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Homeomorphisms of the annulus with a transitive lift

Salvador Addas-Zanata · Fábio Armando Tal

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Abstract Let f be a homeomorphism of the closed annulus A that preserves the orientation, the boundary components and that has a lift $\tilde{f}$ to the infinite strip $\tilde{A}$ which is transitive. We show that, if the rotation numbers of both boundary components of A are strictly positive, then there exists a closed nonempty unbounded set $B^- \subset \tilde{A}$ such that B^- is bounded to the right, the projection of B^- to A is dense, $B^- - (1, 0) \subset B^-$ and $\tilde{f}(B^-) \subset B^-$. Moreover, if p_1 is the projection on the first coordinate of $\tilde{A}$, then there exists $d > 0$ such that, for any $\tilde{z} \in B^-$,

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\tilde{z})) - p_1(\tilde{z})}{n} < -d.$$

In particular, using a result of Franks, we show that the rotation set of any homeomorphism of the annulus that preserves orientation, boundary components, which has a transitive lift without fixed points in the boundary is an interval with 0 in its interior.

Keywords Closed connected sets · Transitivity · Periodic orbits · Compactification

1 Introduction and statements of the main results

In this paper we consider homeomorphisms of the closed annulus $A = S^1 \times [0, 1]$ ($S^1 = \mathbb{R}/\mathbb{Z}$), which preserve orientation and the boundary components. Any lift of f to the universal cover of the annulus $\widetilde{A} = \mathbb{R} \times [0, 1]$, is denoted by $\widetilde{f}$, a homeomorphism which satisfies

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S. Addas-Zanata (✉) · F. A. Tal
Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010,
Cidade Universitária, São Paulo, SP 05508-090, Brazil
e-mail: sazanata@ime.usp.br

F. A. Tal
e-mail: fabiotal@ime.usp.br

$\widetilde{f}(\widetilde{x}+1,\widetilde{y})=\widetilde{f}(\widetilde{x},\widetilde{y})+(1,0)$ for all $(\widetilde{x},\widetilde{y})\in\widetilde{A}$. We study properties of such homeomorphisms when they have a particular lift $\widetilde{f}$ which is transitive.

Our results do not assume the existence of invariant measures of any type for f, yet the importance of studying consequences of transitivity for such mappings is underlined by the results of [3] and [5], as will be explained below.

In order to motivate our hypotheses a little more let us remember that, for any homeomorphism $f: A\to A$ which preserves orientation and the boundary components and for any Borel probability f-invariant measure μ, an invariant called the rotation number of μ, is defined as follows.

Let $p_1:\widetilde{A}\to\mathbb{R}$ be the projection on the first coordinate and let $p:\widetilde{A}\to A$ be the covering mapping. Given f and $\widetilde{f}$, the displacement function $\phi: A\to\mathbb{R}$ is defined as

$$\phi(x,y)=p_1\circ\widetilde{f}(\widetilde{x},\widetilde{y})-\widetilde{x}, \tag{1}$$

for any $(\widetilde{x},\widetilde{y})\in p^{-1}(x,y)$. The rotation number of μ is then given by

$$\rho(\mu)=\int_A \phi(x,y)d\mu.$$

The importance of this definition becomes clear by Birkhoff's ergodic theorem, which states that, for μ almost every $(x,y)\in A$ and for any $(\widetilde{x},\widetilde{y})\in p^{-1}(x,y)$,

$$\rho(x,y)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi\circ f^i(x,y)=\lim_{n\to\infty}\frac{p_1\circ\widetilde{f}^n(\widetilde{x},\widetilde{y})-\widetilde{x}}{n},$$

exists and

$$\int_A \rho(x,y)d\mu=\rho(\mu).$$

Moreover, if f is ergodic with respect to μ, then $\rho(x,y)$ is constant μ-almost everywhere.

Following the usual definition (see [2]), we refer to the set of area, orientation and boundary components preserving homeomorphisms of the annulus, which satisfy $\rho(Leb)=0$ for a certain fixed lift $\widetilde{f}$, as the set of rotationless homeomorphisms. Every time we say that f is a rotationless homeomorphism, a special lift $\widetilde{f}$ is fixed and used to define ϕ.

In [3] it is proved that the transitivity of $\widetilde{f}$ holds for a residual subset of rotationless homeomorphisms of the annulus and the results in [5] suggest (as explained to us by S. Crovisier) that the same statement holds in the C^1 topology.

Our original motivation in this setting was to study a conjecture posed by P. Boyland, which will be explained below.

Boyland's conjecture Given an area, orientation and boundary components preserving homeomorphism of the annulus f, by a result of Franks (see [6]), if there are two f-invariant probability measures μ_1 and μ_2 with $\rho(\mu_1)<\rho(\mu_2)$, then for every rational $\rho(\mu_1)<\frac{p}{q}<\rho(\mu_2)$, there exists a q-periodic orbit for f with this rotation number. So, suppose f is rotationless and there exists a measure with positive rotation number. By a classical result (a version of the Conley-Zehnder theorem for the annulus) there must be fixed points with zero rotation number. The question is: Is it true that in the above situation there must be orbits with negative rotation number?

The above is a very difficult problem, which we did not solve in full generality. We considered the following situation. Suppose f is an orientation and boundary components preserving homeomorphism of the annulus which has a transitive lift $\widetilde{f} : \widetilde{A} \to \widetilde{A}$ (one with a dense orbit). We denote the set of such mappings by $Hom_+^{trans}(A)$. So every time we say $f \in Hom_+^{trans}(A)$ and refer to a lift $\widetilde{f}$ of f, we are always considering a transitive lift (maybe f has more than one transitive lift, we choose any of them and fix it).

A fundamental set for our theorems is the following:

Definition B^- is the union of all unbounded connected components of

$$B = \bigcap_{n \leq 0} \widetilde{f}^n(]-\infty, 0] \times [0, 1]),$$

which a priori, may be empty. Clearly, $B^- \cap \partial\widetilde{A} = \emptyset$ and $\widetilde{f}(B^-) \subset B^-$.

Theorem 1 *Let $f \in Hom_+^{trans}(A)$ and suppose that the rotation numbers of $(f, \widetilde{f})$ restricted to both boundary components of the annulus are strictly positive. Then the closed set B^- defined above is not empty and intersects* $\{0\} \times [0, 1]$.

The above result is essentially due to Birkhoff, see [4]. We decided to present a proof because our setting is different, but the main idea is the same.

In the next theorem, we consider the ω-limit set of B^-, defined as

$$\omega(B^-) = \bigcap_{n=0}^{\infty} \overline{\left(\bigcup_{i=n}^{\infty} \widetilde{f}^i(B^-) \right)}.$$

Theorem 2 *Under the same hypotheses as in Theorem* 1*, the set B^- has no ω-limit, that is,* $\omega(B^-) = \emptyset$.

Thus, iterates of B^- by $\tilde{f}$ converge to the left end of $\widetilde{A}$. The properties of B^- allow us to extend this theorem and obtain a stronger result:

Theorem 3 *Under the same hypotheses as in Theorem* 1*, there exists a real number* $\rho^+(B^-) < 0$ *such that, if $\widetilde{z} \in B^-$, then*

$$\limsup_{n \to \infty} \frac{p_1 \tilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})}{n} \leq \rho^+(B^-) < 0. \tag{2}$$

The last theorem shows that all points in B^- have a "minimum negative velocity" in the strip $\tilde{A}$.

As $\widetilde{f}$ has a dense orbit, so does f and thus every point in the annulus A is non-wandering for f. In this way, theorem 3 together with Franks version of the Poincaré-Birkhoff theorem from [7] implies the following:

Corollary 4 *Let $f \in Hom_+^{trans}(A)$. If $\tilde{f}$ does not have fixed points in the boundary of $\widetilde{A}$, then the rotation set $\rho(\widetilde{f}) = \{\omega \in \mathbb{R} : \omega = \int_A \phi d\mu$, for some f-invariant Borel probability measure $\mu\}$ is a closed interval with* 0 *in its interior and all its rational points are realized by periodic orbits.*

The above corollary implies that Boyland's Conjecture is true for rotationless homeomorphisms of the annulus which have no fixed points in the boundary and have a transitive lift. In a preprint we prove that it is true for C^r-generic ($r \geq 16$) rotationless diffeomorphisms of the annulus f such that $\widetilde{f}$ has no fixed points in $\partial\widetilde{A}$, see [1].

Another important consequence of Theorem 3 is that, even though there are points with rotation number in $]\rho^+(B^-), 0[$, they do not belong to B^-. In particular, if such points have unstable manifolds unbounded to the left, they must also be unbounded to the right.

Our next result, which is harder than Theorems 1, 2 and 3, give more information on the structure of B^-.

Theorem 5 *Under the same hypotheses as in Theorem 1, $p(B^-)$ is dense in A.*

So, there exists a dense subset of A, such that for any point z in this set and for all $\widetilde{z} \in p^{-1}(z)$, expression (2) holds. Clearly, it is still possible that the rotation number of $(f, \widetilde{f})$ at z does not exist.

2 Proof of Theorem 1

As we already said, this is a simple result, which is not new. We decided to present a proof because our setting is a little bit different from the classical one.

In order to introduce the set B^- and show some of its properties, we will sometimes make use of the already mentioned left and right compactification of $\widetilde{A} = \mathbb{R} \times [0, 1]$, denoted by L, R-compactification, that is, we compactify the infinite strip adding two points, L (left end) and R (right end), getting a closed disk, denoted by $\widehat{A}$. Clearly $\widetilde{f}$ induces a homeomorphism $\widehat{f} : \widehat{A} \to \widehat{A}$, such that $\widehat{f}(L) = L$ and $\widehat{f}(R) = R$.

Given a real number a, let

$$V_a = \{a\} \times [0, 1],$$
$$V_a^- =]-\infty, a] \times [0, 1] \quad \text{and} \quad V_a^+ = [a, +\infty[\times[0, 1].$$

Denote the corresponding sets on $\widehat{A}$ by $\widehat{V}_a$, $\widehat{V}_a^-$ and $\widehat{V}_a^+$. We will also denote the sets V_0, V_0^- and V_0^+ simply by V, V^- and V^+, respectively.

If we consider the closed set,

$$\widehat{B} = \bigcap_{n \le 0} \widehat{f}^n(\widehat{V}^-),$$

we get that, $\widehat{f}(\widehat{B}) \subset \widehat{B}$ and $L \in \widehat{B}$. Denote by $\widehat{B}^-$ the connected component of $\widehat{B}$ which contains L, and by B^- the corresponding set on the strip.

Lemma 6 *Let $f : A \to A$ be an orientation and boundary components preserving homeomorphism, and let $\tilde{f} : \widetilde{A} \to \widetilde{A}$ be a fixed lift of f. Suppose that for every $a < 0$ there is a positive integer n such that $\widetilde{f}^n(V) \cap V_a^- \neq \emptyset$. Then $\widehat{B}^- \cap \widehat{V} \neq \emptyset$ (equivalently for the strip: $B^- \cap V \neq \emptyset$).*

Remark As $f \in Hom_+^{trans}(A)$, clearly $(f, \tilde{f})$ satisfy the lemma hypotheses.

Proof The proof of this result in a different context appears in Le Calvez [8] and even in Birkhoff's paper [4], as we already said.

Given $N > 0$, choose a sufficiently small $a < 0$ such that

$$n = \inf\{i > 0 : \widetilde{f}^{-i}(V_a^-) \cap V \neq \emptyset\} > N.$$

The above is true because as $|a|$ becomes larger, it takes more time for an iterate of V to hit V_a^-.

From the definition of n we get that: $\widetilde{f}^{-i}(V_a) \subset V^-$, for $i = 0, 1, \ldots, n-1$ and $\widetilde{f}^{-n}(V_a^-) \cap V \neq \emptyset$. This implies that there exists a simple continuous arc $\Gamma_N \subset \widetilde{f}^{-n}(V_a^-) \cap V^-$, such that $\widehat{\Gamma}_N$ connects L to $\widehat{V}$ (one endpoint of $\widehat{\Gamma}_N$ is L and the other is in $\widehat{V}$). For this arc, if $1 \leq i \leq n$, we get: $\widetilde{f}^i(\Gamma_N) \subset \widetilde{f}^{-n+i}(V_a^-) \subset V^-$. So,

$$\widehat{\Gamma}_N \subset \bigcap_{i=0}^{n} \widehat{f}^{-i}(\widehat{V}^-),$$

which implies, by taking the limit $N \to \infty \Rightarrow n \to \infty$, that $\widehat{\Gamma}_N$ has a convergent subsequence in the Hausdorff topology to a compact connected set $\widehat{\Gamma} \subset \widehat{A}$, which connects L to $\widehat{V}$. From its choice, it is clear that $\widehat{\Gamma} \subset \widehat{B}^-$ and thus the lemma is proved. □

So, from Lemma 6 we know that $B^- \subset \widetilde{A}$ is a closed set, limited to the right ($B^- \subset V^-$), whose connected components (which may be unique) are all unbounded, and at least one of them intersects V. An important point here is that, as the rotation numbers in the boundary components of the annulus are both positive, B and thus B^-, do not intersect $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$ (because $\widetilde{f}(B) \subset B \subset V^-$). This will be used in a fundamental way to prove our theorems.

3 Preliminaries for the proof of Theorem 2

3.1 The ω-limit set of B^-

Here we examine some properties of the set

$$\omega(\widehat{B}^-) = \bigcap_{n=0}^{\infty} \overline{\bigcup_{i=n}^{\infty} \widehat{f}^i(\widehat{B}^-)}, \tag{3}$$

a subset of $\widehat{A}$, and the corresponding set $\omega(B^-) \subset \widetilde{A}$.

Since $\widehat{f}(\widehat{B}^-) \subset \widehat{B}^-$, and since $\widehat{B}^-$ is closed, we have $\omega(\widehat{B}^-) = \bigcap_{n=0}^{\infty} \widehat{f}^n(\widehat{B}^-)$, therefore $\omega(\widehat{B}^-)$ is the intersection of a nested sequence of compact connected sets, and so it is also a compact connected set. Moreover, definition (3) implies the following lemma:

Lemma 7 *$\omega(B^-)$ is a closed, $\widetilde{f}$-invariant set, whose connected components are all unbounded.*

Proof Since $L \in \widehat{B}^-$ and $\widehat{f}(L) = L$, we get that $L \in \omega(\widehat{B}^-)$. This implies, since $\omega(\widehat{B}^-)$ is connected, that each connected component of $\omega(B^-)$ is unbounded. The other properties follow directly from the previous considerations. □

Of course, since B^- is closed, we also have that $\omega(B^-) \subset B^-$, and as such, $\omega(B^-) \cap \mathbb{R} \times \{i\} = \emptyset$, $i \in \{0, 1\}$, and $\omega(B^-) \subset V^-$. It is still possible that $\omega(B^-) = \emptyset$, and this is in fact true, as we show in the proof of theorem 2. For the moment we can make use of the fact that both B^- and $\omega(B^-)$ have similar properties to shorten our proofs.

3.2 Hypothesis satisfied by the set D

Let $D \subset \widetilde{A}$ be a non-empty closed set with the following properties:

- $\widetilde{f}(D) \subset D$;
- $D \subset V^-$;

- Every connected component of D is unbounded;
- $D \cap \mathbb{R} \times \{i\} = \emptyset,\ i \in \{0, 1\}$;
- If $\widetilde{z} \in D$ then $\widetilde{z} - (1, 0) \in D$.

It is easily verified that B^- has these properties, as does $\omega(B^-)$ if it is nonempty, so every result shown for D must hold in the particular cases of interest for us.

3.3 A preliminary result about $p(D) \subset A$

Lemma 8 $\overline{p(D)} \supset S^1 \times \{0\}$, *or* $\overline{p(D)} \supset S^1 \times \{1\}$.

Proof Suppose that the lemma is false. Then, there are points $P_0 \in S^1 \times \{0\}$ and $P_1 \in S^1 \times \{1\}$ such that $\{P_0, P_1\} \cap \overline{p(D)} = \emptyset$. As $\left(\overline{p(D)}\right)^c$ is an open set, there exists $\epsilon > 0$ such that $B_\epsilon(P_i) \cap \overline{p(D)} = \emptyset$, for $i = 0, 1$. As

$$\widetilde{f}(D) \subset D \quad \text{and} \quad p \circ \widetilde{f}(\widetilde{x}, \widetilde{y}) = f \circ p(\widetilde{x}, \widetilde{y})$$

we get that

$$f(p(D)) \subset p(D) \ \Rightarrow\ f(\overline{p(D)}) \subset \overline{p(D)}.$$

Since $\widetilde{f}$ is transitive, it follows that f is also transitive and so there exists $N > 0$ such that $f^{-N}(B_\epsilon(P_0)) \cap B_\epsilon(P_1) \neq \emptyset$.

Now, we must have that

$$f^{-N}(B_\epsilon(P_0)) \cap \overline{p(D)} = \emptyset,$$

for if this was not true, it would imply $B_\epsilon(P_0) \cap f^N(\overline{p(D)}) \neq \emptyset$, which, in turn, would imply $B_\epsilon(P_0) \cap \overline{p(D)} \neq \emptyset$, because $f^N(\overline{p(D)}) \subset \overline{p(D)}$, contradicting the choice of $\epsilon > 0$.

As $f^{-N}(B_\epsilon(P_0)) \cup B_\epsilon(P_1)$ is disjoint from $\overline{p(D)}$, this implies that there exists a simple continuous arc $\gamma \subset \left(f^{-N}(B_\epsilon(P_0)) \cup B_\epsilon(P_1)\right)$, disjoint from $\overline{p(D)}$, such that its endpoints are in $S^1 \times \{0\}$ and in $S^1 \times \{1\}$. So, if $\widetilde{\gamma}$ is a connected component of $p^{-1}(\gamma)$, then $\widetilde{\gamma} - (i, 0) \cap D = \emptyset$ for all integers $i > 0$. And this contradicts the fact that the connected components of D are unbounded to the left. □

4 Proof of Theorem 2

Suppose, by contradiction, that $\omega(B^-) \neq \emptyset$. This implies, by Lemma 8, that either $S^1 \times \{0\} \subset \overline{p(\omega(B^-))}$ or $S^1 \times \{1\} \subset \overline{p(\omega(B^-))}$. Let us assume, without loss of generality, that $S^1 \times \{0\} \subset \overline{p(\omega(B^-))}$.

Since the rotation number of $\widetilde{f}$ restricted to $S^1 \times \{0\}$ is strictly positive, there exists $\sigma > 0$ such that $p_1(\widetilde{f}(\widetilde{x}, 0)) > \widetilde{x} + 2\sigma$ for all $\widetilde{x} \in \mathbb{R}$. Let $\epsilon > 0$ be sufficiently small such that for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times [0, \epsilon]$, $p_1 \circ \tilde{f}(\tilde{x}, \tilde{y}) > \tilde{x} + \sigma$.

As $S^1 \times \{0\} \subset \overline{p(\omega(B^-))}$, there is a real a such that

$$\omega(B^-) \cap \{a\} \times [0, \epsilon] \neq \emptyset. \tag{4}$$

The fact that $\omega(B^-)$ is closed implies that there must be a $\delta \leq \epsilon$ such that $(a, \delta) \in \omega(B^-)$, and such that for all $0 \leq \tilde{y} < \delta$, $(a, \tilde{y}) \notin \omega(B^-)$. In other words, (a, δ) is the "lowest" point of $\omega(B^-)$ in $\{a\} \times [0, \epsilon]$. We denote by v the segment $\{a\} \times [0, \delta[$. Let Ω be the connected component of $\left(\omega(B^-) \cup v\right)^c$ that contains $]-\infty, a[\times\{0\}$.

Proposition 9 *The following inclusion holds:* $\Omega \subset \tilde{f}(\Omega)$.

Proof First, note that the boundary of $\tilde{f}(\Omega)$ is contained in $\omega(B^-) \cup \tilde{f}(v)$. We claim that $\partial \tilde{f}(\Omega) \cap \Omega = \emptyset$. This follows from the two conditions below:

1) By the choice of $\epsilon > 0$, $\tilde{f}(v) \cap v = \emptyset$;
2) As $\omega(B^-)$ is $\tilde{f}$-invariant and does not intersect v, we get that $\omega(B^-) \cap \tilde{f}(v) = \tilde{f}(\omega(B^-)) \cap v = \emptyset$;
   So, as $]-\infty, a[\times\{0\} \subset \tilde{f}(]-\infty, a[\times\{0\})$, we get that $\Omega \subset \tilde{f}(\Omega)$.

□

As Ω is open, the transitivity of $\tilde{f}$ and the last proposition yields that it is dense in the strip. But $\Omega \subset V^-$, a contradiction that proves theorem 2. This same argument is used in the proof of Theorem 5.

5 Proof of Theorem 3

Lemma 10 *There exists an integer* $N_1 > 0$ *such that* $\tilde{f}^{N_1}(B^-) \subset B^- - (1, 0)$.

Proof Theorem 2 shows that $\omega(B^-) = \emptyset$, so there is an integer $N_1 > 0$ such that, $\tilde{f}^{N_1}(B^-) \subset V^-_{-2}$. Thus, $\tilde{f}^{N_1}(B^-)+(1, 0) \subset V^-$ and as each connected component of $\tilde{f}^{N_1}(B^-)+(1, 0)$ is unbounded and this set is positively invariant, it must be the case that $\tilde{f}^{N_1}(B^-)+(1, 0) \subset B^-$.
□

As $\tilde{f}^{N_1}(B^-) \subset B^- - (1, 0)$, for any positive integer k,

$$\tilde{f}^{kN_1}(B^-) \subset B^- - (k, 0) \subset V^-_{-k},$$

and so it follows that, for any point $\tilde{z} \in B^-$,

$$\limsup_{n\to\infty} \frac{p_1(\tilde{f}^n(\tilde{z})) - p_1(\tilde{z})}{n} \leq -\frac{1}{N_1},$$

and this proves Theorem 3.

In the next section we present some technical results used in the proof of Theorem 5.

6 More on the structure of $p(D) \subset A$

Remember that $\tilde{f}$ is transitive and it moves points in $\partial\tilde{A}$ uniformly to the right. For the hypothesis on the set D, see Sect. 3.2.

The next lemma tells us what happens if $p(D)$ is not dense in A.

Lemma 11 *If* $\overline{p(D)} \neq A$, *then* $\overline{p(D)}^c$ *is connected and dense in* A *and, moreover,* $\overline{p(D)}^c$ *contains a homotopically non trivial simple closed curve in the open annulus* $S^1\times]0, 1[$.

Proof The transitivity of f and the inclusion $f(\overline{p(D)}) \subset \overline{p(D)}$ imply that $\left(\overline{p(D)}\right)^c$ is dense. Let E be a connected component of $\left(\overline{p(D)}\right)^c$. Assume by contradiction that there is no simple closed curve $\gamma \subset E$ which is homotopically non trivial as a curve of the annulus.

In this case, $p^{-1}(E)$ is not connected and there exists an open connected set $E_{lift} \subset \widetilde{A}$, such that $E_{lift} \cap (E_{lift} + (i, 0)) = \emptyset$, for all integers $i \neq 0$ and

$$p^{-1}(E) = \bigcup_{i=-\infty}^{+\infty} \left(E_{lift} + (i, 0)\right).$$

As f is transitive and $f^{-1}\left(\left(\overline{p(D)}\right)^c\right) \subset \left(\overline{p(D)}\right)^c$, there exists a first $N > 0$, such that $f^{-N}(E) \subset E$ (in particular, for $i \in \{1, 2, \ldots, N-1\}$, $f^{-i}(E) \cap E = \emptyset$). This means that

$$\begin{gathered} \widetilde{f}^{-i}(E_{lift}) \cap p^{-1}(E) = \emptyset \text{ for all } i > 0 \text{ which is not a} \\ \text{multiple of } N \text{ and } \widetilde{f}^{-N}(E_{lift}) \subset E_{lift} + (i_0, 0), \\ \text{for some fixed integer } i_0, \text{ which implies that} \\ \widetilde{f}^{-k.N}(E_{lift}) \subset E_{lift} + (k.i_0, 0), \text{ for all integers } k > 0. \end{gathered} \tag{5}$$

Suppose $i_0 \geq 0$. As $\widetilde{f}$ has a dense orbit, there exists a point $\widetilde{z} \in E_{lift} - (1, 0)$ such that $\widetilde{f}^l(\widetilde{z}) \in E_{lift}$, for some $l > 0$. But this means that, $\widetilde{f}^{-l}(E_{lift}) \cap (E_{lift} - (1, 0)) \neq \emptyset$, something that contradicts (5), because we assumed that $i_0 \geq 0$. A similar argument implies that i_0 can not be smaller then zero. Therefore, all connected components of $\left(\overline{p(D)}\right)^c$ contain a homotopically non trivial simple closed curve.

Let E be a connected component of $\left(\overline{p(D)}\right)^c$ and $\gamma_E \subset E$ be a homotopically non trivial simple closed curve. Since f is transitive, $f^{-1}(\gamma_E) \cap \gamma_E \neq \emptyset$. Thus $f^{-1}(E) \cap E \neq \emptyset$ and so, since $f^{-1}(\left(\overline{p(D)}\right)^c) \subset \left(\overline{p(D)}\right)^c$, $f^{-1}(E) \subset E$. But E is open and f is transitive, so E is dense and therefore it is the only connected component of $\left(\overline{p(D)}\right)^c$. □

So, let

$$\gamma_E \subset \left(\overline{p(D)}\right)^c \cap interior(A) \tag{6}$$

be a homotopically non trivial simple closed curve and let $\gamma_E^- \supset S^1 \times \{0\}$ and $\gamma_E^+ \supset S^1 \times \{1\}$ be the open connected components of γ_E^c. As $\overline{p(D)} \cap \gamma_E = \emptyset$, we obtain that $\overline{p(D)} \subset \gamma_E^- \cup \gamma_E^+$.

Lemma 12 *Let Γ be a connected component of D. If $\overline{p(D)} \neq A$, we have*:

$$\begin{cases} \textit{if } \overline{p(\Gamma)} \subset \gamma_E^-, \textit{ then } \overline{p(\Gamma)} \supset S^1 \times \{0\} \\ \textit{if } \overline{p(\Gamma)} \subset \gamma_E^+, \textit{ then } \overline{p(\Gamma)} \supset S^1 \times \{1\} \end{cases}.$$

Proof First note that as $\overline{p(\Gamma)}^c$ contains the connected and dense set $E = \overline{p(D)}^c$, it is also connected and dense. Without loss of generality, suppose that $\overline{p(\Gamma)} \subset \gamma_E^-$. This implies that $\overline{p(\Gamma)} \cap S^1 \times \{1\} = \emptyset$. If $\overline{p(\Gamma)}$ does not contain $S^1 \times \{0\}$, then there exists a simple continuous arc λ in the annulus, which avoids $\overline{p(\Gamma)}$ and connects some point $P_0 \in \left(S^1 \times \{0\}\right) \setminus \overline{p(\Gamma)}$ to some point $P_1 \in S^1 \times \{1\}$. This is true because $P_0, P_1 \in \overline{p(\Gamma)}^c$, which is an open connected set. But this means that

$$p^{-1}(\lambda) \cap \Gamma = \emptyset,$$

and this is a contradiction because each connected component of $p^{-1}(\lambda)$ is compact and Γ is connected and unbounded to the left. So, $\overline{p(\Gamma)} \supset S^1 \times \{0\}$. □

7 Proof of Theorem 5

Assume by contradiction that $\overline{p(B^-)} \neq A$. From Lemma 12, we can suppose without loss of generality, that $\overline{p(B^-)} \supset S^1 \times \{0\}$.

Remember that γ_E is a homotopically non-trivial simple closed curve in $S^1\times]0,1[$ contained in $\left(\overline{p(B^-)}\right)^c$ and γ_E^- is the connected component of γ_E^c which contains $S^1 \times \{0\}$, see expression (6). Let $\sigma > 0$ be such that $p_1(\widetilde{f}(\widetilde{x}, 0)) > \widetilde{x} + 2\sigma$ for all $\widetilde{x} \in \mathbb{R}$ and let $\epsilon > 0$ be sufficiently small such that:

- $S^1 \times [0, \epsilon] \subset \gamma_E^-$;
- for all $(\widetilde{x}, \widetilde{y}) \in \mathbb{R} \times [0, \epsilon]$, $p_1 \circ \tilde{f}(\tilde{x}, \widetilde{y}) > \tilde{x} + \sigma$.

As $\overline{p(B^-)} \supset S^1 \times \{0\}$, there exists a real a such that

$$B^- \cap \{a\} \times [0, \epsilon] \neq \emptyset. \tag{7}$$

As we did in the proof of Theorem 2, B^- is closed, so there must be a $\delta \leq \epsilon$ such that $(a, \delta) \in B^-$, and such that for all $0 \leq \tilde{y} < \delta$, $(a, \tilde{y}) \notin B^-$, that is, (a, δ) is the "lowest" point of B^- in $\{a\} \times [0, \epsilon]$. We again denote by v the segment $\{a\} \times [0, \delta[$.

Let Γ_1 be the connected component of B^- that contains (a, δ). As $p(\Gamma_1) \cap \gamma_E^- \neq \emptyset$, we get that $\overline{p(\Gamma_1)} \subset \gamma_E^-$, something that implies the following important facts:

$$dist(p(\Gamma_1),\ S^1 \times \{1\}) > 0$$
$$dist(\Gamma_1, \mathbb{R} \times \{1\}) > 0$$

As in the proof of theorem 2, let us consider the connected component of $(\Gamma_1 \cup v)^c$ which contains $]-\infty, a[\times\{0\}$, denoted Ω. Clearly, $\Omega \subset p^{-1}(\gamma_E^-)$ and if we define

$$\Omega_1 = \bigcup_{n=0}^{\infty} \widetilde{f}^{-n}(\Omega),$$

we get that $\widetilde{f}^{-1}(\Omega_1) \subset \Omega_1$. Moreover, the following is true:

Proposition 13 *Ω_1 is contained in $p^{-1}(\gamma_E^-)$.*

Proof First note that, as the boundary of Ω is contained in $\Gamma_1 \cup v$, for all integers $i > 0$ we have:

$$\partial \left(\widetilde{f}^{-i}(\Omega)\right) \subset \bigcup_{n=0}^{\infty} \widetilde{f}^{-n}(\Gamma_1 \cup v) = \left(\bigcup_{n=0}^{\infty} \widetilde{f}^{-n}(\Gamma_1)\right) \cup \left(\bigcup_{n=0}^{\infty} \widetilde{f}^{-n}(v)\right) \tag{8}$$

Clearly Ω_1 is an open set. Let us show that it is connected. Each set of the form $\widetilde{f}^{-i}(\Omega)$ is connected because $\widetilde{f}$ is a homeomorphism. Also, since $\widetilde{f}^{-i}(]-\infty, a[\times\{0\}) \subset]-\infty, a[\times\{0\}$, we have $\widetilde{f}^{-i}(\Omega) \cap \Omega \neq \emptyset$. But Ω is also open and connected, so Ω_1 must be connected.

For all integers $i > 0$, as $\widetilde{f}^{-i}(\Omega)$ is connected, intersects $\mathbb{R} \times \{0\}$ and is disjoint from $\mathbb{R} \times \{1\}$, if we show that $\left(\cup_{n=0}^{\infty} \widetilde{f}^{-n}(\Gamma_1)\right) \cup \left(\cup_{n=0}^{\infty} \widetilde{f}^{-n}(v)\right) \subset p^{-1}(\gamma_E^-)$, then expression (8) implies that $\widetilde{f}^{-i}(\Omega) \subset p^{-1}(\gamma_E^-)$, which gives: $\Omega_1 \subset p^{-1}(\gamma_E^-)$ and the proof is complete.

Let us analyze first what happens to $\widetilde{f}^{-n}(\Gamma_1)$, for all integers $n > 0$. Fixed an integer $n > 0$, there exists an integer $k = k(n) > 0$, such that $\widetilde{f}^i\left(\widetilde{f}^{-n}(\Gamma_1) - (k, 0)\right) \subset V^-$ for all $i \in \{0, 1, \ldots, n\}$. As $\Gamma_1 - (k, 0) \subset B^-$, we get that $\widetilde{f}^{-n}(\Gamma_1) - (k, 0)$ is also a subset of B^-, so $(\widetilde{f}^{-n}(\Gamma_1) - (k, 0)) \cap p^{-1}(\gamma_E) = \emptyset$, which implies that $\widetilde{f}^{-n}(\Gamma_1) \cap p^{-1}(\gamma_E) = \emptyset$. Lemma 12 implies that $\overline{p(\Gamma_1)} \supset S^1 \times \{0\}$, so we obtain that $\widetilde{f}^{-n}(\Gamma_1) \subset p^{-1}(\gamma_E^-)$.

We are left to deal with $\cup_{n=0}^{\infty}\widetilde{f}^{-n}(v)$. Let us show that $\widetilde{f}^{-1}(v)\subset\Omega$. From the choice of v, $\widetilde{f}^{-1}(v)\cap v=\emptyset$. Also, from the definition of $v=\{a\}\times[0,\delta[$, we get that $\widetilde{f}^{-1}(v)\cap\Gamma_1=\emptyset$ because $v\cap B^-=\emptyset$. Finally, the following inclusions

$$\Omega\supset]-\infty,a[\times\{0\}\quad\text{and}\quad]-\infty,a[\times\{0\}\supset\widetilde{f}^{-1}(]-\infty,a[\times\{0\})$$

imply that $\widetilde{f}^{-1}(v)\cap\Omega\neq\emptyset$ and so $\widetilde{f}^{-1}(v)\subset\Omega\subset p^{-1}(\gamma_E^-)$.

So, $\widetilde{f}^{-2}(v)\subset\widetilde{f}^{-1}(\Omega)$, whose boundary, $\partial\left(\widetilde{f}^{-1}(\Omega)\right)$, is contained in $\widetilde{f}^{-1}(\Gamma_1)\cup\widetilde{f}^{-1}(v)\subset p^{-1}(\gamma_E^-)$. As above, $\widetilde{f}^{-1}(\Omega)$ is connected, intersects $\mathbb{R}\times\{0\}$ and is disjoint from $\mathbb{R}\times\{1\}$, so we get that

$$\widetilde{f}^{-2}(v)\subset\widetilde{f}^{-1}(\Omega)\subset p^{-1}(\gamma_E^-).$$

Thus, $\widetilde{f}^{-3}(v)\subset\widetilde{f}^{-2}(\Omega)$ and an analogous argument implies that $\widetilde{f}^{-3}(v)\subset\widetilde{f}^{-2}(\Omega)\subset p^{-1}(\gamma_E^-)$. An induction shows that

$$\widetilde{f}^{-n}(v)\subset\widetilde{f}^{-n+1}(\Omega)\subset p^{-1}(\gamma_E^-)\text{ for all integers }n\geq 1,$$

and the proposition is proved. □

Since Ω_1 is open and $\widetilde{f}^{-1}(\Omega_1)\subset\Omega_1$, we must have, by the transitivity of $\widetilde{f}$, that Ω_1 is dense. But this contradicts the proposition.

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