Homeomorphisms of the annulus with a transitive lift II

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Abstract

Let f be a homeomorphism of the closed annulus A that preserves the orientation, the boundary components and that has a lift $\tilde{f}$ to the infinite strip $\tilde{A}$ which is transitive. We show that, if the rotation number of $\tilde{f}$ restricted to both boundary components of A is strictly positive, then there exists a closed nonempty connected set $\Gamma \subset \tilde{A}$ such that $\Gamma \subset]-\infty, 0] \times [0,1]$, Γ is unbounded, the projection of Γ to A is dense, $\Gamma - (1,0) \subset \Gamma$ and $\tilde{f}(\Gamma) \subset \Gamma$. Also, if p_1 is the projection on the first coordinate of $\tilde{A}$, then there exists $d > 0$ such that, for any $\tilde{z} \in \Gamma$,

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\tilde{z})) - p_1(\tilde{z})}{n} < -d.$$

Key words: closed connected sets, order, transitivity, stable sets, compactification

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1 Introduction and statements of the main results

In this paper we continue the study started in [1]. There, we considered homeomorphisms f of the closed annulus $A = S^1 \times [0,1]$ (S^1=$\mathbb{R}/\mathbb{Z}$), which preserve the orientation, the boundary components and have a transitive lift $\widetilde{f}$ to the universal cover of the annulus, $\widetilde{A}$ =$\mathbb{R}\times[0,1]$.

Results from [3] and [4] imply that these are reasonable hypotheses, see the explanation below.

First note that, given a homeomorphism $f : A \to A$ which preserves orientation and the boundary components, for any Borel probability f-invariant measure μ, the rotation number of μ, is defined as follows.

Let $p_1 : \widetilde{A} \to\mathbb{R}$ be the projection on the first coordinate and let $p : \widetilde{A} \to A$ be the covering mapping. Fixed f and $\widetilde{f}$, the displacement function $\phi : A\to\mathbb{R}$ is defined as

$$\phi(x, y) = p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) - \widetilde{x}, \tag{1}$$

for any $(\widetilde{x}, \widetilde{y}) \in p^{-1}(x, y)$. The rotation number of μ is then given by

$$\rho(\mu) = \int_A \phi(x, y)d\mu.$$

Following the usual definition (see [2]), we refer to the set of area, orientation and boundary components preserving homeomorphisms of the annulus, which satisfy $\rho(Leb) = 0$ for a certain fixed lift $\widetilde{f}$, as the set of rotationless homeomorphisms. Every time we say that f is a rotationless homeomorphism, a special lift $\widetilde{f}$ is fixed and used to define ϕ.

In [3] it is proved that the transitivity of $\widetilde{f}$ holds for a residual subset of rotationless homeomorphisms of the annulus and the results in [4] suggest that the same statement holds in the C^1 topology.

Here we consider the following situation: Suppose f is an orientation and boundary components preserving homeomorphism of the annulus which has a transitive lift $\widetilde{f} : \widetilde{A}\to \widetilde{A}$ (one with a dense orbit). We denote the set of such mappings by $Hom^{trans}_+(A)$. So every time we say $f \in Hom^{trans}_+(A)$ and refer to a lift $\widetilde{f}$ of f, we are always considering a transitive lift (it is possible that f has more then one transitive lift, we choose any of them and fix it).

Before we state the theorems proved in [1], we need a definition:

Definition : B^- is the subset of

$$B = \bigcap_{n\leq 0} \widetilde{f}^n(]-\infty, 0] \times [0, 1])$$

which contains exactly all unbounded connected components of B and nothing else.

Theorem 1 *: If $f \in Hom_+^{trans}(A)$ and the rotation number of $\widetilde{f}$ restricted to the boundary components of the annulus is strictly positive, then B^- is not empty and in particular, $B^- \cap \{0\} \times [0,1] \neq \emptyset$. Also B^- is positively invariant, closed and does not intersect the boundary of $\widetilde{A}$.*

Theorem 2 *: Under the hypotheses of theorem 1, the ω-limit set of B^-, $\omega(B^-) \stackrel{def.}{=} \bigcap_{n=0}^{\infty} \overline{\left(\bigcup_{i=n}^{\infty} \widetilde{f}^i(B^-) \right)}$ is empty.*

Thus, iterates of B^- by $\tilde{f}$ converge to the left end of $\widetilde{A}$. The properties of B^- allow us to extend this theorem and obtain a stronger result:

Theorem 3 *: Under the hypotheses of theorem 1, there exists a real number $\rho^+(B^-) < 0$ such that, if $\widetilde{z} \in B^-$, then*

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})}{n} \leq \rho^+(B^-) < 0.$$

The last theorem shows that all points in B^- have a "minimum negative velocity" in the strip $\tilde{A}$.

An important consequence of theorem 3 is that, even though there are points with rotation number in $]\rho^+(B^-), 0[$, they do not belong to B^-. In particular, if such points are hyperbolic periodic saddles that have unstable manifolds unbounded to the left, then they must also be unbounded to the right.

Theorem 4 *: Under the hypotheses of theorem 1, $p(B^-)$ is dense in A.*

Proofs of the above results can be found in [1].

Our main objective here is to have a more precise understanding of the structure of B^-. It will be shown that a connected component Γ of B^- can, a priory, be of one of the following types:

1. injective, which means that $\Gamma \cap (\Gamma + (s, 0)) = \emptyset$, for all integers $s \neq 0$;

2. non injective, which means that $\Gamma - (1, 0) \subset \Gamma$.

Our theorems are the following:

Theorem 5 *: Under the hypotheses of theorem 1, there exists a non injective connected component Γ of B^-, such that $p(\Gamma)$ is dense in the annulus, $\widetilde{f}(\Gamma) \subset \Gamma$, and for some positive integer k, $\widetilde{f}^{-1}(\Gamma) \subset \Gamma + (k, 0)$, which implies that $f(p(\Gamma)) = p(\Gamma)$.*

Theorem 6 *: Under the hypotheses of theorem 1, all connected component of B^- are non injective.*

It is worth noticing that, if Γ is the component of B^- described in theorem 5, it is a closed connected $\widetilde{f}$-invariant set which contains infinite negative integer translates of all its points. If we look at the projection of a connected piece of Γ to the annulus, it can not separate the boundary components of A, because f is transitive and, as we will show in proposition 1, Γ^c is connected. Thus, there can not be a compact connected subset of Γ containing both a point P and $P-(1,0)$.

Theorems 3, 4, 5 and 6 above have an interesting consequence. If Γ is as in theorem 5, and we consider the set $\Gamma_{sat}=\bigcup_{i=0}^{\infty}\Gamma+(i,0)$, then Γ_{sat} is dense and connected in the strip, $\tilde{f}(\Gamma_{sat})=\Gamma_{sat}$ and, in a sense, all points in Γ_{sat} converge to the left end of $\widetilde{A}$ through iterations of $\tilde{f}$ with a strictly negative velocity. Therefore Γ_{sat} can be seen as part of a dense "stable manifold of the point L in the L,R-compactification (left and right compactification) of the strip".

2 Preliminaries for the proofs

In this section we state some simple results proved in [1] that will be necessary for our arguments and give a general set of hypotheses that already appeared in [1].

2.1 Hypothesis satisfied by the set D

Let $D\subset\widetilde{A}$ be a non-empty closed set with the following properties:

- $\widetilde{f}(D)\subset D$;
- $D\subset]-\infty,0]\times[0,1]$;
- Every connected component of D is unbounded;
- $D\cap\mathbb{R}\times\{i\}=\emptyset$, $i\in\{0,1\}$;
- If $\widetilde{z}\in D$ then $\widetilde{z}-(1,0)\in D$.

It is easily verified that B^- has these properties, so every result shown for D must hold in the particular case of interest to us. In the proof of theorem 5, we find another set with the properties listed above.

2.2 On the structure of $p(D)\subset A$

Remember that $\widetilde{f}$ is transitive and it moves points in $\partial\widetilde{A}$ uniformly to the right.

Lemma 1 *:* $\overline{p(D)}\supset S^1\times\{0\}$, *or* $\overline{p(D)}\supset S^1\times\{1\}$.

Lemma 2 *: If* $\overline{p(D)}\neq A$, *then* $\overline{p(D)}^c$ *is connected and dense in* A *and, moreover,* $\overline{p(D)}^c$ *contains a homotopically non trivial simple closed curve in the open annulus* $S^1\times]0,1[$.

3 On the structure of $D \subset \widetilde{A}$

Let Γ be a connected component of D, see subsection 2.1. We recall that, by the definition of D, Γ is unbounded and contained in $]-\infty, 0] \times [0,1]$. Before going on, we present a few related definitions. For every $a \in \mathbb{R}$ we let

$$V_a = \{a\} \times [0,1], \ V_a^+ = [a, +\infty[\times[0,1] \text{ and } V_a^- =]-\infty, a] \times [0,1]. \qquad (2)$$

If $a = 0$ we denote V_0 and $V_0^{+(-)}$ simply by V and $V^{+(-)}$. The next proposition is important for the results in this section.

Proposition 1 *: Γ^c is connected.*

Proof:
Clearly, there is one connected component of Γ^c which contains $int(V^+)$, $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$.

So, if by contradiction, we suppose that Γ^c has another connected component, denoted C, contained in V^-, its boundary must be contained in Γ. As $\widetilde{f}^n(\Gamma) \subset V^-$ for all $n \geq 0$, we get that $\widetilde{f}^n(C) \subset V^-$ for all $n \geq 0$. So for every $\widetilde{z} \in C$, $\limsup_{n\to\infty} p_1 \circ \widetilde{f}^n(\widetilde{z}) \leq 0$. As C is an open subset of $\widetilde{A}$, this contradicts the transitivity of $\widetilde{f}$. $\square$

For a connected component Γ of D, let us define

$$m_\Gamma = \sup\{\widetilde{x} \in \mathbb{R} : (\widetilde{x}, \widetilde{y}) \in \Gamma, \text{ for some } \widetilde{y} \in [0,1]\} \leq 0. \qquad (3)$$

Consider the closed connected set $\Gamma \cup \{m_\Gamma\} \times [0,1]$. Its complement has two open unbounded connected components in

$$]-\infty, m_\Gamma[\times[0,1],$$

one of which contains $]-\infty, m_\Gamma[\times\{0\}$ (denoted Γ_{down}) and another one which contains $]-\infty, m_\Gamma[\times\{1\}$ (denoted Γ_{up}). It is possible that $(\Gamma \cup \{m_\Gamma\} \times [0,1])^c$ has other unbounded (to the left) connected components. But only Γ_{up} and Γ_{down} will be of interest to us, because of the following fact, whose proof is an exercise which depends only on the connectivity of Γ (see lemma 4 for a generalization of this result):

Fact 1 *: Given a connected component Γ of D, if Θ is a closed unbounded connected set, which satisfies $\Theta \cap \Gamma = \emptyset$ and $\Theta \subset]-\infty, m_\Gamma[\times[0,1]$, then $\Theta \subset \Gamma_{up}$ or $\Theta \subset \Gamma_{down}$.*

In the following, we will generalize the above construction and present some simple results on the connected components of D. These results will permit us to define an order $\prec$ on the connected components of D. Actually, the next results will hold in a slightly more general context: Any two disjoint closed unbounded connected sets $\Theta_1, \Theta_2 \subset V^-$, which have connected complements will be related by this order, that is either $\Theta_1 \prec \Theta_2$ or $\Theta_2 \prec \Theta_1$. We choose to define the partial order in this setting because, if Γ_1, Γ_2 are connected components of D, then $\widetilde{f}(\Gamma_1)$ and $\widetilde{f}(\Gamma_2)$ may not be, they are just contained in connected components of D. But in this case, if $\Gamma_1 \prec \Gamma_2$, as $\widetilde{f}(\Gamma_1)$ and $\widetilde{f}(\Gamma_2)$ are disjoint closed unbounded connected sets which have connected complements, we will show that $\widetilde{f}^{\pm 1}(\Gamma_1) \prec \widetilde{f}^{\pm 1}(\Gamma_2)$, that is $\widetilde{f}^{\pm 1}$ preserves the order. The proofs of the following results are easy, but some are long and a little boring. For this reason, they will be given in the appendix.

To begin, let us define the set:

$$UnConn = \{\Gamma \subset int(\widetilde{A}) : \Gamma \text{ is a closed unbounded}$$
$$\text{connected set, bounded to the right, which has a connected complement}\}$$

Lemma 3 *: Let $\Gamma \in UnConn$ and $a \in \mathbb{R}$ be such that V_a intersects Γ. Then, $\Gamma^c \cap]-\infty, a[\times[0,1]$ has at least two (open) connected components, one containing $]-\infty, a[\times\{0\}$ and the other containing $]-\infty, a[\times\{1\}$.*

Proof:
Immediate. $\square$

Proposition 2 *: Let $\Gamma \in UnConn$ and let $a \in \mathbb{R}$ be such that V_a intersects Γ. Then, $\Gamma \cap V_a^-$ has at least one unbounded connected component, which intersects V_a.*

Proof:
See the appendix. $\square$

Before going on, let us define the sets $\Gamma_{a,down}$ and $\Gamma_{a,up}$ as follows:

$$\Gamma_{a,down} \text{ (resp. } \Gamma_{a,up}) \text{ is the connected component of } \Gamma^c \cap]-\infty, a] \times [0,1]$$
$$\text{that contains }]-\infty, a[\times\{0\} \text{ (resp. }]-\infty, a[\times\{1\}).$$

If $\Gamma \in UnConn$ intersects some vertical V_a, it is possible that $\Gamma \cap V_a^-$ has more than one unbounded connected component. We denote by

$$\left[\Gamma \cap V_a^-\right] = \text{union of all unbounded connected components of } \Gamma \cap V_a^-. \quad (4)$$

Proposition 3 *: Let $a, b \in \mathbb{R}$ be such that $b < a$ and let Γ be an element of $UnConn$, which intersects V_a. Then $\Gamma_{b,down} \subset \Gamma_{a,down}$ and $\Gamma_{b,up} \subset \Gamma_{a,up}$.*

Proof:
See the appendix. $\square$

Let $\Gamma_1, \Gamma_2 \in UnConn$ be disjoint sets and let V_a be a vertical which intersects Γ_1.

Lemma 4 *: One and only one of the following possibilities must hold:*
$[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$ *or* $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$.

Proof:
See the appendix. $\Box$

The previous results will be used in what follows in order to define an order among disjoint elements of $UnConn$.

Let Γ_1 and Γ_2 be disjoint elements of $UnConn$ and let $a \in \mathbb{R}$ be such that Γ_1 and Γ_2 intersect V_a. We say that $\Gamma_2 \prec_a \Gamma_1$, if $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$.

Lemma 5 *: Given Γ_1, Γ_2 and $a \in \mathbb{R}$ as above, either $\Gamma_2 \prec_a \Gamma_1$ or $\Gamma_1 \prec_a \Gamma_2$.*

Proof:
See the appendix. $\Box$

Finally, in order to present a good definition of order, we need the following two lemmas:

Lemma 6 *: Let Γ_1 and Γ_2 be elements of $UnConn$ such that $\Gamma_1 \cap \Gamma_2 = \emptyset$ and let $a, b \in \mathbb{R}$ be such that Γ_1 and Γ_2 intersect V_a and V_b. Then we have the following:*

$$\begin{array}{ccc} \Gamma_1 \prec_a \Gamma_2 & \Leftrightarrow & \Gamma_1 \prec_b \Gamma_2 \\ \Gamma_2 \prec_a \Gamma_1 & \Leftrightarrow & \Gamma_2 \prec_b \Gamma_1 \end{array}$$

Proof:
See the appendix. $\Box$

So if Γ_1 and Γ_2 are disjoint elements of $UnConn$ and $a \in \mathbb{R}$ is such that $\Gamma_2 \prec_a \Gamma_1$, we simply say that $\Gamma_2 \prec \Gamma_1$.

Also, let us prove the following transitivity lemma:

Lemma 7 *: If $\Gamma_1, \Gamma_2, \Gamma_3$ are elements of $UnConn$ that do not intersect, such that $\Gamma_1 \prec \Gamma_2$ and $\Gamma_2 \prec \Gamma_3$, then $\Gamma_1 \prec \Gamma_3$.*

Proof:
See the appendix. $\Box$

Our next objective is to show that $\widetilde{f}^{\pm 1}$ preserves the order just defined.

Lemma 8 *: Let Γ_1, Γ_2 be disjoint elements of $UnConn$ and suppose $\Gamma_1 \prec \Gamma_2$. Then, $\widetilde{f}^{\pm 1}(\Gamma_1) \prec \widetilde{f}^{\pm 1}(\Gamma_2)$.*

Proof:
See the appendix. $\Box$

We now, for a fixed connected component Γ of D (see subsection 2.1), consider the covering mapping $p\,|_\Gamma$. It may or may not be injective. We examine the consequences in each case:

3.1 The covering mapping $p\mid_\Gamma$ is not injective

This means that there exists $\widetilde{z}\in\widetilde{A}$ and an integer $s>0$ such that $\widetilde{z},\widetilde{z}+(s,0)\in\Gamma$. So, $\Gamma\cap(\Gamma-(s,0))\neq\emptyset$. The last property of D tell us that $\Gamma-(s,0)\subset D$. But this implies that

$$\Gamma-(s,0)\subset\Gamma, \tag{5}$$

because Γ is a connected component of D.

Suppose that $\Gamma-(1,0)$ is not contained in Γ. As $\Gamma-(1,0)\subset D$, we get that $(\Gamma-(1,0))\cap\Gamma=\emptyset$. As $\Gamma-(1,0)$ does not intersect $V_{m_\Gamma}=\{m_\Gamma\}\times[0,1]$, lemma 4 implies that either $\Gamma-(1,0)\subset\Gamma_{down}$ or $\Gamma-(1,0)\subset\Gamma_{up}$. Suppose it is contained in Γ_{up}.

Proposition 4 *: If $\Gamma-(1,0)\subset\Gamma_{up}$, then $\Gamma-(i,0)\subset\Gamma_{up}$ for all integers $i>1$.*

Proof:
See the appendix. □

Therefore, if the map $p\mid_\Gamma$ is not injective, then

$$\Gamma-(1,0)\subset\Gamma. \tag{6}$$

As we already said in the introduction, Γ will be called a non injective component.

3.2 The covering mapping $p\mid_\Gamma$ is injective

This implies that $\Gamma\cap(\Gamma+(s,0))=\emptyset$, for all integers $s\neq0$. In particular, $\Gamma\cap(\Gamma-(1,0))=\emptyset$ and we use this relation to describe the asymptotic behavior of $p(\Gamma)$ around the annulus.

Definition: We say that Γ is a down component of D if $\Gamma\prec\Gamma-(1,0)$ and, analogously, Γ is an up component if $\Gamma-(1,0)\prec\Gamma$.

Lemma 9 *: If $\Gamma\subset D$ is a down component, then $dist(\Gamma,\ \mathbb{R}\times\{1\})>0$ and analogously, if $\Gamma\subset D$ is an up component, then $dist(\Gamma,\ \mathbb{R}\times\{0\})>0$.*

Proof:
See the appendix. □

In both cases we will say that Γ is an injective component.

4 Proof of theorem 5

Let $\epsilon>0$ be such that for all $(\widetilde{x},\widetilde{y})\in\mathbb{R}\times\{[0,\epsilon]\cup[1-\epsilon,1]\}$, $p_1\circ\widetilde{f}(\tilde{x},\widetilde{y})>\tilde{x}+\sigma$, for a certain fixed $\sigma>0$. Such $\epsilon>0$ exists since $\widetilde{f}$ moves points in the boundary of $\widetilde{A}$ uniformily to the right.

As theorem 4 says that $\overline{p(B^-)} = A$, there exists a real b such that

$$B^- \cap \{b\} \times [0, \epsilon] \neq \emptyset.$$

In the following we will consider the "lowest" component of B^- in $\{b\} \times [0, \epsilon]$.

First, remember that as B^- is closed, there must be a $0 < \delta \leq \epsilon$ such that $(b, \delta) \in B^-$, and for all $0 \leq \tilde{y} < \delta, (b, \tilde{y}) \notin B^-$, that is, (b, δ) is the "lowest" point of B^- in $\{b\} \times [0, \epsilon]$. We denote by v the segment $\{b\} \times [0, \delta[$ and by Γ_1 the connected component of B^- that contains (b, δ).

Definition: Let Ω be the open connected component of $(\Gamma_1 \cup v)^c$, which contains $]-\infty, b[\times\{0\}$.

Proposition 5 *:* $\widetilde{f}(\Gamma_1) \cap v = \emptyset$.

Proof:
Immediate. □

Proposition 6 *: If* $\Gamma_1 \cap \widetilde{f}(\Gamma_1) = \emptyset$ *and* $\Gamma_1 \prec \widetilde{f}(\Gamma_1)$, *then* $\widetilde{f}(v) \cap \Gamma_1 = \emptyset$.

Proof:
As $\widetilde{f}(\Gamma_1) \subset B^-$, either $\Gamma_1 \supset \widetilde{f}(\Gamma_1)$ or $\Gamma_1 \cap \widetilde{f}(\Gamma_1) = \emptyset$. So, if $\Gamma_1 \prec \widetilde{f}(\Gamma_1)$, we get by lemma 8 that $\widetilde{f}^{-1}(\Gamma_1) \prec \Gamma_1$, so $\left[\widetilde{f}^{-1}(\Gamma_1) \cap V_b^-\right] \subset \Gamma_{1b,down}$.

As $closure(\Omega) \supset closure(\Gamma_{1b,down})$, we get that $\left[\widetilde{f}^{-1}(\Gamma_1) \cap V_b^-\right] \subset closure(\Omega)$. If $\widetilde{f}^{-1}(\Gamma_1) \cap v \neq \emptyset$, then consider an element $\Upsilon \in \left[\widetilde{f}^{-1}(\Gamma_1) \cap closure(\Omega)\right] =$ $\{$unbounded connected components of $\widetilde{f}^{-1}(\Gamma_1) \cap closure(\Omega)\}$. The connectivity of $\widetilde{f}^{-1}(\Gamma_1)$ and the fact that $\widetilde{f}^{-1}(\Gamma_1) \cap \Gamma_1 = \emptyset$ imply that Υ intersects v. As $\Upsilon \subset V^-$ and $\widetilde{f}(\Upsilon) \subset \Gamma_1$, we get that $\Upsilon \subset B^-$, something in contradiction with $B^- \cap v = \emptyset$. So $\widetilde{f}^{-1}(\Gamma_1) \cap v = \emptyset$. □

Lemma 10 *: Either* $\Gamma_1 \supset \widetilde{f}(\Gamma_1)$ *or* $\widetilde{f}(\Gamma_1) \prec \Gamma_1$, *that is,* $\tilde{f}(\Gamma_1)$ *is not above* Γ_1.

Proof:
Suppose $\Gamma_1 \prec \tilde{f}(\Gamma_1)$. Clearly, as $\partial\Omega \subset \Gamma_1 \cup v$, $\tilde{f}(\Gamma_1) \cap \Gamma_1 = \emptyset$, $v \cap \tilde{f}(v) = \emptyset$ and $\tilde{f}(\Gamma_1) \cap v = \tilde{f}(v) \cap \Gamma_1 = \emptyset$ (see the previous proposition), we obtain

$$\tilde{f}(\partial\Omega) \cap \Omega = \emptyset.$$

As $]-\infty, b[\times\{0\} \subset \tilde{f}(]-\infty, b[\times\{0\})$, we get that $\Omega \subset \tilde{f}(\Omega)$. But, since Ω is open and limited to the right, this contradicts the transitivity of $\widetilde{f}$. □

So, lemma 10 implies that either $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or $\tilde{f}(\Gamma_1) \prec \Gamma_1$. In order to analyze the two previous possibilities, we have to consider all possible "shapes" for Γ_1 :

1) Γ_1 is an injective down component;
2) Γ_1 is an injective up component;
3) Γ_1 is a non-injective component;

Now we will exclude some of the above possibilities, but before we present an useful proposition.

Proposition 7 *: If Γ is a connected component of B^-, injective or non-injective, which satisfies $\widetilde{f}(\Gamma) \cap \Gamma = \emptyset$ and $\widetilde{f}(\Gamma) \prec \Gamma$, then $\widetilde{f}^n(\Gamma) \cap \Gamma = \emptyset$ and $\widetilde{f}^n(\Gamma) \prec \Gamma$ for all integers $n > 0$.*

Proof:

By contradiction, suppose there exists some $n_0 > 1$ (the smallest one) such that $\widetilde{f}^{n_0}(\Gamma) \cap \Gamma \neq \emptyset$. This means that Γ, $\widetilde{f}(\Gamma), \widetilde{f}^2(\Gamma), ..., \widetilde{f}^{n_0-1}(\Gamma)$ are disjoint closed connected subsets of the strip $\widetilde{A}$, each of them having a connected complement and $\widetilde{f}^{n_0}(\Gamma) \subset \Gamma$. As $\widetilde{f}(\Gamma_1) \prec \Gamma_1$, lemmas 7 and 8 imply that

$$\widetilde{f}^{n_0-1}(\Gamma) \prec ... \prec \widetilde{f}^2(\Gamma) \prec \widetilde{f}(\Gamma) \prec \Gamma. \tag{7}$$

On the other hand, as $\widetilde{f}^{n_0}(\Gamma) \cap \widetilde{f}^{n_0-1}(\Gamma) = \emptyset$, lemma 8 implies that $\widetilde{f}^{n_0}(\Gamma) \prec \widetilde{f}^{n_0-1}(\Gamma)$. So, $\widetilde{f}^{n_0}(\Gamma) \cap \left(\widetilde{f}^{n_0-1}(\Gamma)\right)_{down}$ has an unbounded connected component. As $\widetilde{f}^{n_0}(\Gamma) \subset \Gamma$ and $\widetilde{f}^{n_0-1}(\Gamma) \cap \Gamma = \emptyset$, using lemma 4 we get that $\Gamma \prec \widetilde{f}^{n_0-1}(\Gamma)$, a contradiction with expression (7). So, $\widetilde{f}^n(\Gamma) \cap \Gamma = \emptyset$ for all integers $n > 0$. The other implication follows from lemma 8. $\square$

Lemma 11 *: Suppose that we are in case 1 or in case 3 and $\tilde{f}(\Gamma_1) \prec \Gamma_1$ (see the end of page 8). Then there exists a vertical $V_r = \{r\} \times [0, 1]$ and a sequence $n_i \overset{i\to\infty}{\to} \infty$ such that $\widetilde{f}^{n_i}(\Gamma_1) \cap V_r \neq 0$ for all i.*

Proof:

The proof is naturally divided into two parts.

- Suppose that case 1 holds;

As Γ_1 is a down component, $\Gamma_1 \prec \Gamma_1 - (1, 0)$. Lemma 8 tell us that

$$\widetilde{f}(\Gamma_1) \prec \widetilde{f}(\Gamma_1) - (1, 0). \tag{8}$$

A simple analysis shows that in both cases, $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or $\tilde{f}(\Gamma_1) \prec \Gamma_1$, we get that

$$\widetilde{f}(\Gamma_1) + (k, 0) \prec \Gamma_1 \tag{9}$$

for all integers $k > 0$. Moreover, as $v \cap B^- = \emptyset$, the following holds: $\widetilde{f}(\Gamma_1) \subset \Gamma_1 \cup \Omega$.

Our objective now is to show that for all integers $n > 0$ and $k > 0$, the following is true: $(\widetilde{f}^n(\Gamma_1) + (k, 0)) \cap \Gamma_1 = \emptyset$ and $\widetilde{f}^n(\Gamma_1) + (k, 0) \prec \Gamma_1$.

This clearly follows from the properties of the order and expression (9).

- Suppose that case 3 holds and $\tilde{f}(\Gamma_1) \prec \Gamma_1$;

As $\widetilde{f}(\Gamma_1) \prec \Gamma_1$, $closure(\Gamma_{1b,down}) \subset closure(\Omega)$ and $\partial\Omega \subset \Gamma_1 \cup v$ does not intersect $\widetilde{f}(\Gamma_1)$, we get that $\widetilde{f}(\Gamma_1) \subset \Omega$.

If for some integers $n_0 > 0$ and $k_0 > 0$, $(\widetilde{f}^{n_0}(\Gamma_1) + (k_0, 0)) \cap \Gamma_1 \neq \emptyset$, then $\widetilde{f}^{n_0}(\Gamma_1) \cap (\Gamma_1 - (k_0, 0)) \neq \emptyset \Rightarrow \widetilde{f}^{n_0}(\Gamma_1) \cap \Gamma_1 \neq \emptyset \Rightarrow \widetilde{f}^{n_0}(\Gamma_1) \subset \Gamma_1$, which, using proposition 7, implies that $\widetilde{f}(\Gamma_1) \subset \Gamma_1$, a contradiction. So, for all integers $n > 0$ and $k > 0$, as $\widetilde{f}^n(\Gamma_1) + (k, 0) \supset \widetilde{f}^n(\Gamma_1)$ and

$$\tilde{f}^n(\Gamma_1) \prec ... \prec \tilde{f}(\Gamma_1) \prec \Gamma_1 \text{ (see proposition 7),}$$

we get that $(\widetilde{f}^n(\Gamma_1) + (k, 0)) \cap \Gamma_1 = \emptyset$ and $\widetilde{f}^n(\Gamma_1) + (k, 0) \prec \Gamma_1$.

Now the proof continues in a similar way in both the injective and non injective cases: We only use that for all integers $n > 0$ and $k > 0$, the following is true:

$$(\widetilde{f}^n(\Gamma_1) + (k, 0)) \cap \Gamma_1 = \emptyset$$
$$\text{and}$$
$$\widetilde{f}^n(\Gamma_1) + (k, 0) \prec \Gamma_1$$

Let us fix some $k' > 0$ in a way that $\widetilde{f}(\Gamma_1) + (k', 0)$ intersects v. The reason why such a k' exists is the following: As $(\widetilde{f}(\Gamma_1) + (k, 0)) \cap \Gamma_1 = \emptyset$ and $\widetilde{f}(\Gamma_1) + (k, 0) \prec \Gamma_1$ for all integers $k > 0$, we get that $\left[(\widetilde{f}(\Gamma_1) + (k, 0)) \cap V_b^-\right] \subset closure(\Gamma_{1b,down}) \subset closure(\Omega)$. And as $\partial\Omega \subset \Gamma_1 \cup v$ and $\overline{\Omega} \subset]-\infty, 0] \times [0, 1]$, we get that if $k' > 0$ is sufficiently large in a way that $\widetilde{f}(\Gamma_1) + (k', 0)$ intersects $\{1\} \times [0, 1]$, then $\widetilde{f}(\Gamma_1) + (k', 0)$ intersects the boundary of Ω in the only possible place, v. Denote by Γ^* an unbounded connected component of $(\widetilde{f}(\Gamma_1) + (k', 0)) \cap closure(\Omega)$.

By the choice of $k' > 0$ and the connectivity of $\widetilde{f}(\Gamma_1) + (k', 0)$, we get that Γ^* is not contained in B^- because it intersects v. So, there exists a positive integer $a_1 > 0$ such that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0, +\infty[\times[0, 1]$. Remember that $\widetilde{f}^{a_1}(\Gamma^*) \subset \widetilde{f}^{a_1+1}(\Gamma_1) + (k', 0) \prec \Gamma_1$ and so $\widetilde{f}^{a_1}(\Gamma^*) \prec \Gamma_1$. As above, the fact that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0, +\infty[\times[0, 1]$ implies that $\widetilde{f}^{a_1}(\Gamma^*) \cap closure(\Omega)$ has an unbounded connected component, Γ^{**} which intersects v. So, Γ^{**} is not contained in B^- and thus there exists an integer $a_2 > 0$ such that $\widetilde{f}^{a_2}(\Gamma^{**}) \subset \widetilde{f}^{a_2+a_1+1}(\Gamma_1) + (k', 0)$ intersects $]0, +\infty[\times[0, 1]$. In exactly the same way as above, we obtain an unbounded connected component of $\widetilde{f}^{a_2}(\Gamma^{**}) \cap closure(\Omega)$, denoted Γ^{***} which intersects v. So, Γ^{***} is not contained in B^- and there exists an integer $a_3 > 0$ such that $\widetilde{f}^{a_3}(\Gamma^{***})$ intersects $]0, +\infty[\times[0, 1]$ and so on.

Thus, if we define $n_i = a_1 + a_2 + ... + a_i + 1$, we get that $n_i \overset{i\to\infty}{\to} \infty$ and for all $i \geq 1$, $\widetilde{f}^{n_i-1}(\widetilde{f}(\Gamma_1) + (k', 0)) \supset \widetilde{f}^{n_i-1}(\Gamma^*) = \widetilde{f}^{a_2+...+a_i}(\widetilde{f}^{a_1}(\Gamma^*)) \supset \widetilde{f}^{a_2+...+a_i}(\Gamma^{**}) \supset ... \supset \widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*...*}})$ and $\widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*...*}})$ intersects $]0, +\infty[\times[0, 1]$. So,

$$\widetilde{f}^{n_i}(\Gamma_1) \text{ intersects } V_0 - (k', 0) = V_{-k'}$$

and the lemma is proved. $\square$

Thus, in case 1 and in case 3 if $\tilde{f}(\Gamma_1) \prec \Gamma_1$, lemma 11 implies that $\omega(B^-)$ is not empty. And this is a contradiction with theorem 2. So, either Γ_1 is an injective up component or Γ_1 is an non-injective component and $\tilde{f}(\Gamma_1) \subset \Gamma_1$. In the remainder of this section, we show that it is not possible for Γ_1 to be an injective up component.

Lemma 12 *: If Γ_1 is an injective up component, then $dist(\Gamma_1, \mathbb{R} \times \{1\}) = 0$.*

Proof:
As Γ_1 is an injective up component, lemma 9 implies that

$$dist(\Gamma_1, \mathbb{R} \times \{0\}) > 0.$$

So, if by contradiction we suppose that $dist(\Gamma_1, \mathbb{R}\times\{1\}) > 0$, there exists $\epsilon_1 > 0$ such that $\Gamma_1 \cap \mathbb{R} \times \{[0, \epsilon_1] \cup [1 - \epsilon_1, 1]\} = \emptyset$.

Since $\tilde{f}$ is transitive, f is transitive and thus there is a point $z \in S^1 \times [1 - \epsilon_1/2, 1]$ and an integer $n > 0$ such that $f^{-n}(z) \in S^1 \times [0, \epsilon_1/2]$. We know that $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or $\tilde{f}(\Gamma_1) \prec \Gamma_1$, so by proposition 7 and lemmas 7 and 8 we get that

$$\tilde{f}^n(\Gamma_1) \subset \Gamma_1 \ or \ \ \tilde{f}^n(\Gamma_1) \prec \Gamma_1.$$

Now let $d \in \mathbb{R}$ be such that $\widetilde{f}^{-i}(V_d) \subset V^-_{m_{\Gamma_1}-1}$ (m_{Γ_1} is defined in expression (3)) for $i = 0, 1, ..., n$ and $\tilde{f}^n(\Gamma_1) \cap V^-_d \subset \Gamma_1 \cup \Gamma_{1\ down}$, see proposition 10 of the appendix.

Let $\widetilde{z} \in V^-_d$ be a point such that $p(\widetilde{z}) = z$ and let k be the vertical line segment that has as extremes $\widetilde{z}$ and a point $\widetilde{z}_1$ in $\mathbb{R} \times \{1\}$.

As $\widetilde{f}^{-n}(\widetilde{z}) \in \mathbb{R} \times [0, \epsilon_1/2] \cap V^-_{m_{\Gamma_1}-1}$, we obtain that $\widetilde{f}^{-n}(\widetilde{z}) \in \Gamma_{1\ down}$. As $\widetilde{f}^{-n}(V_d) \subset V^-_{m_{\Gamma_1}-1}$, we get that $\widetilde{f}^{-n}(k) \cap V_{m_{\Gamma_1}} = \emptyset$. Since $\widetilde{f}^{-n}(\widetilde{z}_1) \notin \Gamma_{1\ down}$ and $\tilde{f}^{-n}(\tilde{z}) \in \Gamma_{1\ down}$, and since k is connected, $\tilde{f}^{-n}(k) \cap \partial(\Gamma_{1\ down}) \neq \emptyset$. But $\partial(\Gamma_{1\ down}) \subset \Gamma_1 \cup V_{m_{\Gamma_1}}$, so $\widetilde{f}^{-n}(k) \cap \Gamma_1 \neq \emptyset$, which implies that $k \cap \widetilde{f}^n(\Gamma_1) \neq \emptyset$ and this is a contradiction because $k \subset V^-_d \cap \Gamma_{1\ up}$ and $\tilde{f}^n(\Gamma_1) \cap V^-_d \subset \Gamma_1 \cup \Gamma_{1\ down}$. $\Box$

Thus if Γ_1 is an injective up component, then $\overline{p(\Gamma_1)}$ intersects $S^1 \times \{1\}$. So consider a real c such that

$$\Gamma_1 \cap \{c\} \times [1 - \epsilon, 1] \neq \emptyset,$$

where $\epsilon > 0$ was defined in the beginning of this section. As we did before, as B^- is closed, there must be a $0 < \mu \leq \epsilon$ such that $(c, 1 - \mu) \in B^-$, and for all $1 - \mu \leq \tilde{y} < 1, (c, \tilde{y}) \notin B^-$, that is, $(c, 1 - \mu)$ is the "highest" point of B^- in $\{c\} \times [1 - \epsilon, 1]$. We denote by w the segment $\{c\}\times]1 - \mu, 1]$.

Let Γ_2 be the connected component of B^- that contains $(c, 1 - \mu)$. An argument analogous to the one which implies that Γ_1 can not be an injective down component implies that Γ_2 can not be an injective up component, so if Γ_1 is injective up, $\Gamma_1 \neq \Gamma_2$ and thus $\Gamma_1\cap\Gamma_2 = \emptyset$. In this way, Γ_2 is either non-injective or an injective down component. In the second case, as $dist(\Gamma_2, \mathbb{R} \times \{1\}) > 0$ (see lemma 9), it is not possible that $\Gamma_1 \prec \Gamma_2$. But this implies that $\Gamma_2 \prec \Gamma_1$ and so Γ_2 intersects v, which is a contradiction with the definition of v. So Γ_2 is a non-injective component. By exactly the same reasoning applied to Γ_1, we must have $\tilde{f}(\Gamma_2) \subset \Gamma_2$.

The following lemma concludes the proof of theorem 5, because either Γ_1 is a non-injective component and $\tilde{f}(\Gamma_1) \subset \Gamma_1$ or, in case Γ_1 is an injective up component, Γ_2 is non-injective and $\tilde{f}(\Gamma_2) \subset \Gamma_2$.

Lemma 13 *: If Γ is a non-injective component of B^- such that $\tilde{f}(\Gamma) \subset \Gamma$, then $\overline{p(\Gamma)} = A$.*

Proof:

First of all, note that the set Γ has all the properties required for the set D in subsection 2.1, so lemma 1 implies that either $\overline{p(\Gamma)} \supset S^1 \times \{0\}$ or $\overline{p(\Gamma)} \supset S^1 \times \{1\}$. Let us suppose, without loss of generality, that

$$\overline{p(\Gamma)} \supset S^1 \times \{0\}.$$

Lemma 2 shows that, if $p(\Gamma)$ is not dense in A, then there exists a simple closed curve $\gamma \subset interior(A)$, which is homotopically non trivial and such that $\overline{p(\Gamma)} \cap \gamma = \emptyset$. But since $p(\Gamma)$ is connected, we must have $\Gamma \subset p^{-1}(\gamma^-)$, where γ^- is the connected component of γ^c which contains $S^1 \times \{0\}$.

As Γ is closed and $S^1 \times \{0\} \subset \overline{p(\Gamma)}$, we can find a point $(c', \delta') \in \Gamma$ such that:

1. $\delta' < \epsilon$, where $\epsilon > 0$ was defined in the beginning of this section;
2. if $v' = c' \times [0, \delta'[$, then $\Gamma \cap v' = \emptyset$ and $\overline{v'} \subset p^{-1}(\gamma^-)$.

Now, let us choose Ω' as the connected component of $(\Gamma \cup v')^c$ that contains $]-\infty, c'[\times\{0\}$ and consider the following set:

$$\Omega_{sat} = \overset{\infty}{\underset{n=0}{\cup}} \widetilde{f}^{-n}(\Omega')$$

First note that, as the boundary of Ω' is contained in $\Gamma \cup v'$, for all integers $n > 0$ we have:

$$\partial\left(\widetilde{f}^{-n}(\Omega')\right) \subset \left(\overset{\infty}{\underset{i=0}{\cup}} \widetilde{f}^{-i}(\Gamma)\right) \cup \left(\overset{\infty}{\underset{i=0}{\cup}} \widetilde{f}^{-i}(v')\right) \tag{10}$$

Clearly Ω_{sat} is an open set. Let us show that it is connected. Each set of the form $\widetilde{f}^{-i}(\Omega')$ is connected because $\widetilde{f}$ is a homeomorphism. Also, since $\widetilde{f}^{-i}(]-\infty, c'[\times\{0\}) \subset]-\infty, c'[\times\{0\}$, we have $\widetilde{f}^{-i}(\Omega') \cap \Omega' \neq \emptyset$. But Ω' is also open and connected, so Ω_{sat} must be connected.

For all integers $n > 0$, as $\widetilde{f}^{-n}(\Omega')$ is connected, intersects $\mathbb{R} \times \{0\}$ and is disjoint from $\mathbb{R} \times \{1\}$, if we show that $\left(\overset{\infty}{\underset{i=0}{\cup}} \widetilde{f}^{-i}(\Gamma)\right) \cup \left(\overset{\infty}{\underset{i=0}{\cup}} \widetilde{f}^{-i}(v')\right) \subset p^{-1}(\gamma^-)$, then expression (10) implies that $\widetilde{f}^{-n}(\Omega') \subset p^{-1}(\gamma^-)$, which gives: $\Omega_{sat} \subset p^{-1}(\gamma^-)$ and the proof is complete.

Let us prove that $\widetilde{f}^{-i}(\Gamma) \subset p^{-1}(\gamma^-)$ for all integers $i > 0$. This follows from the following:

Proposition 8 *: There exists an integer $k > 0$ such that $\widetilde{f}^{-1}(\Gamma) \subset \Gamma + (k, 0)$.*

Proof:

Since $\tilde{f}(\Gamma) \subset \Gamma$, we get $\Gamma \subset \tilde{f}^{-1}(\Gamma)$. As $\Gamma \subset V^{-}$, $\tilde{f}^{-1}(\Gamma)$ is limited to the right, so there exists an integer $k > 0$ such that $\tilde{f}^{-1}(\Gamma) - (k, 0) \subset V^{-}$. If $i \geq 1$,

$$\tilde{f}^{i}\left(\tilde{f}^{-1}(\Gamma) - (k, 0)\right) = \tilde{f}^{i-1}(\Gamma) - (k, 0) \subset \Gamma - (k, 0) \subset \Gamma.$$

So $\tilde{f}^{-1}(\Gamma) - (k, 0)$ is contained in B^{-}. As $\Gamma \subset \tilde{f}^{-1}(\Gamma) \Rightarrow \Gamma - (k, 0) \subset \tilde{f}^{-1}(\Gamma) - (k, 0)$. So, $\tilde{f}^{-1}(\Gamma) - (k, 0)$ intersects Γ because $\Gamma \supset \Gamma - (k, 0)$, therefore $\tilde{f}^{-1}(\Gamma) - (k, 0)$ (which is clearly connected) is contained in Γ. $\square$

The above proposition implies that for any integer $i \geq 1$, $\widetilde{f}^{-i}(\Gamma) \subset \Gamma + (i.k, 0) \subset p^{-1}(\gamma^{-})$.

We are left to deal with $\overset{\infty}{\underset{i=0}{\cup}} \widetilde{f}^{-i}(v')$. Let us show that $\widetilde{f}^{-1}(v') \subset \Omega'$. From the choice of v', $\widetilde{f}^{-1}(v') \cap v' = \emptyset$ and $\widetilde{f}^{-1}(v') \cap \Gamma = \emptyset$ because $v' \cap \Gamma = \emptyset$ and $\widetilde{f}(\Gamma) \subset \Gamma$. Finally, the following inclusions

$$\Omega' \supset]-\infty, c'[\times\{0\} \text{ and }]-\infty, c'[\times\{0\} \supset \widetilde{f}^{-1}(]-\infty, c'[\times\{0\})$$

imply that $\widetilde{f}^{-1}(v') \cap \Omega' \neq \emptyset$ and so $\widetilde{f}^{-1}(v') \subset \Omega' \subset p^{-1}(\gamma^{-})$.

Thus, $\widetilde{f}^{-2}(v') \subset \widetilde{f}^{-1}(\Omega')$, whose boundary, $\partial\left(\widetilde{f}^{-1}(\Omega')\right)$, is contained in $\widetilde{f}^{-1}(\Gamma) \cup \widetilde{f}^{-1}(v') \subset p^{-1}(\gamma^{-})$. As above, $\widetilde{f}^{-1}(\Omega')$ is connected, intersects $\mathbb{R} \times \{0\}$ and is disjoint from $\mathbb{R} \times \{1\}$, so we get that

$$\widetilde{f}^{-2}(v') \subset \widetilde{f}^{-1}(\Omega') \subset p^{-1}(\gamma^{-}).$$

Thus, $\widetilde{f}^{-3}(v') \subset \widetilde{f}^{-2}(\Omega')$ and an analogous argument implies that $\widetilde{f}^{-3}(v') \subset \widetilde{f}^{-2}(\Omega') \subset p^{-1}(\gamma^{-})$. An induction shows that

$$\widetilde{f}^{-n}(v') \subset \widetilde{f}^{-n+1}(\Omega') \subset p^{-1}(\gamma^{-}) \text{ for all integers } n \geq 1,$$

and the lemma is proved because the above implies that $\tilde{f}^{-1}(\Omega_{sat}) \subset \Omega_{sat} \subset p^{-1}(\gamma^{-})$ and this contradicts the transitivity of $\tilde{f}$. $\square$

As we know that at least one element of the set $\{\Gamma_1, \Gamma_2\}$ is non-injective and positively invariant, the above lemma implies that Γ_1 or Γ_2 must have a dense projection to the annulus. One more thing can be said, which will be important in the proof of the next theorem:

Proposition 9 *: Both Γ_1 and Γ_2 are non-injective components.*

Proof:

If the proposition is not true for Γ_1, then as we already proved, Γ_1 must be an injective up component. As Γ_2 does not cross v, and Γ_1 does not cross w, by the definitions of v and w, it must be the case that $\Gamma_1 \prec \Gamma_2$ and so

$$dist(\Gamma_2, \mathbb{R} \times \{0\}) > 0,$$

something that contradicts lemma 13. So Γ_1 is a non-injective component. To conclude the proof we have to note that Γ_1 and Γ_2 have analogous properties, Γ_1 is the connected component of B^- that contains the lowest point of B^- in $\{b\} \times [0, \epsilon]$ and Γ_2 is the connected component of B^- that contains the "highest" point of B^- in $\{c\} \times [1-\epsilon, 1]$. So Γ_2 must also be a non-injective component. $\square$

Summarizing, the above results prove that for $\epsilon > 0$ satisfying the condition stated in the beginning of this section and for every vertical segment u of the form $\{l\}\times[0,\epsilon]$ (or $\{l\}\times[1-\epsilon,1]$) which intersects B^-, the "lowest" (or "highest") component of B^- in u must be non-injective, $\widetilde{f}$-positively invariant and dense when projected to A.

5 Proof of theorem 6

Without loss of generality, suppose $\Gamma \subset B^-$ is an injective down connected component. Consider a vertical $v = \{c\} \times [0, \epsilon[$, such that:

$$\begin{array}{l} 1)\ 0 < \epsilon \leq \epsilon',\ \text{where } \epsilon' \text{ is such that } \forall(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times [0, \epsilon'], \\ p_1 \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) > \widetilde{x} + \sigma,\ \text{for some } \sigma > 0; \\ 2)\ v \subset \Gamma_{down}; \\ 3)\ v \cap B^- \neq \emptyset. \end{array} \tag{11}$$

The above is possible because $\overline{p(B^-)} = A$ and so there exists a real $c < m_\Gamma$ such that $\{c\}\times[0,\epsilon'[\cap B^- \neq \emptyset$, where ϵ' comes from 1) of (11). As Γ is an injective component, from the previous theorem we get that the "lowest" component of B^- in $\{c\} \times [0, \epsilon'[$ can not be Γ. So, if $\{c\} \times [0, \epsilon'[$ do not intersect Γ, then let $v = \{c\} \times [0, \epsilon'[$, otherwise let $0 < \epsilon < \epsilon'$ be such that $v = \{c\} \times [0, \epsilon[\subset \Gamma_{down}$ and $(c, \epsilon) \in \Gamma$.

Denote by Θ the "lowest" connected component of B^- in v and by

$$w = \{c\} \times [0, \delta[\subset v$$

the vertical such that $w \cap B^- = \emptyset$ and $(c, \delta) \in \Theta$. From theorem 5 we know that Θ satisfies the following conditions:

$$\begin{array}{l} i)\ \Theta \text{ is non-injective;} \\ ii)\ \widetilde{f}(\Theta) \subset \Theta; \\ iii)\ \overline{p(\Theta)} = A; \end{array} \tag{12}$$

If $\Theta \prec \Gamma$, then proposition 10 implies the existence of a real number d such that $\Theta \cap V_d^- \subset \Gamma_{down}$. As $dist(\Gamma, \mathbb{R}\times\{1\}) > 0$, we get that $dist(\Theta, \mathbb{R}\times\{1\}) > 0$, something that contradicts property iii) of expression (12).

So we can assume that $\Gamma \prec \Theta$. As in some of the previous results, let Ω be the connected component of $(\Theta \cup w)^c$ that contains $]-\infty, c[\times\{0\}$. We know that

$$closure(\Omega) \supset closure(\Theta_{c,down}) \tag{13}$$

and, as $\Gamma \prec \Theta$, $[\Gamma \cap V_c^-] \subset \Theta_{c,down}$. So, using expression (13) we get that $\Gamma \subset \Omega$ because Γ is connected, $\Gamma \cap \Omega \neq \emptyset$ and $\Gamma \cap \partial\Omega \subset \Gamma \cap (\Theta \cup w) = \emptyset$. The rest of our proof will be divided in two steps:

Step 1: Here we are going to prove that for all integers $n > 0$ and $k \geq 0$, $\widetilde{f}^n(\Gamma) + (k,0)$ is disjoint from Θ and $\widetilde{f}^n(\Gamma) \subset \Omega \Rightarrow \widetilde{f}^n(\Gamma) + (k,0) \prec \Theta$.

As $\Gamma \prec \Theta$, we get for any integer $n > 0$, that either $\widetilde{f}^n(\Gamma) \subset \Theta$ or $\widetilde{f}^n(\Gamma) \cap \Theta = \emptyset$ and $\widetilde{f}^n(\Gamma) \prec \Theta$ (because $\widetilde{f}(\Theta) \subset \Theta$), which implies that $\widetilde{f}^n(\Gamma) \subset \Omega$. To begin, suppose $\widetilde{f}(\Gamma) \subset \Theta$. This means that $\widetilde{f}^{-1}(\Theta) \supset \Gamma$ and so, if $\widetilde{f}^{-1}(\Theta) \cap V = \emptyset$ (remember that $V = \{0\} \times [0,1]$), then $\widetilde{f}^{-1}(\Theta)$ is contained in a connected component of B^-, that is, $\widetilde{f}^{-1}(\Theta) = \Gamma$, a contradiction because Γ is injective and Θ is not. So, $\widetilde{f}^{-1}(\Theta) \cap V \neq \emptyset$. Let Γ' be the connected component of $\widetilde{f}^{-1}(\Theta) \cap V^-$ that contains Γ. The fact that $\widetilde{f}^{-1}(\Theta)$ is connected implies that Γ' intersects V, is contained in B^- and contains Γ. So, $\Gamma' = \Gamma$ and this is a contradiction because $\Gamma \subset \Omega$ and $\Omega \cap V = \emptyset$. So, $\widetilde{f}(\Gamma) \cap \Theta = \emptyset \Rightarrow \widetilde{f}(\Gamma) \prec \Theta \Rightarrow \widetilde{f}(\Gamma) \subset \Omega$.

Now note that $\widetilde{f}(\Gamma)$ is itself a connected component of B^-. In order to prove the previous assertion, denote by Π the connected component of B^- that contains $\widetilde{f}(\Gamma)$. If $\widetilde{f}(\Gamma) \neq \Pi$, then $\widetilde{f}^{-1}(\Pi) \supset \Gamma$ is not a connected component of B^-, so $\widetilde{f}^{-1}(\Pi) \cap V \neq \emptyset$, which means that $\widetilde{f}^{-1}(\Pi) \cap w \neq \emptyset$ because $\widetilde{f}^{-1}(\Pi) \cap \Theta = \emptyset$ and $\widetilde{f}^{-1}(\Pi) \cap \Omega \neq \emptyset$. If we denote by Γ^* the connected component of $\widetilde{f}^{-1}(\Pi) \cap \Omega$ that contains Γ, then as $\widetilde{f}^{-1}(\Pi)$ is connected, Γ^* intersects w and is contained in B^-, a contradiction.

So, $\widetilde{f}(\Gamma) = \Pi \subset \Omega$ and an induction using the above argument implies that for every integer $n > 0$:

1) $\widetilde{f}^n(\Gamma) \cap \Theta = \emptyset$;
2) $\widetilde{f}^n(\Gamma)$ is a connected component of B^-;
3) $\widetilde{f}^n(\Gamma) \subset \Omega$;
4) $\widetilde{f}^n(\Gamma) \prec \Theta$;

As $\Gamma \subset B^-$ is an injective down connected component, the same holds for $\widetilde{f}^n(\Gamma)$ (for any integer $n > 0$). So the assertion from step 1 holds.

Step 2: Here we perform the same construction as we did in lemma 11, see it for more details.

Let us fix some $k' > 0$ in a way that $\widetilde{f}(\Gamma) + (k',0)$ intersects w. Denote by Γ^* an unbounded connected component of $(\widetilde{f}(\Gamma) + (k',0)) \cap closure(\Omega)$. By the choice of $k' > 0$ and the connectivity of $\widetilde{f}(\Gamma) + (k',0)$, we get that Γ^* is not contained in B^- because it intersects w. So, there exists a positive integer $a_1 > 0$ such that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0,+\infty[\times[0,1]$.

Step 1 implies that $\widetilde{f}^{a_1}(\widetilde{f}(\Gamma) + (k',0)) = \widetilde{f}^{a_1+1}(\Gamma) + (k',0)$ does not intersect Θ and is smaller then it the order $\prec$. So, as

$$\Gamma^* \subset \widetilde{f}(\Gamma) + (k',0) \;\Rightarrow\; \widetilde{f}^{a_1}(\Gamma^*) \cap \Theta = \emptyset \text{ and } \widetilde{f}^{a_1}(\Gamma^*) \prec \Theta.$$

The fact that $\widetilde{f}^{a_1}(\Gamma^*)$ intersects $]0,+\infty[\times[0,1]$ implies that $\widetilde{f}^{a_1}(\Gamma^*)\cap closure(\Omega)$ has an unbounded connected component, Γ^{**} which intersects w. So, Γ^{**} is not contained in B^- and thus there exists an integer $a_2>0$ such that $\widetilde{f}^{a_2}(\Gamma^{**})\subset \widetilde{f}^{a_2+a_1+1}(\Gamma)+(k',0)$ intersects $]0,+\infty[\times[0,1]$. In exactly the same way as above, we obtain an unbounded connected component of $\widetilde{f}^{a_2}(\Gamma^{**})\cap closure(\Omega)$, denoted Γ^{***} which intersects w. So, Γ^{***} is not contained in B^- and there exists an integer $a_3>0$ such that $\widetilde{f}^{a_3}(\Gamma^{***})$ intersects $]0,+\infty[\times[0,1]$ and so on.

Thus, if we define $n_i=a_1+a_2+...+a_i+1$, we get that $n_i \overset{i\to\infty}{\to} \infty$ and for all $i\geq 1$, $\widetilde{f}^{n_i-1}(\widetilde{f}(\Gamma)+(k',0))\supset \widetilde{f}^{n_i-1}(\Gamma^*)=\widetilde{f}^{a_2+...+a_i}(\widetilde{f}^{a_1}(\Gamma^*))\supset \widetilde{f}^{a_2+...+a_i}(\Gamma^{**})\supset ...\supset \widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*...*}})$ and $\widetilde{f}^{a_i}(\Gamma^{\overset{i-times}{*...*}})$ intersects $]0,+\infty[\times[0,1]$. So,

$$\widetilde{f}^{n_i}(\Gamma) \text{ intersects } V-(k',0)=V_{-k'}$$

and this contradicts theorem 2 and thus proves theorem 6.

6 Appendix

Here we prove the results from section 3 used to define the order between elements of $UnConn$.

Proof of proposition 2:

As we did in [1], let us consider the L,R-compactification of $\widetilde{A}=\mathbb{R}\times[0,1]$, denoted by $\widehat{A}$ (we compactify $\widetilde{A}$ by adding two points to it, L and R, the left and right ends, respectively, getting a closed disk). For every object (point, set, etc) in $\widetilde{A}$, we denote the corresponding object in $\widehat{A}$ by putting a $\widehat{}$ on it.

Let $z_n\in\Gamma\cap V_a^-$ be a sequence such that $p_1(z_n)\overset{n\to\infty}{\to}-\infty$, or equivalently, $\widehat{A}\ni\widehat{z}_n\overset{n\to\infty}{\to}L$.

Note that $\widehat{\Gamma}$ is connected, it intersects $\widehat{V}_a$ and contains L, so each $\widehat{z}_n$ belongs to a connected component of $\widehat{\Gamma}\cap\widehat{V}_a^-$, denoted $\widehat{\Gamma}_n$, which intersects $\widehat{V}_a$. Let $\widehat{\Gamma}_{n_i}$ be a convergent subsequence in the Hausdorff topology, $\widehat{\Gamma}_{n_i}\overset{n\to\infty}{\to}\widehat{\Gamma}^*$. This means that, given any open neighborhood $\widehat{N}$ of $\widehat{\Gamma}^*$, for all sufficiently large i, $\widehat{\Gamma}_{n_i}$ is contained in $\widehat{N}$. So $\widehat{\Gamma}^*$ must contain L and must intersect $\widehat{V}_a$. Suppose that $\widehat{\Gamma}^*$ is not contained in $\widehat{\Gamma}$. This means that there exists $\widehat{P}\in\widehat{\Gamma}^*$, with $\widehat{P}\notin\widehat{\Gamma}$. As $\widehat{\Gamma}$ is closed, for some $\epsilon_0>0$, $B_{\epsilon_0}(\widehat{P})\cap\widehat{\Gamma}=\emptyset$, where $B_{\epsilon_0}(\widehat{P})=\{\widehat{z}\in\widehat{A}: d_{Euclidean}(\widehat{z},\widehat{P})<\epsilon_0\}$ and $d_{Euclidean}(\bullet,\bullet)$ is the usual Euclidean distance in $\widehat{A}$. But as $\widehat{\Gamma}_{n_i}\overset{n\to\infty}{\to}\widehat{\Gamma}^*$ in the Hausdorff topology, for all sufficiently large i, $\widehat{\Gamma}^*\subset(\epsilon_0/2-neighborhood\ of\ \widehat{\Gamma}_{n_i})$. Thus we get that $d_{Euclidean}(\widehat{P},\widehat{\Gamma})\leq d_{Euclidean}(\widehat{P},\widehat{\Gamma}_{n_i})<\epsilon_0/2$, something that contradicts the choice of $\widehat{P}\in\widehat{\Gamma}^*$. So $\widehat{\Gamma}^*\subset\widehat{\Gamma}$ and the proposition is proved because although Γ^* may not be connected, it must contain an unbounded connected component which intersects V_a. $\square$

Proof of proposition 3:

Let $z\in\Gamma_{b,down}$. This means that there exists a simple continuous arc θ which connects z to a point $z_0\in]-\infty,b[\times\{0\}$, $\theta\cap\Gamma=\emptyset$ and $\theta\subset\Gamma_{b,down}\subset]-\infty,b[\times[0,1]$. As $a>b$, $\theta\cap\partial\Gamma_{a,down}\subset\theta\cap\Gamma=\emptyset$. As $z_0\in\Gamma_{a,down}$, we get that $\theta \subset \Gamma_{a,down}$, which implies that $\Gamma_{b,down} \subset \Gamma_{a,down}$. The other inclusion is proved in a similar way. $\square$

Proof of lemma 4:

Without loss of generality, let us suppose that $\Gamma_1, \Gamma_2 \subset V^-$. We first prove that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down} \cup \Gamma_{1a,up}$. Suppose this is not the case. Then, there exists an unbounded connected component of $\Gamma_2 \cap V_a^-$, denoted Γ_2^*, contained in a connected component of $\Gamma_1^c \cap]-\infty, a[\times[0,1]$, different from $\Gamma_{1a,down}$ and $\Gamma_{1a,up}$. Denote this component by $\Gamma_{1a,mid}$. Fix some $P \in \Gamma_2^*$. As $P \notin \Gamma_1$, there exists $\epsilon > 0$ such that $B_\epsilon(P) \cap \Gamma_1 = \emptyset$. Now, let $\alpha' \subset \mathbb{R}\times]0,1[$ be a simple continuous arc which connects P to $(1, 0.5)$, totally contained in Γ_1^c, which is an open connected set that contains P and $(1, 0.5)$. Moreover, as $]0, +\infty[\times[0,1] \subset \Gamma_1^c$, we can take α' so that it does not intersect $]1, +\infty[\times\{0.5\}$. Finally, let α be a simple continuous arc given by $[1, +\infty[\times\{0.5\}$ plus a continuous part of α', whose endpoints are $(1, 0.5)$ and some point in Γ_2^*, so that $\alpha \cap \Gamma_2^*$ consists of only its end point (clearly, this end point may not be P).

Properties of $\alpha \cup \Gamma_2^*$:

- $\alpha \cup \Gamma_2^*$ is a closed, connected set, disjoint from $\mathbb{R} \times \{0, 1\}$;
- $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are in different connected components of $(\alpha \cup \Gamma_2^*)^c$;
- α is limited to the left, that is, there exists a number $M > 0$ such that, for all points $\widetilde{z}$ in α, $p_1(\widetilde{z}) > -M$;
- $(\alpha \cup \Gamma_2^*) \cap \Gamma_1 = \emptyset$;

Let us choose $b < a$ such that $\alpha \subset V_{b+1/2}^+$. By proposition 3, $\Gamma_{1b,down} \subset \Gamma_{1a,down}$ and $\Gamma_{1b,up} \subset \Gamma_{1a,up}$, so we get that $\Gamma_2^* \cap (\Gamma_{1b,down} \cup \Gamma_{1b,up}) = \emptyset$. Now let $\beta_0 \subset \Gamma_{1b,down}$ and $\beta_1 \subset \Gamma_{1b,up}$ be simple continuous arcs which satisfy the following:

- β_0 connects a point of $]-\infty, b[\times\{0\}$ to a point of Γ_1;
- β_1 connects a point of $]-\infty, b[\times\{1\}$ to a point of Γ_1;

So the following conditions hold

$$(\beta_0 \cup \beta_1) \cap \Gamma_2^* = \emptyset \text{ and } (\beta_0 \cup \beta_1) \subset V_b^- \;\Rightarrow\; (\beta_0 \cup \beta_1) \cap \alpha = \emptyset$$

and thus

$$(\beta_0 \cup \Gamma_1 \cup \beta_1) \cap (\alpha \cup \Gamma_2^*) = \emptyset,$$

something that contradicts the fact that $(\beta_0 \cup \Gamma_1 \cup \beta_1)$ is a closed connected set and the "**Properties of** $\alpha \cup \Gamma_2^*$" listed above. So, $[\Gamma_2 \cap V_a^-] \subset (\Gamma_{1a,down} \cup \Gamma_{1a,up})$.

Suppose now that for some $\Gamma_2^*, \Gamma_2^{**} \in [\Gamma_2 \cap V_a^-]$, we have $\Gamma_2^* \subset \Gamma_{1a,down}$ and $\Gamma_2^{**} \subset \Gamma_{1a,up}$. In the same way as above, there exists a simple continuous arc $\alpha \subset \mathbb{R}\times]0,1[$ which contains $[1, +\infty[\times\{0.5\}$ and connects some point of Γ_1 to

$(1, 0.5)$, in a way that $\alpha \subset \Gamma_2^c$ and α intersects Γ_1 only at its end point. Clearly, $(\alpha \cup \Gamma_1)$ is a closed connected set, which satisfies: $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are in different connected components of $(\alpha \cup \Gamma_1)^c$.

Again, as above, let us choose $b < a$ such that $\alpha \subset V_{b+1}^+$. Proposition 2 implies that $\left[\Gamma_2^* \cap V_b^-\right]$ and $\left[\Gamma_2^{**} \cap V_b^-\right]$ are non-empty. From what we did above, we get that $\left[\Gamma_2^* \cap V_b^-\right] \cup \left[\Gamma_2^{**} \cap V_b^-\right] \subset \Gamma_{1b,down} \cup \Gamma_{1b,up}$.

If, $\left[\Gamma_2^* \cap V_b^-\right] \cap \Gamma_{1b,up} \neq \emptyset \Rightarrow \Gamma_2^* \cap \Gamma_{1b,up} \neq \emptyset$, which implies, by proposition 3, that $\Gamma_2^* \cap \Gamma_{1a,up} \neq \emptyset$, a contradiction. So, $\left[\Gamma_2^* \cap V_b^-\right] \cap \Gamma_{1b,up} = \emptyset$ and a similar argument gives $\left[\Gamma_2^{**} \cap V_b^-\right] \cap \Gamma_{1b,down} = \emptyset$. So, $\left[\Gamma_2^* \cap V_b^-\right] \subset \Gamma_{1b,down}$ and $\left[\Gamma_2^{**} \cap V_b^-\right] \subset \Gamma_{1b,up}$. Thus, there exists a simple continuous arc β_0 contained in $\Gamma_{1b,down}$ which connects a point of Γ_2^* to some point in $]-\infty, b[\times\{0\}$. Similarly, there exists a simple continuous arc β_1 contained in $\Gamma_{1b,up}$ which connects a point of Γ_2^{**} to some point in $]-\infty, b[\times\{1\}$. But $(\beta_0 \cup \Gamma_2 \cup \beta_1)$ is a closed connected set and by construction of β_0 and β_1,

$$(\beta_0 \cup \Gamma_2 \cup \beta_1) \cap (\alpha \cup \Gamma_1) = \emptyset,$$

which is a contradiction, completing the proof of the lemma. □

Proof of lemma 5:

As in the previous lemma, let us suppose that $\Gamma_1, \Gamma_2 \subset V^-$. From lemma 4, either $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$ or $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$. In the first possibility, $\Gamma_2 \prec_a \Gamma_1$. So we are left to show that, if $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$, then $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$, which means that $\Gamma_1 \prec_a \Gamma_2$.

Thus, let us suppose that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$ and $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,up}$. If we arrive at a contradiction, the lemma will be proved.

The argument here is very similar to the one used in the proof of lemma 4. First, choose a simple continuous arc $\alpha \subset \mathbb{R}\times]0,1[$ which contains $[1,+\infty[\times\{0.5\}$ and connects some point of Γ_1 to $(1, 0.5)$, in a way that $\alpha \subset \Gamma_2^c$ and α intersects Γ_1 only at its end point. Clearly, $(\alpha \cup \Gamma_1)$ is a closed connected set, which satisfies: $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ are in different connected components of $(\alpha \cup \Gamma_1)^c$.

As $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$, there exists an element of $[\Gamma_2 \cap V_a^-]$, denoted Γ_2^*, which by definition is closed, connected, unbounded to the left and is contained in $\Gamma_{1a,up}$. Again, let us choose $b < a$ such that $\alpha \subset V_{b+1}^+$.

As in the end of the proof of lemma 4, we get that $\left[\Gamma_2^* \cap V_b^-\right] \subset \Gamma_{1b,up}$. So, there exists a simple continuous arc β_1 contained in $\Gamma_{1b,up}$ which connects a point of Γ_2^* to some point in $]-\infty, b[\times\{1\}$. Clearly, $\beta_1 \cap \Gamma_1 = \emptyset$.

As $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,up}$, an argument similar to the one used to prove proposition 2 implies:

Proposition 10 *: There exists a real number $c \leq b$, such that $(\Gamma_1 \cap V_c^-) \cap \Gamma_{2a,down} = \emptyset$.*

Proof:

Suppose by contradiction, that the proposition is not true. Then, there is a sequence of points $z_n \in \Gamma_1 \cap \Gamma_{2a,down}$, such that $p_1(z_n) \overset{n\to\infty}{\to} -\infty$, or equivalently,

$\widehat{A} \ni \widehat{z}_n \overset{n\to\infty}{\to} L$. Now, the proof goes exactly as in proposition 2 and we thus obtain an unbounded connected component of $\Gamma_1 \cap \Gamma_{2a,down}$, a contradiction with $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,up}$. $\Box$

Now let us look at $\Gamma_{2c,down} \subset \Gamma_{2b,down}$, where c comes from proposition 10. Clearly, $\Gamma_1 \cap \Gamma_{2c,down} = \emptyset$. So, there exists a simple continuous arc β_0 which connects a point of Γ_2 to some point in $]-\infty, c[\times\{0\}$, in a way that $\beta_0 \cap \Gamma_2$ is one extreme of β_0, denoted m_0, and $\beta_0 \backslash \{m_0\} \subset \Gamma_{2c,down}$, which implies that $\beta_0 \cap \Gamma_1 = \emptyset$ and $\beta_0 \cap \alpha = \emptyset$. So, $(\beta_0 \cup \Gamma_2 \cup \beta_1)$ is a closed connected set, which intersects $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. And, by construction

$$(\beta_0 \cup \Gamma_2 \cup \beta_1) \cap (\alpha \cup \Gamma_1) = \emptyset,$$

a contradiction. So if $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$, then $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$, which implies that $\Gamma_1 \prec_a \Gamma_2$ and the lemma is proved. $\Box$

Proof of lemma 6:

Suppose that $b < a$ and $\Gamma_2 \prec_a \Gamma_1 \Leftrightarrow [\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,down}$. Proposition 10 tells us that $(\Gamma_2 \cap V_a^-) \cap \Gamma_{1a,up}$ is a limited set. So, as $\Gamma_{1b,up} \subset \Gamma_{1a,up}$, $\left[\Gamma_2 \cap V_b^-\right]$ must be contained in $\Gamma_{1b,down}$, which means that $\Gamma_2 \prec_b \Gamma_1$. The other implications are proved in a similar way. $\Box$

Proof of lemma 7:

Let $a \in \mathbb{R}$ be such that Γ_1, Γ_2 and Γ_3 intersect V_a. Then, $[\Gamma_1 \cap V_a^-] \subset \Gamma_{2a,down}$ and $[\Gamma_2 \cap V_a^-] \subset \Gamma_{3a,down}$. In the proof of lemma 5, we proved that if Θ and Λ are disjoint elements of $UnConn$ and $a \in \mathbb{R}$ is such that Θ and Λ intersect V_a then, $[\Theta \cap V_a^-] \subset \Lambda_{a,down}$ implies $[\Lambda \cap V_a^-] \subset \Theta_{a,up}$. So, $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$ and $[\Gamma_3 \cap V_a^-] \subset \Gamma_{2a,up}$. Now, using proposition 10, let us choose $b \leq a$ such that the following inclusions hold:

$$\begin{array}{l} \Gamma_3 \cap V_b^- \subset \Gamma_{2b,up} \\ \Gamma_1 \cap V_b^- \subset \Gamma_{2b,down} \\ \Gamma_2 \cap V_b^- \subset \Gamma_{3b,down} \end{array} \tag{14}$$

Finally, let us prove that $\Gamma_{2b,down} \subset \Gamma_{3b,down}$.

If this is not the case, then there exists a simple continuous arc $\alpha \subset \Gamma_{2b,down}$ that connects a point from $]-\infty, b[\times\{0\}$ to a point $P \notin \Gamma_{3b,down}$. Thus α intersects Γ_3, a contradiction with expression (14). So, $\Gamma_1 \cap V_b^- \subset \Gamma_{2b,down} \subset \Gamma_{3b,down}$, which implies that $\left[\Gamma_1 \cap V_b^-\right] \subset \Gamma_{3b,down} \Leftrightarrow \Gamma_1 \prec \Gamma_3$ and the lemma is proved. $\Box$

Proof of lemma 8:

Suppose that $\widetilde{f}(\Gamma_2) \prec \widetilde{f}(\Gamma_1)$. As $\Gamma_1 \prec \Gamma_2$, for any $a \in \mathbb{R}$ such that Γ_1 and Γ_2 intersect V_a, the proof of lemma 5 implies that $[\Gamma_2 \cap V_a^-] \subset \Gamma_{1a,up}$. From proposition 10, there exists a sufficiently small $b < 0$ such that:

$$\begin{array}{l} \Gamma_1 \cap V_b^- \subset \Gamma_{2b,down} \\ \Gamma_2 \cap V_b^- \subset \Gamma_{1b,up} \\ \widetilde{f}(\Gamma_2) \cap V_b^- \subset \widetilde{f}(\Gamma_1)_{b,down} \end{array} \tag{15}$$

Let $c < b$ be such that $\widetilde{f}^{\pm 1}(V_c) \cap V_b = \emptyset$. From our previous results, we get that

$$\begin{array}{l} \Gamma_1 \cap V_c^- \subset \Gamma_{2c,down} \\ \Gamma_2 \cap V_c^- \subset \Gamma_{1c,up} \\ \widetilde{f}(\Gamma_2) \cap V_c^- \subset \widetilde{f}(\Gamma_1)_{c,down} \end{array}.$$

So, there exists a simple continuous arc $\alpha \subset \widetilde{f}(\Gamma_1)_{c,down} \subset V_c^-$ that connects a point from $]-\infty, c[\times\{0\}$ to a point $P \in \widetilde{f}(\Gamma_2)$. From the choice of c, $\widetilde{f}^{-1}(\alpha) \subset V_b^-$ and it connects a point from $]-\infty, b[\times\{0\}$ to $\widetilde{f}^{-1}(P) \in \Gamma_2$. As $\alpha \subset \widetilde{f}(\Gamma_1)_{c,down}$, $\alpha \cap \widetilde{f}(\Gamma_1) = \emptyset$, so $\widetilde{f}^{-1}(\alpha) \cap \Gamma_1 = \emptyset$. Thus, $\widetilde{f}^{-1}(\alpha) \subset \Gamma_{1b,down}$, which implies that $\Gamma_2 \cap \Gamma_{1b,down} \neq \emptyset$ and this contradicts (15). So, $\widetilde{f}(\Gamma_1) \prec \widetilde{f}(\Gamma_2)$. The other implication is proved in an analogous way. $\square$

Proof of proposition 4:

Suppose there exists $s_0 > 1$ (the smallest one) such that $\Gamma - (s_0, 0) \subset \Gamma$. This means that $\Gamma, \Gamma - (1,0), ..., \Gamma - (s_0 - 1, 0)$ are all disjoint.

As $\Gamma - (1,0) \subset \Gamma_{up}$ (which implies that $\Gamma \prec (\Gamma - (1,0))$), we get that $(\Gamma - (s,0)) \cap (\Gamma - (s+1,0)) = \emptyset$ and $\Gamma - (s,0) \prec \Gamma - (s+1,0)$, for all integers $s > 0$. So, in particular, using lemma 7, we obtain the following implications:

$$\begin{array}{l} 1)\ \Gamma \prec \Gamma - (1,0) \prec \Gamma - (2,0) \prec \Gamma - (3,0) \prec ... \prec \Gamma - (s_0 - 1, 0) \\ 2)\ \Gamma - (s_0 - 1, 0) \prec \Gamma - (s_0, 0). \end{array} \tag{16}$$

So, as $\Gamma - (s_0, 0) \subset \Gamma$ and $\Gamma \cap (\Gamma - (s_0 - 1, 0)) = \emptyset$, we get from 2) of (16) that $\Gamma - (s_0 - 1, 0) \prec \Gamma$, a contradiction with 1) of (16). Thus for all integers $i > 0$, $\Gamma \cap (\Gamma - (i,0)) = \emptyset$ and so

$$\Gamma \prec \Gamma - (1,0) \prec \Gamma - (2,0) \prec \Gamma - (3,0) \prec ... \prec \Gamma - (i,0).$$

But this clearly implies that $\Gamma - (i,0) \subset \Gamma_{up}$, because $(\Gamma - (i,0)) \cap V_{m_\Gamma} = \emptyset$. $\square$

Proof of lemma 9:

In both cases, the proof is analogous, so suppose Γ is a down component. This means that $\Gamma - (1,0)$ is contained in Γ_{up}.

Thus, for any $\widetilde{x} < m_\Gamma$ (see expression (3) for the definition of m_Γ), if we consider the segment $\{\widetilde{x}\} \times [0, \widetilde{y}^*]$, where

$$\widetilde{y}^* = \widetilde{y}^*(\widetilde{x}) = \sup\{\widetilde{y} \in]0,1[: \Gamma \cap \{\widetilde{x}\} \times [0, \widetilde{y}] = \emptyset\}, \tag{17}$$

we get that $(\Gamma - (1,0)) \cap \{\widetilde{x}\} \times [0, \widetilde{y}^*] = \emptyset$.

Now, consider a point $(m_\Gamma - 1, \widetilde{y}_\Gamma) \in \Gamma - (1,0)$ and a simple continuous arc $\gamma \subset int(\widetilde{A})$, such that:

i) $\gamma \cap (\Gamma - (1,0)) = (m_\Gamma - 1, \widetilde{y}_\Gamma)$

ii) $\gamma \cap \Gamma = \emptyset$

iii) the endpoints of γ are $(m_\Gamma - 1, \widetilde{y}_\Gamma)$ and $(m_\Gamma + 1, 0.5)$

iv) $\gamma \cap \{m_\Gamma + 1\} \times [0,1] = (m_\Gamma + 1, 0.5)$

As $\Gamma - (1,0) \subset \Gamma_{up}$ and $(\Gamma \cup (\Gamma - (1,0)))^c$ is connected, it is possible to choose γ as above.

The complement of the closed connected set $(\Gamma-(1,0))\cup\gamma\cup\{m_\Gamma+1\}\times[0,1]$ has exactly two connected components in $]-\infty,m_\Gamma+1[\times[0,1]$, one containing $]-\infty,m_\Gamma+1[\times\{0\}$, denoted $((\Gamma-(1,0))\cup\gamma)_{down}$ and the other containing $]-\infty,m_\Gamma+1[\times\{1\}$, denoted $((\Gamma-(1,0))\cup\gamma)_{up}$. Note that this construction is not unique, because we may have more then one point in $(\Gamma-(1,0))\cap\{m_\Gamma-1\}\times[0,1]$. Nevertheless, for any such choice, $\Gamma\subset((\Gamma-(1,0))\cup\gamma)_{down}$. This follows from the fact that, for any

$$\widetilde{x}<\min\{m_\Gamma,\min\{\widetilde{x}\in\mathbb{R}:(\widetilde{x},\widetilde{y})\in\gamma,\text{ for some }0<\widetilde{y}<1\}\}-10,$$

the segment $\{\widetilde{x}\}\times[0,\widetilde{y}^*]$ (see (17)) does not intersect $(\Gamma-(1,0))\cup\gamma\cup\{m_\Gamma+1\}\times[0,1]$ and $(\widetilde{x},\widetilde{y}^*)\in\Gamma$.

Now suppose, by contradiction, that $dist(\Gamma,\,\mathbb{R}\times\{1\})=0$. As Γ is closed and $\Gamma\cap\mathbb{R}\times\{1\}=\emptyset$, we get that for every

$$M\le M_0=\min\{m_\Gamma-10,\min\{\widetilde{x}\in\mathbb{R}:(\widetilde{x},\widetilde{y})\in\gamma,\text{ for some }0<\widetilde{y}<1\}\}-10$$

there exists $\epsilon>0$ such that if $\widetilde{z}\in\Gamma\cap\mathbb{R}\times[1-\epsilon,1]$, then $p_1(\widetilde{z})<M$. So, for M_0 and $\epsilon>0$ as above, let us choose a point $\widetilde{z}_0\in\Gamma\cap\mathbb{R}\times[1-\epsilon,1]$ such that $p_1(\widetilde{z}_0)\ge p_1(\widetilde{z})$ for all $\widetilde{z}\in\Gamma\cap\mathbb{R}\times[1-\epsilon,1]$ and

$$\begin{gathered}dist(\widetilde{z}_0,\mathbb{R}\times\{1\})<dist(\widetilde{z},\mathbb{R}\times\{1\})\text{ for all}\\ \widetilde{z}\in\Gamma\cap\{p_1(\widetilde{z}_0)\}\times[1-\epsilon,1]\text{ with }\widetilde{z}\neq\widetilde{z}_0.\end{gathered}$$

Intuitively, if we start going left from $\{m_\Gamma\}\times[0,1]$, $\widetilde{z}_0$ is the point of Γ with largest possible $\widetilde{x}$ and $\widetilde{y}$ coordinates, that belongs to $\mathbb{R}\times[1-\epsilon,1]$.

Now consider a closed vertical segment l contained in $\mathbb{R}\times[1-\epsilon,1]$, starting at $\widetilde{z}_0$ and ending at $\mathbb{R}\times\{1\}$. By construction of l, $l\cap\Gamma=\widetilde{z}_0$. As $\Gamma\subset((\Gamma-(1,0))\cup\gamma)_{down}$ and $l\cap(\gamma\cup\{m_\Gamma+1\}\times[0,1])=\emptyset$, we get that $l\cap(\Gamma-(1,0))\neq\emptyset$. So, there exists $\widetilde{z}_1\in l\cap(\Gamma-(1,0))$ which implies that $\widetilde{z}_1+(1,0)\in(l+(1,0))\cap\Gamma$. And this contradicts the choice of $\widetilde{z}_0$. $\square$

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