nu^-, \mu^+, \mu^-$ are associated to $K(s, q)$.

Following Le Calvez [**16**], we say that the triplet (s, q, p) is positive (respectively negative) for T if $\widehat{T}^q \circ K(s, q)$ is above (9) (respectively below (10)) the graph.

For all $T \in TQ$, we have

$$T(\phi, I) = (\phi', I') \Leftrightarrow I = g(\phi, \phi') \quad \text{and} \quad I' = g'(\phi, \phi'), \tag{11}$$

where g and g' are differentiable mappings from $\mathbb{R}^2$ to $\mathbb{R}$ with some special properties (see the proof of Lemma 1).

If $T, T^* \in TQ$, we say that $T \leq T^*$ if $g^* \leq g$ and $g' \leq g^{*\prime}$, where (g, g') is associated to T and $(g^*, g^{*\prime})$ to T^* (as in (11)).

PROPOSITION 1. *If* (s, q, p) *is a positive (respectively negative) triplet of* T *and if* $T \leq T^*$ *(respectively* $T \geq T^*$*), then* (s, q, p) *is a positive (respectively negative) triplet of* T^*.

The above proposition is a consequence of the following result, also proved in [**16**]. Note that $h_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $h_\alpha(\phi, I) = (\phi, I + \alpha)$.

LEMMA 5. *Let* $T \in \mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2)$ *and* $(\phi_0, I_0) \in \mathbb{R}^2$. *Denote the points of the orbit of* (ϕ_0, I_0) *by* $(\phi_n, I_n) = T^n(\phi_0, I_0)$, $n \geq 0$, *and consider the mapping* $h_\alpha \circ T \circ h_\alpha$, *for* $\alpha > 0$. *Then, for any* $n \geq 1$, *the image by* $(h_\alpha \circ T \circ h_\alpha)^n$ *of the lower vertical* $\{\phi_0\} \times]-\infty, I_0 - \alpha]$ *meets the upper vertical* $\{\phi_n\} \times [I_n + \alpha, +\infty[$. *Moreover, the intersection point whose second coordinate is the larger may be written* $(h_\alpha \circ T \circ h_\alpha)^n(\phi_0, I_0')$, *where* $(h_\alpha \circ T \circ h_\alpha)^n(\{\phi_0\} \times]-\infty, I_0'[)$ *is located to the left of the vertical passing through* (ϕ_n, I_n).

2.2. *Results on the theory of cocycles of ergodic transformation groups.* Now, we present some results and ideas from the theory of cocycles of ergodic transformation groups. A fundamental reference in this subject is the book of Schmidt [**22**]. In particular, our presentation will be directed towards the kind of application we need, so the definitions and results will be stated with no generality. Another fundamental result for us is due to Atkinson [**6**].

Given a homeomorphism $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, a $\overline{T}$-invariant ergodic probability measure μ and a continuous function $\phi : \mathrm{T}^2 \to \mathbb{R}$, we define the cocycle for $\overline{T}$ given by ϕ to be the function $a : \mathbb{Z} \times \mathrm{T}^2 \to \mathbb{R}$ given by

$$a(n, \overline{x}) = \begin{cases} \sum_{i=0}^{n-1} \phi \circ \overline{T}^i(\overline{x}), & \text{for } n > 0, \\ 0, & \text{for } n = 0, \\ -a(-n, \overline{T}^n(\overline{x})), & \text{for } n < 0. \end{cases} \tag{12}$$

The skew-product extension of $\overline{T}$, determined by ϕ, is given by the following mapping $V : \mathrm{T}^2 \times \mathbb{R} \to \mathrm{T}^2 \times \mathbb{R}$:

$$V(\overline{x}, \alpha) = (\overline{T}(\overline{x}), \alpha + \phi(\overline{x})). \tag{13}$$

So, the powers of V can be expressed as

$$V^n(\overline{x}, \alpha) = (\overline{T}^n(\overline{x}), \alpha + a(n, \overline{x})).$$

We say that the cocycle a is recurrent, if and only if, for every $B \in \sigma_B(\mathrm{T}^2) = \{$*Borel* σ-algebra of $\mathrm{T}^2\}$ with $\mu(B) > 0$ and every $\epsilon > 0$, there is an $n \neq 0$ such that

$$\mu(B \cap \overline{T}^{-n}(B) \cap \{\overline{x} : |a(n, \overline{x})| < \epsilon\}) > 0.$$

Now we present a result from [**6**].

THEOREM 4. *Suppose that* $(\mathrm{T}^2, \sigma_B(\mathrm{T}^2), \mu)$ *is a non-atomic probability space,* $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$ *is a homeomorphism ergodic with respect to* μ *and* $\phi : \mathrm{T}^2 \to \mathbb{R}$ *is a continuous function such that* $\int_{\mathrm{T}^2} \phi(\overline{x})\, d\mu = 0$*. Then, the cocycle* $a(n, \overline{x})$ *(see (12)) is recurrent.*

It is easy to see that the skew-product V (see (13)) is invariant under the product measure $\mu \times \lambda$, where λ is the Lebesgue measure in $\mathbb{R}$. The problem here is that the space $\mathrm{T}^2 \times \mathbb{R}$ is not compact, so we need to work a little more in order to get some kind of recurrence for V. An important definition for this purpose is the following (see Schmidt [**22**, Ch. 1]).

We say that the skew-product V is conservative if for every $A \in \sigma_B(\mathrm{T}^2 \times \mathbb{R})$ with $\mu \times \lambda(A) > 0$ and for $\mu \times \lambda$-almost everywhere $(\overline{x}, \alpha) \in A$, the set

$$\left[\bigcup_{n \in \mathbb{Z}} V^n(\overline{x}, \alpha) \right] \cap A$$

is infinite. At last, we present a theorem relating the concepts of recurrence and conservativeness (see [**22**, Ch. 5]).

THEOREM 5. *Suppose* $(\mathrm{T}^2, \sigma_B(\mathrm{T}^2), \mu)$ *is a non-atomic probability space, the homeomorphism* $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$ *is ergodic with respect to* μ *and the cocycle* $a(n, \overline{x})$ *(see (12)) is recurrent. Then the skew-product* $V(\overline{x}, \alpha) = (\overline{T}(\overline{x}), \alpha + a(1, \overline{x}))$ *is conservative.*

So, Theorems 4 and 5 imply that for any continuous function $\phi : \mathrm{T}^2 \to \mathbb{R}$ such that $\int_{\mathrm{T}^2} \phi(\overline{x})\, d\mu = 0$, the cocycle $a(n, \overline{x})$ is recurrent and the skew-product $V(\overline{x}, \alpha)$ is conservative. An equivalent way to say that a skew-product V as in (13) is conservative is the following.

LEMMA 6. *If a skew-product V as in (13) is conservative, then, given any $B \in \sigma_B(\mathrm{T}^2)$, with $\mu(B) > 0$ and any $\delta > 0$, for μ-almost everywhere $\overline{x} \in B$ we have*

$$\overline{T}^n(\overline{x}) \in B \quad \text{and} \quad |a(n, \overline{x})| < \delta, \quad \text{for infinitely many } n \in \mathbb{Z}.$$

Proof. Immediate from the definitions. □

2.3. *Some results previously obtained by the author.* Finally, we present some results from [**2, 3, 5**] that are used in this paper. The first result, we recall, was presented in an informal way in Definition 7 of §1.

THEOREM 6. *To each mapping $T \in TQ$, we can associate a maximal closed interval $\rho_V(T) = [\rho_V^-, \rho_V^+]$, possibly degenerated to a single point, such that for every $\omega \in]\rho_V^-, \rho_V^+[$ there is a compact $\overline{T}$-invariant set $\overline{Q}_\omega \subset \mathrm{T}^2$ with $\rho_V(\overline{x}) = \omega$, for all $\overline{x} \in \overline{Q}_\omega$. If ω is a rational number p/q, then $\overline{Q}_\omega$ is a q-periodic orbit. In fact, in this case there are at least two periodic orbits.*

For a proof, see [**3**, Appendix, Theorem 6] and [**2**, Theorem 5]. The next results are proved in [**5**].

THEOREM 7. *The functions $\rho_V^+, \rho_V^- : TQ \to \mathbb{R}$ are continuous in the C^1-topology.*

This result trivially implies that given $p/q \in]\rho_V^-, \rho_V^+[$, the periodic orbits with this vertical rotation number cannot be destroyed by arbitrarily small perturbations applied to $\overline{T}$. See also [**8**, Theorem 2.3].

LEMMA 7. *Given $T \in TQ$, let $f : \mathbb{R} \to \mathbb{R}$ be given by: $f(\alpha) = \rho_V^+(T_\alpha)$. Then f is a non-decreasing function of α.*

Remember that $T_\alpha(\phi, I) = T(\phi, I) + (0, \alpha)$.

3. *Proofs*

3.1. *Proof of Theorem 1.* Let $G : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $G(x) = T^q(x) - (0, p)$ and let $\widehat{G}$ be the cylinder mapping induced by G. As $\rho_V^+(T) = p/q$, if $\widehat{G}$ has no fixed points, then

$$\widehat{G}(C(0, q)) \cap C(0, q) = \emptyset. \tag{14}$$

Let U be the unbounded component of $C(0, q)^c$ which contains the lower end of the cylinder. From (14), without loss of generality, we can suppose that $\widehat{G}(\text{closure}(U)) \subset U$. Now we have two possibilities:

(1) there exists $N \geq 1$ such that $\widehat{G}^N(\text{closure}(U)) \subset \text{closure}(U) - (0, 1)$;

(2) for every $n \geq 1$, there exists $\widehat{z}_n \in \text{closure}(U) \cap \{\text{closure}(U) - (0, 1)\}^c$ such that $\widehat{G}^i(\widehat{z}_n) \notin \text{closure}(U) - (0, 1)$ for $0 \leq i \leq n$.

The first possibility easily implies that $\rho_V^+(T) \leq p/q - 1/Nq$, which is not true. So condition (2) is satisfied. As the $\widehat{z}_n$ are contained in a compact set, let $\widehat{z}^*$ be an accumulation point of the sequence $(\widehat{z}_n)_{n\geq 1}$. Clearly, the positive orbit of $\widehat{z}^*$ is bounded, so consider the ω-limit set of $\widehat{z}^*$, $\omega(\widehat{z}^*, \widehat{G})$. It is a $\widehat{G}$-invariant, compact subset of the cylinder. To conclude the proof, we note that every continuous mapping of a compact metric space has an invariant minimal subset (see [**13**, Proposition 3.3.6]). □

3.2. *Proof of Theorem 2.* Let $\eta > 0$ be a fixed number and suppose that $\widehat{G}$ does not have periodic points. From the previous theorem, we know that there is a compact, $\widehat{G}$-invariant minimal subset of the cylinder, which we call $\widehat{A}$.

Now we prove a lemma which will be used in the proofs of Theorems 2 and 3. It is a consequence of Lemma 5 and the ideas in [**16**].

LEMMA 8. *Given $\delta > 0$, suppose that there exist $z_0 = (\phi_0, I_0) \in \mathbb{R}^2$, $n \geq 2$ and $s, m \in \mathbb{Z}$ such that $T^n(z_0) - (s, m) \in B_\delta(z_0)$. Let*

$$M = \max\{\text{Lipschitz}(T), \text{Lipschitz}(T^{-1})\}$$

and β be a minimal value of the angle between a vertical and the pre-image of a vertical. As $T \in TQ$ is a twist mapping, $\beta > 0$. If

$$\alpha > \max\{(\sin\beta)^{-1}M^2\delta, (1 + \cot\beta)M\delta\},$$

then the image by $(h_\alpha \circ T \circ h_\alpha)^n$ of the lower vertical $\{\phi_0\} \times]-\infty, I_0 - \alpha]$ meets the upper vertical $\{\phi_0 + s\} \times [I_0 + m, +\infty[$. In particular, the maps $\mu^-, \mu^+, \nu^-, \nu^+$ associated to the set $C_\alpha(s, n)$ satisfy $\nu^+(\phi_0) > \mu^-(\phi_0) + m$.

Proof. First, let us denote $z_i = (\phi_i, I_i) = T^i(z_0)$, for all $0 \leq i \leq n$. The distance between z_{n-1} and $T^{-1}(z_0 + (s, m))$ is at most $M\delta$ and the distance between $T^{-1}(z_0 + (s, m))$ and $T^{-1}(z_0 + (s, m - \alpha))$ is at least $M^{-1}\alpha$. Using the definition of β and the inequality $M^{-1}\alpha \sin\beta > M\delta$, we get that $T^{-1}(z_0 + (s, m - \alpha))$ is located to the right of the vertical passing through z_{n-1}. From the other inequality, $\alpha > (1 + \cot\beta)M\delta$, we get that $T^{-1}(\{\phi_0 + s\} \times \mathbb{R})$ intersects the vertical $\{\phi_{n-1}\} \times \mathbb{R}$ in a point whose second coordinate is smaller than $I_{n-1} + \alpha$. See Figure 2 for a picture of this situation (in the picture we use η instead of α). These two facts imply that $(h_\alpha \circ T \circ h_\alpha)^{-1}(\{\phi_0 + s\} \times [I_0 + m, +\infty[)$ meets $\{\phi_{n-1}\} \times \mathbb{R}$ in a point whose second coordinate is smaller than $I_{n-1} + \alpha$. Using Lemma 5, we obtain that $(h_\alpha \circ T \circ h_\alpha)^{n-1}(\{\phi_0\} \times]-\infty, I_0 - \alpha])$ meets $(h_\alpha \circ T \circ h_\alpha)^{-1}(\{\phi_0 + s\} \times [I_0 + m, +\infty[)$, and the proof is complete. □

Now, choose $\delta > 0$ such that $\max\{(\sin\beta)^{-1}M^2\delta, (1 + \cot\beta)M\delta\} < \eta/4$. As $\widehat{A}$ is minimal, let us fix some $\widehat{z_0} \in \widehat{A}$ such that $\widehat{G}^n(\widehat{z_0}) \in B_\delta(\widehat{z_0})$, for some $n \geq 2$. This implies that

$$T^{nq}(z_0) - (s, np) \in B_\delta(z_0),$$

for any $z_0 \in \pi^{-1}(\widehat{z_0})$ and for some $s \in \mathbb{Z}$. If we apply Lemma 8, we get that the triplet (s, nq, np) is non-negative for $h_{\eta/4} \circ T \circ h_{\eta/4}$. As $\rho_V^- < p/q$ and we are supposing that $\widehat{G}$ has no periodic points, the triplet (s, nq, np) is negative for T. So, there are two possibilities:

(1) $\overline{h_{\eta/4} \circ T \circ h_{\eta/4}}$ has an nq-periodic point with $\rho_V = p/q$;
(2) the triplet (s, nq, np) is positive for $h_{\eta/4} \circ T \circ h_{\eta/4}$.

Let us analyze the second possibility. As (s, nq, np) is negative for T and positive for $h_{\eta/4} \circ T \circ h_{\eta/4}$, there exists $\alpha \in [0, \eta/4]$ such that $\overline{h_\alpha \circ T \circ h_\alpha}$ has a periodic point of type (s, nq, np). Otherwise, (the following argument was taken from [**16**]) the upper semi-continuity in the Hausdorff topology of the mappings

$$\alpha \to C_\alpha(s, nq) \quad \text{and} \quad \alpha \to (h_\alpha \widehat{\circ T \circ} h_\alpha)^{nq} \circ C_\alpha(s, nq)$$

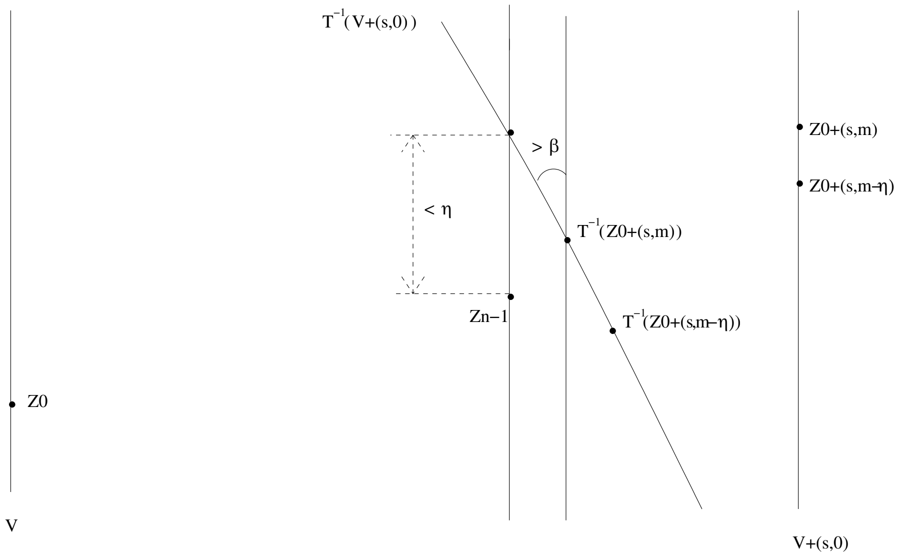

FIGURE 2. Diagram explaining the proof of Lemma 8.

and the connectivity of $[0, \eta/4]$ imply that (s, nq, np) is a positive triplet for every mapping $h_\alpha \circ T \circ h_\alpha$, $\alpha \in [0, \eta/4]$, or it is a negative triplet for every mapping $h_\alpha \circ T \circ h_\alpha$, $\alpha \in [0, \eta/4]$. And this is a contradiction. So, in both possibilities, there exists $\alpha \in [0, \eta/4]$ such that $\overline{h_\alpha \circ T \circ h_\alpha}$ has a periodic point of type (s, nq, np). Now, we note that

$$h_\alpha \circ (h_\alpha \circ T \circ h_\alpha) \circ h_{-\alpha}(\phi, I) = T(\phi, I) + (0, 2\alpha) = T_{2\alpha}(\phi, I). \tag{15}$$

So, if $\overline{h_\alpha \circ T \circ h_\alpha}$ has a periodic point of type (s, nq, np) for some $\alpha \in [0, \eta/4]$, then $\overline{T}_{2\alpha}$ has an nq-periodic point with $\rho_V = p/q$. □

3.3. *Proof of Corollary 1.* If $\overline{z} \in \mathrm{T}^2$ is periodic for $\overline{T}$, then the corollary hypothesis implies that

$$\rho_V(\overline{z}) = \lim_{n\to\infty} \frac{p_2 \circ T^n(z) - p_2(z)}{n} < \frac{p}{q},$$

for any $z \in p^{-1}(\overline{z})$. As $\rho_V^- < p/q$, this means that for every $s \in \mathbb{Z}$ and $n \in \mathbb{N}^*$, the triplet (s, nq, np) is negative for T, so it is also negative for $T_{-\alpha}$, for all $\alpha > 0$. Given $\epsilon > 0$, we know from Lemma 7 that $\rho_V^+(T_{-\epsilon}) \le p/q$. Suppose that $\rho_V^+(T_{-\epsilon}) = p/q$. If we apply Theorem 2 to $T_{-\epsilon}$ we get that there exists $\alpha \in [0, \epsilon/4]$ such that $T_{-\epsilon+\alpha}(\phi, I) = T(\phi, I) + (0, -\epsilon + \alpha)$ induces a mapping $\overline{T}_{-\epsilon+\alpha}$ on the torus with an n^*q-periodic orbit $\overline{Q}_{p/q}$ ($n^* \ge 1$) that has $\rho_V(\overline{Q}_{p/q}) = p/q$. This means that for some $s^* \in \mathbb{Z}$, the triplet (s^*, n^*q, n^*p) is non-negative for $T_{-\epsilon+\alpha}$. As $-\epsilon + \alpha < 0$, this is a contradiction. □

3.4. *Proof of Corollary 2.* As there is a neighborhood $T \in \mathcal{U} \subset TQ$ such that for any $T^* \in \mathcal{U}$, $\rho_V^+(T^*) \ge p/q$, we get from the previous lemma that there exist $s^* \in \mathbb{Z}$ and $n^* \in \mathbb{N}^*$ such that $\overline{T}$ has a periodic point of type (s^*, n^*q, n^*p).

Now, suppose that for all $s \in \mathbb{Z}$ and $n \in \mathbb{N}^*$, we have the following inequalities:

$$\nu^+(\widehat{\phi}) - \mu^-(\widehat{\phi}) - np \leq 0, \tag{16}$$

for all $\widehat{\phi} \in S^1$, where μ^-, ν^+ are the mappings associated to $C(s, nq)$ for T. If we remember [**16**, Proposition 3, p. 466] we get, for any $\alpha < 0$, that

$$\nu_\alpha^+(\widehat{\phi}) - \mu_\alpha^-(\widehat{\phi}) - np < 0, \tag{17}$$

for all $\widehat{\phi} \in S^1$, where $\mu_\alpha^-, \nu_\alpha^+$ are the mappings associated to $C(s, nq)$ for T_α. If $|\alpha|$ is sufficiently small ($\alpha < 0$), then $T_\alpha, T_{2\alpha} \in \mathcal{U}$ and so Lemma 7, together with the corollary hypothesis, implies that $\rho_V^+(T_\alpha) = \rho_V^+(T_{2\alpha}) = p/q$. As expression (17) is true for all $s \in \mathbb{Z}$ and $n \in \mathbb{N}^*$, we get that $\overline{T}_\alpha$ has no periodic points with vertical rotation number $\rho_V = p/q$. Then, Corollary 1 implies that $\rho_V^+(T_{2\alpha}) < p/q$, a contradiction. So, as $\rho_V^- < p/q$, there exists $s^* \in \mathbb{Z}$ and $n^* \in \mathbb{N}^*$ such that

$$\begin{aligned} &\nu^+(\widehat{\phi}_0) - \mu^-(\widehat{\phi}_0) - n^* p < 0, \\ &\nu^+(\widehat{\phi}_1) - \mu^-(\widehat{\phi}_1) - n^* p > 0, \end{aligned} \tag{18}$$

for some $\widehat{\phi}_0, \widehat{\phi}_1 \in S^1$, where μ^-, ν^+ are the mappings associated to $C(s^*, n^*q)$ for T. To conclude the proof, we just have to remember that the mappings

$$\widehat{T} \to C(s^*, n^*q) \quad \text{and} \quad \widehat{T} \to \widehat{T}^{n^*q} \circ C(s^*, n^*q)$$

are upper semi-continuous in the Hausdorff topology. So, an expression similar to (18) holds for every mapping sufficiently C^1-close to T. Also, the connectivity of $C(s^*, n^*q)$ implies the existence of at least two periodic points of type (s^*, n^*q, n^*p) for all mappings in *TQ* such that (18) holds. See also [**8**, Theorem 2.3]. □

3.5. *Proof of Lemma 1.* In order to show that for any $\omega \in \mathbb{R}$, the set $(\rho_V^+)^{-1}(\omega)$ is a path-connected subset of *TQ*, we will show that there exists an isotopy $T_t \in (\rho_V^+)^{-1}(\omega)$, $t \in [0, 1]$, between any given $T \in (\rho_V^+)^{-1}(\omega)$ and the following mapping:

$$D_\omega : \begin{cases} \phi' = \phi + I, \\ I' = I + \omega. \end{cases}$$

This is enough, because $D_\omega \in (\rho_V^+)^{-1}(\omega)$. As $T \in (\rho_V^+)^{-1}(\omega) \subset TQ$, it can be written as:

$$T : \begin{cases} \phi' = \phi + I + h_\phi(\phi, I), \\ I' = I + h_I(\phi, I), \end{cases}$$

where h_ϕ and h_I are bi-periodic functions. So the mappings (g, g') associated to T (see (11)) satisfy the following properties:

(1) $g(\phi, \phi') = \phi' - \phi + m(\phi, \phi')$ and $g'(\phi, \phi') = \phi' - \phi + m'(\phi, \phi')$, where m and m' are bi-periodic functions;

(2) $\partial_{\phi'} g(\phi, \phi') = 1 + \partial_{\phi'} m(\phi, \phi') > 0$ and $\partial_\phi g'(\phi, \phi') = -1 + \partial_\phi m'(\phi, \phi') < 0$.

Our isotopy T_t is given by

$$T_t(\phi, I) = T_t^*(\phi, I) + (0, \alpha(t)), \tag{19}$$

where $T_t^* \in TQ$ is the mapping induced by the following pair $(g_t^*, g_t^{*\prime})$:

$$g_t^*(\phi, \phi') = \phi' - \phi + (1-t).m(\phi, \phi'),$$
$$g_t^{*\prime}(\phi, \phi') = \phi' - \phi + (1-t).m'(\phi, \phi').$$

It is clear that for $t \in [0, 1]$, $(g_t^*, g_t^{*\prime})$ induce a twist mapping, because property (2) above implies that $\partial_{\phi'} m(\phi, \phi') > -1$ and $\partial_\phi m'(\phi, \phi') < 1$. So, $\partial_{\phi'} g_t^*(\phi, \phi') > 0$ and $\partial_\phi g_t^{*\prime}(\phi, \phi') < 0$. Thus, T_t^* is the lift of an isotopy between T and D_0 ($T_0^* = T$ and $T_1^* = D_0$).

We still have to prove that the function $\alpha : [0, 1] \to \mathbb{R}$ that appears in (19), which is chosen in a way that $T_t \in (\rho_V^+)^{-1}(\omega)$ for all $t \in [0, 1]$, can be chosen to be continuous. Clearly, $\alpha(0) = 0$ and $\alpha(1) = \omega$.

Let us define $T_{t,\alpha}^*(\phi, I) = T_t^*(\phi, I) + (0, \alpha)$ and let

$$A_\omega = \{(t, \alpha) \in [0, 1] \times \mathbb{R} : \rho_V^+(T_{t,\alpha}^*) = \omega\}.$$

Theorem 7 implies that A_ω is a closed set. As it is trivially bounded, A_ω is a compact subset of $[0, 1] \times \mathbb{R}$. Also, for each $t \in [0, 1]$, as $\rho_V^+(T_{t,\alpha}^*)$ is a non-decreasing function of α (see Lemma 7) and $\rho_V^+(T_{t,\alpha}^*) \overset{\alpha\to\pm\infty}{\to} \pm\infty$, there exists a closed interval $[\alpha^-(t), \alpha^+(t)]$ (maybe degenerated to a point) such that $\rho_V^+(T_{t,\alpha}^*) = \omega$, if and only if $\alpha \in [\alpha^-(t), \alpha^+(t)]$. So, $A_\omega \cap \{t\} \times \mathbb{R} = [\alpha^-(t), \alpha^+(t)]$. We will conclude the proof by showing that $\alpha^-(t)$ and $\alpha^+(t)$ are continuous functions of t. This clearly implies that A_ω contains the graph of a continuous function $\alpha : [0, 1] \to \mathbb{R}$ such that $\alpha(0) = 0$ and $\alpha(1) = \omega$.

Suppose that $\alpha^-(t)$ is not continuous. Then, there exists $\bar{t} \in [0, 1], \epsilon > 0$ and a sequence $t_i \overset{i\to\infty}{\to} \bar{t}$ such that $\alpha^-(t_i) > \alpha^-(\bar{t}) + \epsilon$, for all $i \in \mathbb{N}$. If we choose a sufficiently small $\delta > 0$, then the following is true for any $t_i \in [\bar{t} - \delta, \bar{t} + \delta]$:

$$\begin{aligned} T_{t_i}^*\left(\phi, I - \frac{\epsilon}{4}\right) + \left(0, \alpha^-(\bar{t}) - \frac{\epsilon}{4}\right) &\le T_{\bar{t}}^*(\phi, I) + (0, \alpha^-(\bar{t})) \\ &\le T_{t_i}^*\left(\phi, I + \frac{\epsilon}{4}\right) + \left(0, \alpha^-(\bar{t}) + \frac{\epsilon}{4}\right). \end{aligned}$$

The above expression implies (see the proof of Lemma 7) that:

$$\begin{aligned} &\rho_V^+\left(T_{t_i}^*\left(\phi, I - \frac{\epsilon}{4}\right) + \left(0, \alpha^-(\bar{t}) - \frac{\epsilon}{4}\right)\right) \\ &\quad \le \omega \le \rho_V^+\left(T_{t_i}^*\left(\phi, I + \frac{\epsilon}{4}\right) + \left(0, \alpha^-(\bar{t}) + \frac{\epsilon}{4}\right)\right). \end{aligned}$$

So, if we remember Theorem 7, expression (15), which means that for any given $\alpha \in \mathbb{R}$, $\rho_V^+(T(\phi, I + \alpha) + (0, \alpha)) = \rho_V^+(T(\phi, I) + (0, 2\alpha))$ and Lemma 7, we get that there exists $\tau \in [-\epsilon/2, \epsilon/2]$ such that $\rho_V^+(T_{t_i,\alpha^-(\bar{t})+\tau}^*) = \omega$. Thus, $\alpha^-(\bar{t}) + \tau \ge \alpha^-(t_i) > \alpha^-(\bar{t}) + \epsilon$, which is a contradiction.

The proof of the continuity of $\alpha^+(t)$ is analogous, so we omit it. $\square$

3.6. *Proof of Theorem 3.* Given $\epsilon > 0$ and $T \in TQ$ such that $\rho_V(T) = [\rho_V^-, \omega]$, with $\omega \notin \mathbb{Q}$, we are going to prove that $\rho_V^+(T_\epsilon) > \omega$. Our proof will be divided into two steps.

Step 1. In this part we prove that for any given $\delta > 0$, there exists a rational number $m/n > \omega$ and a point $\widehat{x} \in S^1 \times \mathbb{R}$ such that:

- $\widehat{T}^n(\widehat{x}) - (0, m) \in B_\delta(\widehat{x})$;
- $\rho_V(\widehat{x}) = \omega$.

If we remember [**19**, Theorem 2.4], we get that there is an ergodic $\overline{T}$-invariant measure μ_ω with vertical rotation number $\rho_V(\mu_\omega) = \omega$, which means that if we define a function $\phi : \mathrm{T}^2 \to \mathbb{R}$ by

$$\phi(\overline{x}) = p_2 \circ T(x) - p_2(x) - \omega, \quad \text{for any } x \in p^{-1}(\overline{x}), \tag{20}$$

then

$$\int_{\mathrm{T}^2} \phi(\overline{x})\, d\mu_\omega = 0. \tag{21}$$

The measure μ_ω is trivially non-atomic, because it is ergodic and $\omega \notin \mathbb{Q}$.

Now, let $\overline{A} \subset \mathrm{T}^2$ be the following set: $\overline{A} = \{\overline{x} \in \mathrm{supp}(\mu_\omega) : \overline{x}$ is recurrent and $\rho_V(\overline{x}) = \omega\}$, which has full μ_ω-measure, $\mu_\omega(\overline{A}) = 1$, and is $\overline{T}$-invariant, $\overline{T}(\overline{A}) = \overline{A}$.

Using expression (21), we get from Theorems 4 and 5 that the cocycle $a(n, \overline{x})$ and the skew-product $V(\overline{x}, \alpha)$ associated to $\overline{T}$ and ϕ (see (20)) are, respectively, recurrent and conservative.

Choose any $\overline{x}_0 \in A$. As $\overline{x}_0$ is recurrent, there exists $n_0 > 0$ such that $\overline{T}^{n_0}(\overline{x}_0) \in B_{\delta/8}(\overline{x}_0)$. This means that there exists $m_0 \in \mathbb{Z}$, such that for all $\widehat{x_0} \in S^1 \times \mathbb{R}$, lift of $\overline{x}_0$, we have

$$\widehat{T}^{n_0}(\widehat{x_0}) - (0, m_0) \in B_{\delta/8}(\widehat{x_0}).$$

If $m_0/n_0 > \omega$, then we are done. So, suppose that $m_0/n_0 < \omega$. Let $C = n_0\omega - m_0 > 0$ and let $0 < \overline{\delta}_1 < \delta/8$ be such that for all $\overline{z} \in B_{\overline{\delta}_1}(\overline{x}_0)$, $\overline{T}^{n_0}(\overline{z}) \in B_{\delta/8}(\overline{x}_0)$. Now define $\delta_1 = \min\{\overline{\delta}_1/10, C/10\}$. As $\overline{x}_0 \in \overline{A}$, we get that $\mu_\omega(B_{\delta_1}(\overline{x}_0)) > 0$, which implies by Lemma 6 that there exists $\overline{z} \in B_{\delta_1}(\overline{x}_0) \cap \overline{A}$ and $n_1 > n_0$ such that $\overline{T}^{n_1}(\overline{z}) \in B_{\delta_1}(\overline{x}_0)$ and $|a(n_1, \overline{z})| < \delta_1$. And this means that there exists $m_1 \in \mathbb{Z}$ such that for any $\widehat{z} \in S^1 \times \mathbb{R}$, lift of $\overline{z}$, we have

$$\widehat{T}^{n_1}(\widehat{z}) - (0, m_1) \in B_{2\delta_1}(\widehat{z}), \tag{22}$$

$$|a(n_1, \overline{z})| = \left| \sum_{i=0}^{n_1-1} \phi \circ \overline{T}^i(\overline{z}) \right| = |p_2 \circ \widehat{T}^{n_1}(\widehat{z}) - p_2(\widehat{z}) - n_1\omega| < \delta_1. \tag{23}$$

Expressions (22) and (23) imply that

$$|n_1\omega - m_1| < 3\delta_1 < C = n_0\omega - m_0. \tag{24}$$

From (24), we get the following inequality:

$$\omega < \frac{m_1 - m_0}{n_1 - n_0}.$$

Defining $\overline{w} = \overline{T}^{n_0}(\overline{z}) \in B_{\delta/8}(\overline{x}_0)$, the choice of $\delta_1 > 0$ implies that

$$\widehat{T}^{n_1 - n_0}(\widehat{w}) - (0, m_1 - m_0) \in B_{\delta_1 + \delta/8}(\widehat{w}) \subset B_{\delta/4}(\widehat{w}).$$

To finish this step, we just have to notice that as $\overline{z} \in \overline{A}$ and $\overline{T}(\overline{A}) = \overline{A}$, then $\overline{w} = \overline{T}^{n_0}(\overline{z}) \in \overline{A}$ and so $\rho_V(\overline{w}) = \omega$.

Step 2. Choose some $\delta > 0$ such that

$$\max\{(\sin\beta)^{-1}M^2\delta, (1+\cot\beta)M\delta\} < \frac{\epsilon}{4} \tag{25}$$

and apply step 1 in order to get a point $x \in \mathbb{R}^2$ and a rational number $m/n > \omega$ such that

$$T^n(x) - (s, m) \in B_\delta(x), \tag{26}$$

for some $s \in \mathbb{Z}$.

Now, as in the proof of Theorem 2, if we apply Lemma 8 we get that the triplet (s, n, m) is non-negative for $h_{\epsilon/4} \circ T \circ h_{\epsilon/4}$. In fact, there are two possibilities:

(1) there exists $x_{m/n} \in \mathbb{R}^2$ such that $(h_{\epsilon/4} \circ T \circ h_{\epsilon/4})^n(x_{m/n}) = x_{m/n} + (s, m)$;

(2) the triplet (s, n, m) is positive for $h_{\epsilon/4} \circ T \circ h_{\epsilon/4}$.

In the case when the second possibility is satisfied, let U be the unbounded component of $C(s, n)^c$ which contains the upper end of the cylinder. As (s, n, m) is positive for $h_{\epsilon/4} \circ T \circ h_{\epsilon/4}$, we get that $\widehat{(h_{\epsilon/4} \circ T \circ h_{\epsilon/4})}^n(\text{closure}(U)) \subset U + (0, m)$. This means that there exists a constant $C \in \mathbb{R}$ such that for all $x \in \mathbb{R}^2$ and $i > 0$,

$$p_2 \circ (h_{\epsilon/4} \circ T \circ h_{\epsilon/4})^{i.n}(x) - p_2(x) - i.m > C,$$

which implies that, at every point where the vertical rotation number is defined, it is larger than m/n. And we know from [**19**, Theorem 2.4] that there is a point $\overline{x}^+ \in \mathrm{T}^2$, such that $\rho_V(\overline{x}^+) = \rho_V^+(h_{\epsilon/4} \circ T \circ h_{\epsilon/4})$, and thus

$$\rho_V(\overline{x}^+) \geq \frac{m}{n} > \omega.$$

So, both possibilities imply the existence of points with vertical rotation number $\rho_V \geq m/n$. Now, we just have to remember expression (15) in order to conclude that $\rho_V^+(T_\epsilon) \geq m/n > \omega$. □

3.7. *Proof of Corollary 3.* First of all, we remember the proof of Lemma 7, which says that given $T_1, T_2 \in TQ$, if $T_1 \leq T_2$, then $\rho_V^+(T_1) \leq \rho_V^+(T_2)$. The hypothesis of the corollary implies that, given $\epsilon > 0$, there exists a T^* sufficiently C^1-close to T, such that $\rho_V^+(T^*) \neq \omega^+$ and the following is true:

$$h_{-\epsilon/2} \circ T \circ h_{-\epsilon/2} \leq T^* \leq h_{\epsilon/2} \circ T \circ h_{\epsilon/2}.$$

Suppose that $\rho_V^+(T^*) < \omega^+$. Then, $\rho_V^+(h_{-\epsilon/2} \circ T \circ h_{-\epsilon/2}) \leq \rho_V^+(T^*) < \omega^+$. The other case is analogous. To conclude, we remember that $\rho_V^+(h_{\alpha/2} \circ T \circ h_{\alpha/2}) = \rho_V^+(T_\alpha)$, for any $\alpha \in \mathbb{R}$. Now we look at the second part of the corollary. One implication is trivial. Let us prove the other. Suppose that there exists $\epsilon > 0$ such that $\rho_V^+(T_\alpha) = \omega^+$, for all $\alpha \in [-\epsilon, \epsilon]$, and there exists a sequence $T_i \overset{i\to\infty}{\to} T$, in the C^1-topology, such that $\rho_V^+(T_i) \neq \omega^+$. If $i_0 > 0$ is sufficiently large, then

$$h_{-\epsilon/2} \circ T \circ h_{-\epsilon/2} \leq T_{i_0} \leq h_{\epsilon/2} \circ T \circ h_{\epsilon/2},$$

which implies that $\omega^+ = \rho_V^+(T_{-\epsilon}) \leq \rho_V^+(T_{i_0}) \leq \rho_V^+(T_\epsilon) = \omega^+$, and this is in contradiction to the choice of T_{i_0}. □

4. *Consequences, applications and some motivation for the previous results*
First of all, we present a consequence of Theorem 3.

COROLLARY 5. *Let* $T \in TQ$ *be such that* $\rho_V(T) = [\rho_V^-, \omega]$*, with* $\omega \notin \mathbb{Q}$ *and* $\rho_V^- < \omega$*. Consider the following one-parameter family:* $T_t(\phi, I) = T(\phi, I) + (0, t)$*. Then, there exists* $\epsilon > 0$*, such that* $\rho_V(T_t) = [\rho_V^-(t), \rho_V^+(t)]$ *satisfies* $\rho_V^-(t) - \omega < 0$ *and* $\rho_V^+(t) - \omega > 0$*, for any* $t \in]0, \epsilon]$*. So as* t *goes from* 0 *to* ϵ*, infinitely many non-degenerated periodic points are created.*

Remark. We say that a certain set of periodic points is non-degenerated if it cannot be destroyed by arbitrarily small perturbations applied to the mapping.

Proof. This proof is a trivial consequence of Theorems 3, 6 and 7. □

In particular, the above corollary implies that a mapping $T \in TQ$ such that $\rho_V(\overline{T}) = [\rho_V^-, \omega]$, with $\omega \notin \mathbb{Q}$ and $\rho_V^- - \omega < 0$, is a bifurcation point of the set TQ. By an arbitrarily small perturbation (in the same topology of the mapping) one creates infinitely many periodic points.

If $T \in TQ$ induces an analytic area-preserving mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, for instance as the standard mapping (see (1)), such that the above corollary hypothesis is satisfied, then for any $\epsilon > 0$, there exists $\alpha \in [0, \epsilon]$ such that $\overline{T}_\alpha$ is not fully ergodic. To see this, note that Corollary 5 implies that as t goes from 0 to ϵ, lots of non-degenerated periodic points are created. And, right after the creation of some of these points (with the same vertical rotation number p/q), we have two possibilities.

(i) There are infinitely many periodic points with vertical rotation number p/q. In this case, the proof of [**3**, Theorem 3] implies that non-trivial invariant sets appear (by 'non-trivial' we mean that the set does not have zero or full Lebesgue measure).

(ii) There are only finitely many periodic points with vertical rotation number p/q. In this case, we proceed as follows. As $\overline{T}$ is area-preserving and the periodic points with vertical rotation number p/q are isolated and non-trivial as a set, some of them have negative topological index and some of them have positive topological index. In the area-preserving setting, the only positive index allowed is 1 (see [**17**]). Preservation of area also implies that in a sufficiently small neighborhood of the periodic points, the diffeomorphism is the time-one mapping of a formal vector field (defined by a formal series), see [**20**]. So, we are able to apply results from [**10**], which say that at least in a topological sense, the dynamics near the fixed points can be obtained as in the vector field setting, gluing a finite number of sectors. As we are supposing that area is preserved by the diffeomorphism, there cannot be elliptic, expanding or attracting sectors. As the topological index of the point is 1 and the eigenvalues are both near 1 because the point has just been created, we get that it must be a center.

So, if for some parameter $k > 0$, the supremum of the vertical rotation number of the standard mapping (1) is irrational, then arbitrarily close to it in the analytic topology there are mappings that are not fully ergodic. Unfortunately, we do not know if the perturbation performed could be restricted to the standard family (just a change in the value of k). See [**9**] for more information on this problem.

The case when $\rho_V^+(T)$ is a rational number is not completely clear. Let $T \in TQ$ induce an analytic area-preserving mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$, such that $\rho_V(T) = [\rho_V^-, p/q]$, with $p/q \in \mathbb{Q}$ and $\rho_V^- - p/q < 0$. As we do not have an analogue of Corollary 5, there are two possibilities:

(1) there exists $\epsilon > 0$, such that $\rho_V(\overline{T}_t) = [\rho_V^-(t), \rho_V^+(t)]$ satisfies one of the following:
  (1a) $\rho_V^-(t) - p/q < 0$ and $\rho_V^+(t) - p/q > 0$, for all $t \in]0, \epsilon]$,
  (1b) $\rho_V^-(t) - p/q < 0$ and $\rho_V^+(t) - p/q < 0$, for all $t \in [-\epsilon, 0[$;

(2) there exists $\epsilon > 0$, such that $\rho_V(\overline{T}_t) = [\rho_V^-(t), \rho_V^+(t)]$ satisfies: $\rho_V^-(t) - p/q < 0$ and $\rho_V^+(t) = p/q$, for all $t \in [-\epsilon, \epsilon]$.

Case (1) implies that, as in the irrational case, there exists $t \in \mathbb{R}$ arbitrarily small, such that $\overline{T}_t$ is not fully ergodic. The second possibility is harder to study. We believe that it implies the existence of non-trivial invariant sets with vertical rotation number $\rho_V = p/q$ (for instance, like elliptic islands). If this is true, it is also associated with non-ergodicity.

5. *Open question*

Here we present a conjecture that in a certain sense is the closure of this work. The setting is the following. Let $T \in TQ$ be such that $\rho_V(T) = [\rho_V^-, p/q]$, with $p/q \in \mathbb{Q}$ and $\rho_V^- - p/q < 0$. Then we have the following.

CONJECTURE. *Given $T \in TQ$ as above, it is not possible that $\rho_V^+(T_\alpha) \neq p/q$, for all $\alpha \neq 0$.*

If this conjecture is true, as $\rho_V^+(T_\alpha)$ is a non-decreasing function of α, there exists $\epsilon > 0$ such that one of the following possibilities must hold:

(1) $T \in \partial(\rho_V^+)^{-1}(p/q)$ and $\rho_V^+(T_\alpha) = p/q$, for $\alpha \in [0, \epsilon]$;
(2) $T \in \partial(\rho_V^+)^{-1}(p/q)$ and $\rho_V^+(T_\alpha) = p/q$, for $\alpha \in [-\epsilon, 0]$;
(3) $T \in \mathrm{int}(\rho_V^+)^{-1}(p/q)$.

One of the indications that this conjecture is true will be explained in the rest of this section.

Let $T \in TQ$ be written as

$$T : \begin{cases} \phi' = T_\phi(\phi, I), \\ I' = I + h_I(\phi), \end{cases}$$

where $h_I(\phi)$ is a periodic function. This is true for the standard mapping and for F (see (2)). Suppose that $\rho_V(T) = [\rho_V^-, 1]$, with $\rho_V^- < 1$. Then, we get that for some $\phi_0 \in \mathbb{R}$, $h_I(\phi_0) \geq 1$. As $\rho_V^- < 1$, there exists $\phi_1 \in \mathbb{R}$, such that $h_I(\phi_1) < 1$. If $h_I(\phi_0) > 1$, then the mappings μ^-, ν^+ associated to $C(0, 1)$ satisfy (as $C(0, 1)$ is a graph over S^1, $\mu^- = \mu^+$ and $\nu^- = \nu^+$)

$$\nu^+(\widehat{\phi_0}) - \mu^-(\widehat{\phi_0}) - 1 > 0$$

and

$$\nu^+(\widehat{\phi_1}) - \mu^-(\widehat{\phi_1}) - 1 < 0.$$

As $C(0, 1)$ is a connected set, there exists $\widehat{\phi_*}$ such that $\nu^+(\widehat{\phi_*}) - \mu^-(\widehat{\phi_*}) - 1 = 0$. Finally, as in the proof of Corollary 2, or [**8**, Theorem 2.3] , we get that some fixed points for $\overline{T}$ with vertical rotation number $\rho_V = 1$ cannot be destroyed by arbitrarily small perturbations.

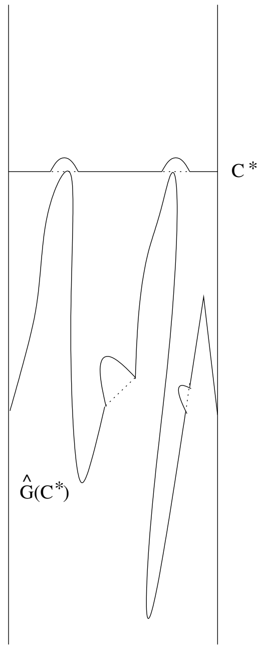

FIGURE 3. Diagram showing the construction of C^*.

So, as $\rho_V^+(T_\alpha) \leq 1$ for any $\alpha < 0$, there exists $\epsilon > 0$ such that $\rho_V^+(T_\alpha) = 1$, for all $\alpha \in [-\epsilon, 0]$.

Now, suppose that $h_I(\phi) \leq 1$, for all $\phi \in \mathbb{R}$. Let $C \subset S^1 \times \mathbb{R}$ be a curve of the following form:

$$C = \{(\widehat{\phi}, \widehat{I}) \in S^1 \times \mathbb{R} : \widehat{I} = c\},$$

where $c \in \mathbb{R}$ is chosen in such a way that C does not contain fixed points for $\widehat{G}(\phi, I) \stackrel{\text{def}}{=} \widehat{T}(\phi, I) - (0, 1)$. Clearly, this is always possible. As $\rho_V(T) = [\rho_V^-, 1]$, there exists $\phi_0 \in \mathbb{R}$ such that $h_I(\phi_0) = 1$, which means that

$$\widehat{G}(C) \subset \text{closure}(C^-) = C^- \cup C,$$

where $C^{-(\text{respectively}+)}$ is the open connected component of C^c that is below (respectively above) C. Now, suppose that

$$\{\widehat{G}(C) \cap C\} \cap \{\widehat{G}^{-1}(C) \cap C\} = \emptyset. \tag{27}$$

This is trivially true for the standard mapping and for F, because in both cases $\widehat{G}(C) \cap C$ is a single point, which is not fixed by the choice of C. As the sets in (27) are compact and disjoint, there exists a neighborhood $V \subset C$ of $\widehat{G}(C) \cap C$, disjoint from $\widehat{G}^{-1}(C) \cap C$. Now, we are going to perform some changes in C in order to get a new curve C^*, disjoint from its image by $\widehat{G}$. Define C^* as the homotopically non-trivial simple closed curve obtained from $C \backslash V$ together with some small arcs contained in C^+ (the end points of these arcs are in C). These arcs must be chosen sufficiently close to C, so that their images by $\widehat{G}$ are contained in C^-. See Figure 3 for a picture of this situation.

So we have the following equation for C^*:

$$\widehat{G}(C^*) \subset C^{*-},$$

which persists under small perturbations applied to T. So, the supremum of the vertical rotation number of all mappings in a sufficiently small neighborhood of T is less than or equal to 1. In particular, there exists $\epsilon > 0$ such that $\rho_V^+(T_\alpha) = 1$, for $\alpha \in [0, \epsilon]$. That is, the above conjecture is true in this particular situation.

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