

Area-preserving diffeomorphisms of the torus whose rotation sets have non-empty interior

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Abstract

In this paper we consider $C^{1+\epsilon}$ area-preserving diffeomorphisms of the torus f , either homotopic to the identity or to Dehn twists. We suppose that f has a lift $\tilde{f}$ to the plane such that its rotation set has interior and prove, among other things that if zero is an interior point of the rotation set, then there exists a hyperbolic $\tilde{f}$ -periodic point $\tilde{Q} \in \mathbb{R}^2$ such that $\overline{W^u(\tilde{Q})}$ intersects $W^s(\tilde{Q} + (a, b))$ for all integers (a, b) , which implies that $\overline{W^u(\tilde{Q})}$ is invariant under integer translations. Moreover, $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})}$ and $\tilde{f}$ restricted to $\overline{W^u(\tilde{Q})}$ is invariant and topologically mixing. Each connected component of the complement of $\overline{W^u(\tilde{Q})}$ is a disk with diameter uniformly bounded from above. If f is transitive, then $\overline{W^u(\tilde{Q})} = \mathbb{R}^2$ and $\tilde{f}$ is topologically mixing in the whole plane.

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1 Introduction and main results

One of the most well understood chapters of dynamics of surface homeomorphisms is the case of the torus. Any orientation preserving homeomorphism f of the torus can be associated in a canonical way with a two by two matrix A with integer coefficients and determinant one. Depending on this matrix, there are basically three types of maps (if we are allowed to consider iterates of f and coordinate changes):

1. for some integer $n > 0$, A^n is the identity and in this case, f^n is said to be homotopic to the identity;
2. for some integer $n > 0$, A^n is, up to a conjugation, equal to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, for some integer $k \neq 0$. If this is the case, f^n is said to be homotopic to a Dehn twist;
3. A is hyperbolic, that is, A has real eigenvalues λ and μ and $|\mu| < 1 < |\lambda|$;

This paper concerns to the first two cases, namely we suppose throughout the paper that the matrix A associated to f is either the identity or it is equal to $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, for some integer $k \neq 0$. In these cases, it is possible to consider a rotation set which, roughly speaking, "measures" how orbits in the torus rotate with respect to the homology (below we will present precise definitions). It is a generalization of the rotation number of an orientation preserving circle homeomorphism to this two-dimensional context and a lot of work has been done on this subject. For instance, one wants to show connections between dynamical properties of f and geometric properties of the rotation set, see [13], [28] and [27] for maybe the first references on this problem and also which sets can be realized as rotation sets of a torus homeomorphism, see for instance [23] and [24].

Our initial motivation was the following question: Is the "complexity" of the map f , in some sense, shared by its lift to the plane? The objective was to look at an area-preserving homeomorphism f of the torus with a rotation set with non-empty interior. And then give some consequences of these hypotheses to

the lift of f to the plane, denoted $\tilde{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, used to compute the rotation set. In some cases, we could assume other hypothesis on f , for instance like being transitive and we wanted to know what happened with $\tilde{f}$. Unfortunately, to prove something interesting, we had to leave the continuous world and assume $C^{1+\epsilon}$ differentiability (for some $\epsilon > 0$), so that we could use Pesin theory. Apart from our main theorems, this paper has some lemmas that may have interest by themselves. For instance, it is known since [13], [9] and [3] that rational points in the interior of the rotation set are realized by periodic orbits. One of our results says that given a rational rotation vector in the interior of the rotation set, there is a hyperbolic periodic point which realizes this rotation vector and it has, what Kwapisz and Swanson [25] called a rotary horseshoe, that is, the union of its stable and unstable manifolds contains a homotopically non-trivial simple closed curve in the torus. This sort of property is useful to prove most of the results in this paper. In particular, given two rational vectors in the interior of the rotation set, there are hyperbolic periodic points which realize these rotation vectors and the stable manifold of one of them intersects the unstable manifold of the other periodic point (and vice versa) in a topologically transverse way (see definition 9).

We have similar results for maps homotopic to the identity and homotopic to Dehn twists. Although some proofs are slightly different, most of them work in both cases, with just simple adjustments. Only when the ideas involved are different, we will present separate proofs in the Dehn twist case. Studying maps homotopic to Dehn twists is certainly of great interest, since for instance the well-known Chirikov standard map, $S_{M,k} : \mathbb{T}^2 \rightarrow \mathbb{T}^2$, given by

$$S_{M,k}(x, y) = (x + y + k \sin(2\pi x) \bmod 1, y + k \sin(2\pi x) \bmod 1), \quad (1)$$

where $k > 0$ is a parameter, is homotopic to $(x, y) \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} (\bmod 1)^2$. And, maybe some of the hardest questions in surface dynamics refer to the standard map. For instance, one of them, which has some relation with the results proved in this paper is the following:

Conjecture : Is there $k > 0$ such that $S_{M,k}$ is transitive?

This is an very difficult question. Just to give an idea, Pedro Duarte [10] proved that for a residual set of large values of k , $S_{M,k}$ has lots of elliptic islands, therefore it is not transitive.

Finally, we shall point out that in [21] and [15] results with more or less the same flavor as the ones from this paper have been shown to be true for area preserving torus homeomorphisms homotopic to the identity, using different techniques.

Before presenting our results, we need some definitions.

Definitions:

1. Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $p : \mathbb{R}^2 \longrightarrow T^2$ and $\pi : \mathbb{R}^2 \longrightarrow S^1 \times \mathbb{R}$ be the associated covering maps. Coordinates are denoted as $(\tilde{x}, \tilde{y}) \in \mathbb{R}^2$, $(\hat{x}, \hat{y}) \in S^1 \times \mathbb{R}$ and $(x, y) \in T^2$.
2. Let $Diff_0^{1+\epsilon}(T^2)$ be the set of $C^{1+\epsilon}$ (for some $\epsilon > 0$) area preserving diffeomorphisms of the torus homotopic to the identity and let $Diff_0^{1+\epsilon}(\mathbb{R}^2)$ be the set of lifts of elements from $Diff_0^{1+\epsilon}(T^2)$ to the plane. Maps from $Diff_0^{1+\epsilon}(T^2)$ are denoted f and their lifts to the plane are denoted $\tilde{f}$.
3. Let $Diff_k^{1+\epsilon}(T^2)$ be the set of $C^{1+\epsilon}$ (for some $\epsilon > 0$ and some integer $k \neq 0$) area preserving diffeomorphisms of the torus homotopic to a Dehn twist $(x, y) \longrightarrow (x + ky \bmod 1, y \bmod 1)$ and let $Diff_k^{1+\epsilon}(S^1 \times \mathbb{R})$ and $Diff_k^{1+\epsilon}(\mathbb{R}^2)$ be the sets of lifts of elements from $Diff_k^{1+\epsilon}(T^2)$ to the cylinder and plane. As defined above, maps from $Diff_k^{1+\epsilon}(T^2)$ are denoted f and their lifts to the vertical cylinder and plane are respectively denoted $\hat{f}$ and $\tilde{f}$.
4. Let $p_{1,2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the standard projections; $p_1(\tilde{x}, \tilde{y}) = \tilde{x}$ and $p_2(\tilde{x}, \tilde{y}) = \tilde{y}$.
5. Given $f \in Diff_0^{1+\epsilon}(T^2)$ and a lift $\tilde{f} \in Diff_0^{1+\epsilon}(\mathbb{R}^2)$, the so called rotation set of $\tilde{f}$, $\rho(\tilde{f})$, can be defined as follows (see [28]):

$$\rho(\tilde{f}) = \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} \left\{ \frac{\tilde{f}^n(\tilde{z}) - \tilde{z}}{n} : \tilde{z} \in \mathbb{R}^2 \right\}} \quad (2)$$

This set is a compact convex subset of $\mathbb{R}^2$ and it was proved in [13] and [28] that all points in its interior are realized by compact f -invariant subsets of T^2 , which are periodic orbits in the rational case. We say that some vector $\rho \in \rho(\tilde{f})$ is realized by a compact f -invariant subset $K \subset T^2$ if and only if, for all $z \in K$ and any $\tilde{z} \in p^{-1}(z)$ we have

$$\lim_{n \rightarrow \infty} \frac{\tilde{f}^n(\tilde{z}) - \tilde{z}}{n} = \rho. \quad (3)$$

The above limit, when it exists, is known as the rotation vector of the point z . So when we say that some point ρ in $\rho(\tilde{f})$ is realized by a compact f -invariant subset of T^2 we mean that the rotation vector exists at all points of this set and is equal to ρ .

6. Given $f \in Diff_k^{1+\epsilon}(T^2)$ and a lift $\tilde{f} \in Diff_k^{1+\epsilon}(\mathbb{R}^2)$, the so called vertical rotation set of $\tilde{f}$, $\rho_V(\tilde{f})$, can be defined as follows, see [9], [1] and [3]:

$$\rho_V(\tilde{f}) = \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} \left\{ \frac{p_2 \circ \tilde{f}^n(\tilde{z}) - p_2(\tilde{z})}{n} : \tilde{z} \in \mathbb{R}^2 \right\}} \quad (4)$$

This set is a closed interval (maybe a single point, but never empty) and it was proved in [9], [1] and [3] that all numbers in its interior are realized by compact f -invariant subsets of T^2 , which are periodic orbits in the rational case, in exactly the same sense as above. The only difference is that to define the vertical rotation number of a point $z \in T^2$, we consider the p_2 -projection of the expression (3). In other words we say that the vertical rotation number of z is equal to some number ω if for any $\tilde{z} \in p^{-1}(z)$,

$$\lim_{n \rightarrow \infty} \frac{p_2 \circ \tilde{f}^n(\tilde{z}) - p_2(\tilde{z})}{n} = \omega.$$

7. A connected simply connected open subset D of the torus is called an open disk. Note that in this case, for any connected component $\tilde{D}$ of $p^{-1}(D)$ and any pair of integers $(a, b) \neq (0, 0)$, we have

$$\tilde{D} \cap (\tilde{D} + (a, b)) = \emptyset \text{ and } p^{-1}(D) = \bigcup_{i, j \in \text{integers}} \tilde{D} + (i, j).$$

8. We say that an open disk $D \subset \mathbb{T}^2$ is unbounded if a connected component $\tilde{D}$ of $p^{-1}(D)$ is unbounded. Clearly, if $\tilde{D}'$ is another connected component of $p^{-1}(D)$, as there exists a pair of integers (a, b) such that $\tilde{D} = \tilde{D}' + (a, b)$, all connected components of $p^{-1}(D)$ are unbounded.
9. For any hyperbolic periodic point P , we say that its unstable manifold $W^u(P)$ (or stable manifold $W^s(P)$) has a topologically transverse intersection with some closed connected set K if and only if, for some compact connected piece λ of some branch of $W^u(P)$ (or $W^s(P)$), there exists a rectangle R such that $R \setminus \lambda$ has exactly two connected components, R_{left} and R_{right} and there is a connected component of $K \cap R_{left}$ which intersects λ and another side of R_{left} . An analogous condition is assumed for R_{right} , namely there is a connected component of $K \cap R_{right}$ which intersects λ and another side of R_{right} . By rectangle we mean that R is a topological closed disk and the boundary of R is a simple closed curve divided into four C^1 simple arcs (the sides): two are very small and C^1 -transversal to λ and the other two are C^1 -close to λ , see figure 1. We use the following notation: $W^u(P) \pitchfork K$. It is easy to see, using Hartman-Grobman theorem, that if P is a hyperbolic fixed point for some map f and $W^u(P) \pitchfork K$, then given $\epsilon > 0$ and a compact piece κ of $W^s(P)$, there exists $n(\epsilon, \kappa) > 0$ such that the ϵ -neighborhood of $f^{-n}(K)$ contains κ for all $n > n(\epsilon, \kappa)$. So, if Q is another hyperbolic fixed point for f , we say that $W^u(P) \pitchfork W^s(Q)$ when there exists a closed connected piece α of $W^s(Q)$ such that $W^u(P) \pitchfork \alpha$. In this case $\overline{W^u(P)} \supset \overline{W^u(Q)}$, which implies that if $W^u(Q) \pitchfork K$ for some closed connected set K , then $W^u(P) \pitchfork K$ (this is more or less a C^0 version of the λ -lemma and some of its consequences).

Now we present our main results.

Lemma 1 : *Suppose f belongs to $Diff_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0, 0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $Diff_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. Then there exists $\tilde{Q} \in \mathbb{R}^2$, which is a hyperbolic periodic point for $\tilde{f}$ such that for some pair of integers $(a, b) \neq (0, 0)$, a, b coprimes, $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (a, b))$ (note that $W^s(\tilde{Q} + (a, b)) =$*

$W^s(\tilde{Q}) + (a, b)$. In particular, if $Q = p(\tilde{Q})$ then $W^s(Q) \cup Q \cup W^u(Q)$ contains a homotopically non-trivial simple closed curve in T^2 .

This lemma is easier in case f is transitive, see lemma 24. In the general case, we have to work with pseudo-Anosov maps isotopic to f relative to certain finite invariant sets and apply Handel's shadowing [17] and [6], and other technical results on Pesin theory [8].

Lemma 2 : *Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(T^2)$ and $\text{int}(\rho(\tilde{f}))$ is not empty or f belongs to $\text{Diff}_k^{1+\epsilon}(T^2)$ and $\text{int}(\rho_V(\tilde{f}))$ is not empty. If f is transitive, then f can not have a periodic open disk. In the general case, there exists $M = M(f) > 0$ such that if $D \subset T^2$ is a f -periodic open disk, then for any connected component $\tilde{D}$ of $p^{-1}(D)$, $\text{diam}(\tilde{D}) < M$.*

So if the rotation set has interior, a diffeomorphism of the torus homotopic to the identity or to a Dehn twist can only have bounded periodic open disks and the bound in their diameters (from above) is uniform.

Theorem 3 : *Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(T^2)$, it is transitive and $(\frac{p}{q}, \frac{r}{q}) \in \text{int}(\rho(\tilde{f}))$. Then, $\tilde{f}^q(\bullet) - (p, r)$ has a hyperbolic periodic point $\tilde{Q}$ such that $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \mathbb{R}^2$.*

Theorem 4 : *Suppose f belongs to $\text{Diff}_k^{1+\epsilon}(T^2)$, it is transitive, $\frac{p}{q} \in \text{int}(\rho_V(\tilde{f}))$ and s is any integer number. Then, $\tilde{f}^q(\bullet) - (s, p)$ has a hyperbolic periodic point $\tilde{Q}$ such that $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \mathbb{R}^2$.*

Corollary 5 : *Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(T^2)$ and $(0, 0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(T^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If f is transitive, then $\tilde{f}$ is topologically mixing.*

The next result is a version of theorems 3 and 4 to the general case.

Theorem 6 : Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. Then, $\tilde{f}$ has a hyperbolic periodic point $\tilde{Q}$ such that for any pair of integers (a,b) , $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (a,b))$, so $\overline{W^u(\tilde{Q})} = \overline{W^u(\tilde{Q})} + (a,b)$ and $\overline{W^s(\tilde{Q})} = \overline{W^s(\tilde{Q})} + (a,b)$. Moreover, $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})}$, this set is $\tilde{f}$ -invariant ($\tilde{f}(\overline{W^u(\tilde{Q})}) = \overline{W^u(\tilde{Q})}$) and all connected components of the complement of $\overline{W^u(\tilde{Q})}$ are open disks with diameter uniformly bounded from above, f -periodic when projected to the torus.

Corollary 7 : Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If $\tilde{D} \subset \mathbb{R}^2$ is a wandering open disk, then for all integers n , $\text{diam}(\tilde{f}^n(\tilde{D}))$ is uniformly bounded from above and there exists a f -periodic open disk $D_+ \subset \mathbb{T}^2$ such that $D_+ \supset D = p(\tilde{D})$.

Let us consider now the following subset of $\mathbb{T}^2$,

$$R.I.(f) \stackrel{\text{def.}}{=} p(\overline{W^u(\tilde{Q})}) = \overline{W^u(p(\tilde{Q}))},$$

where the point $\tilde{Q}$ appears in theorem 6. This set is called region of instability of f . It is clearly compact, connected and f -invariant. And from theorem 6 we obtain the following:

Corollary 8 : If f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $\{(\frac{p}{q}, \frac{r}{q}), (0,0)\} \in \text{int}(\rho(\tilde{f}))$ then $\tilde{f}^q(\bullet) - (p,r)$ has a hyperbolic periodic point $\tilde{P}$ such that for any pair of integers (a,b) , $W^u(\tilde{P}) \pitchfork W^s(\tilde{P} + (a,b))$ and $R.I.(f) = p(\overline{W^u(\tilde{P})})$. In case f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $\{\frac{p}{q}, 0\} \in \text{int}(\rho_V(\tilde{f}))$, then, for all integers s , $\tilde{f}^q(\bullet) - (s,p)$ has a hyperbolic periodic point $\tilde{P}$ such that for any pair of integers (a,b) , $W^u(\tilde{P}) \pitchfork W^s(\tilde{P} + (a,b))$ and $R.I.(f) = p(\overline{W^u(\tilde{P})})$. Moreover, if $\tilde{Q}$ is the $\tilde{f}$ -periodic point that appears in the statement of theorem 6, then $W^u(p(\tilde{P})) \pitchfork W^s(p(\tilde{Q}))$ and $W^u(p(\tilde{Q})) \pitchfork W^s(p(\tilde{P}))$.

We believe this is an interesting description of the region of instability of an area-preserving diffeomorphism of the torus. Moreover, a result analogous to corollary 5 holds. The map $\tilde{f}$ restricted to $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})}$ is topologically

mixing, that is for every $\tilde{z}_1, \tilde{z}_2 \in \overline{W^u(\tilde{Q})}$ and $\epsilon > 0$, there exists $N > 0$ such that

$$\tilde{f}^n(B_\epsilon(\tilde{z}_1)) \cap B_\epsilon(\tilde{z}_2) \neq \emptyset, \forall n \geq N.$$

Clearly, many interesting and fundamental questions remain open: Is f transitive when restricted to $R.I.(f)$? Does f has a dense set of periodic points in $R.I.(f)$? What about $\tilde{f}$ restricted to $p^{-1}(R.I.(f))$? Is it transitive? Does it has a dense set of periodic points?

One consequence of the above results, that may be useful in many situations will be presented. For this, we need more definitions.

In the homotopic to the identity case, given a vector $(\cos(\theta), \sin(\theta))$, we define

$$B_\theta = \{\tilde{z} \in \mathbb{R}^2 : \langle \tilde{f}^n(\tilde{z}), (\cos(\theta), \sin(\theta)) \rangle \geq 0 \text{ for all integers } n \geq 0\}$$

and let B_θ^∞ be the union of all unbounded connected components of B_θ . In case f is homotopic to a Dehn twist, it only makes sense to define the sets

$$B_{S(\text{or } N)} = \{\tilde{z} \in \mathbb{R}^2 : p_2 \circ \tilde{f}^n(\tilde{z}) \leq (\geq) 0 \text{ for all integers } n \geq 0\}$$

and let $B_{S(\text{or } N)}^\infty$ be the union of all unbounded connected components of $B_{S(\text{or } N)}$.

In lemma 2 of [5] we proved that, if $(0, 0) \in \rho(\tilde{f})$ or $0 \in \rho_V(\tilde{f})$, then B_θ^∞ , B_S^∞ and B_N^∞ are non-empty, closed subsets of the plane, positively invariant under $\tilde{f}$. It is easy to see that their omega-limits satisfy the following (because $\tilde{f}(B_\theta^\infty) \subset B_\theta^\infty$ and $\tilde{f}(B_{S(N)}^\infty) \subset B_{S(N)}^\infty$) :

$$\begin{aligned} \omega(B_\theta^\infty) &= \bigcap_{i \geq 0} \tilde{f}^i(B_\theta^\infty) = \bigcap_{i \in \text{integers}} \tilde{f}^i(B_\theta^\infty) \\ \omega(B_{S(N)}^\infty) &= \bigcap_{i \geq 0} \tilde{f}^i(B_{S(N)}^\infty) = \bigcap_{i \in \text{integers}} \tilde{f}^i(B_{S(N)}^\infty) \end{aligned}$$

And we know that if $\omega(B_\theta^\infty) = \emptyset$, then all points $\tilde{z} \in B_\theta^\infty$ satisfy (see lemma 10 of [4])

$$\left\langle \left(\frac{\tilde{f}^n(\tilde{z}) - \tilde{z}}{n} \right), (\cos(\theta), \sin(\theta)) \right\rangle > c_\theta > 0,$$

for some constant c_θ and all $n > 0$ suff. large. Analogously, if $\omega(B_{S(N)}^\infty) = \emptyset$, then all points $\tilde{z} \in B_{S(N)}^\infty$ satisfy

$$\frac{p_2 \circ \tilde{f}^n(\tilde{z}) - p_2(\tilde{z})}{n} < c_S < 0 \text{ (resp. } > c_N > 0),$$

for some constant c_S (c_N) and all $n > 0$ suff. large.

Theorem 9 : Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. Then, for all $\theta \in [0, 2\pi]$, $\omega(B_\theta^\infty) = \emptyset$ and $\omega(B_S^\infty) = \omega(B_N^\infty) = \emptyset$.

A simple corollary of the above result, lemma 2 and the ideas in the proof of theorem 6 is the following:

Corollary 10 : Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If f is transitive, then $\overline{p(B_S^\infty)} = \overline{p(B_N^\infty)} = \overline{p(B_\theta^\infty)} = \mathbb{T}^2$, for all $\theta \in [0, 2\pi]$. And for a general f , any of the following sets $\left(\overline{p(B_S^\infty)}\right)^c$, $\left(\overline{p(B_N^\infty)}\right)^c$ and $\left(\overline{p(B_\theta^\infty)}\right)^c$ is the union of f -periodic open disks, with diameters uniformly bounded from above when lifted to the plane.

The hypothesis of zero being an interior point of the rotation set is essential in theorem 9, as the following example shows. Consider the standard map S_{M,k_0} for a parameter $k_0 > 0$ which has a rotational invariant curve γ . Now, if $\epsilon > 0$ is sufficiently small, the map $S_{M,k_0}^\epsilon : \mathbb{T}^2 \rightarrow \mathbb{T}^2$,

$$S_{M,k_0}^\epsilon(x, y) = (x + y + k_0 \sin(2\pi x) \bmod 1, y + k_0 \sin(2\pi x) + \epsilon \bmod 1)$$

still has fixed points of zero vertical rotation number (because S_{M,k_0} has a hyperbolic fixed point with zero vertical rotation number) and, as the vertical rotation number of Lebesgue measure

$$\int_{\mathbb{T}^2} (k_0 \sin(2\pi x) + \epsilon) d\text{Leb} = \epsilon > 0,$$

we get that $\rho_V(\tilde{S}_{M,k_0}^\epsilon)$ is an interval which has zero as the left extreme. Note that $\tilde{S}_{M,k_0}^\epsilon : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is given by

$$\tilde{S}_{M,k_0}^\epsilon(\tilde{x}, \tilde{y}) = (\tilde{x} + \tilde{y} + k_0 \sin(2\pi \tilde{x}), \tilde{y} + k_0 \sin(2\pi \tilde{x}) + \epsilon).$$

This happens because the extremes of the vertical rotation interval of $\tilde{S}_{M,k_0}^\epsilon$ are continuous non-decreasing functions of $\epsilon > 0$ by theorem 10 and lemma 2 of [2].

Now we will show that $\omega(B_S^\infty)$ and $\omega(B_N^\infty)$ of $\tilde{S}_{M,k_0}^\epsilon$ are both non-empty. Birkhoff's invariant curve theorem implies that γ projects injectively on the horizontal direction, so if we consider the cylinder diffeomorphism induced by $\tilde{S}_{M,k_0}^\epsilon$, denoted $\hat{S}_{M,k_0}^\epsilon : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$, we get that the closure of the region below $\hat{\gamma}$ (denoted $\hat{\gamma}^-$) is invariant under $(\hat{S}_{M,k_0}^\epsilon)^{-1}$ and the closure of the region above $\hat{\gamma}$ (denoted $\hat{\gamma}^+$) is invariant under $\hat{S}_{M,k_0}^\epsilon$. Note that $\hat{\gamma}$ is just a lift of γ to the cylinder. Moreover, as $\hat{S}_{M,k_0}^\epsilon$ has fixed points above and below $\hat{\gamma}$, the sets

$$\bigcap_{i \geq 0} (\hat{S}_{M,k_0}^\epsilon)^i(\hat{\gamma}^+) \text{ and } \bigcap_{i \geq 0} (\hat{S}_{M,k_0}^\epsilon)^{-i}(\hat{\gamma}^-)$$

are closed, non-empty, $\hat{S}_{M,k_0}^\epsilon$ -invariant and their connected components are unbounded. Clearly if we consider their inverse images under the projection $\pi : \mathbb{R}^2 \rightarrow S^1 \times \mathbb{R}$, and take adequate vertical translations, we get that both $\omega(B_S^\infty)$ and $\omega(B_N^\infty)$ are non-empty.

This paper is organized as follows. In the second section we present some background results we use, with references and in the third section we prove our results. Just to avoid confusions, let us state that everytime we say a point $Q \in T^2$ is f -periodic of period m and hyperbolic, we not only mean that $f^m(Q) = Q$, but also that $Df^m(P)$ has positive eigenvalues, which implies that all branches of the stable and unstable manifolds at Q are f^m -invariant.

2 Ideas involved in the proofs

The main tools used in our proofs come from two different theories. We will try to give a superficial description on each of them below.

2.1 Nielsen-Thurston theory of classification of homeomorphisms of surfaces and some results on subshifts of finite type

The following is a brief summary of this powerful theory. For more information and proofs see [29], [12] and [16].

Let M be a compact, connected oriented surface possibly with boundary, and $f : M \rightarrow M$ be a homeomorphism. Two homeomorphisms are said to be isotopic if they are homotopic via homeomorphisms. In fact, for closed orientable surfaces, all homotopic pairs of homeomorphisms are isotopic [11].

There are two basic types of homeomorphisms which appear in the Nielsen-Thurston classification : the finite order homeomorphisms and the pseudo-Anosov ones.

A homeomorphism f is said to be of finite order if $f^n = id$ for some $n \in \mathbb{N}$. The least such n is called the order of f . Finite order homeomorphisms have zero topological entropy.

A homeomorphism f is said to be pseudo-Anosov if there is a real number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ such that $f(\mathcal{F}^s) = \lambda^{-1}\mathcal{F}^s$ and $f(\mathcal{F}^u) = \lambda\mathcal{F}^u$. Pseudo-Anosov homeomorphisms preserve area, are topologically transitive, have positive topological entropy, and have Markov partitions [12].

A homeomorphism f is said to be reducible by a system

$$C = \bigcup_{i=1}^n C_i$$

of disjoint simple closed curves $C_1, \dots, C_n$ (called reducing curves) if

- (1) $\forall i$, C_i is not homotopic to a point, nor to a component of ∂M ,
- (2) $\forall i \neq j$, C_i is not homotopic to C_j ,
- (3) C is invariant under f .

Theorem 11 : *If the Euler characteristic $\chi(M) < 0$, then every homeomorphism $f : M \rightarrow M$ is isotopic to a homeomorphism $\phi : M \rightarrow M$ such that either*

- (a) ϕ is of finite order,
- (b) ϕ is pseudo-Anosov, or
- (c) ϕ is reducible by a system of curves C .

Maps ϕ as in theorem 11 are called Thurston canonical forms for f .

Now two applications of theorem 11 that are important to us will be presented. The first is due to Llibre and Mackay [27] and Franks [13], and the second is due to Doeff [9] and myself [1]:

Theorem 12 : *If f is a homeomorphism of the torus homotopic to the identity and $\tilde{f}$ is a lift of f to the plane such that $\rho(\tilde{f})$ has interior, then for any three non-collinear rational vectors $\rho_1, \rho_2, \rho_3 \in \text{int}(\rho(\tilde{f}))$, there are periodic orbits Q_1, Q_2 and Q_3 that realize these rotation vectors such that $f|_{\mathbb{T}^2 \setminus \{Q_1 \cup Q_2 \cup Q_3\}}$ is isotopic to a pseudo-Anosov homeomorphism ϕ of $\mathbb{T}^2 \setminus \{Q_1 \cup Q_2 \cup Q_3\}$.*

In this case we say that f is isotopic to ϕ relative to $Q_1 \cup Q_2 \cup Q_3$. It means that the isotopy acts on the set $Q_1 \cup Q_2 \cup Q_3$ exactly in the same way as f does. The map $\phi: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is then said to be pseudo-Anosov relative to the finite invariant set $Q_1 \cup Q_2 \cup Q_3$ because it satisfies all of the properties of a pseudo-Anosov homeomorphism except that the associated stable and unstable foliations may have 1-prong singularities at points in $Q_1 \cup Q_2 \cup Q_3$.

Theorem 13 : *If f is a homeomorphism of the torus homotopic to a Dehn twist and $\tilde{f}$ is a lift of f to the plane such that $\rho_V(\tilde{f})$ has interior, then for any two different rationals $\rho_1, \rho_2 \in \text{int}(\rho_V(\tilde{f}))$, there are periodic orbits Q_1 and Q_2 that realize these rotation numbers such that f is isotopic relative to $Q_1 \cup Q_2$ to a homeomorphism ϕ of $\mathbb{T}^2$, which is pseudo-Anosov relative to $Q_1 \cup Q_2$.*

In theorem 12, given a rational vector $\rho = (\frac{p}{q}, \frac{r}{q})$ such that the integers p, r, q have no common factors, the chosen periodic orbit Q which realizes this rotation vector must have period q . The existence of such an orbit follows from the main theorem in [13]. An analogous remark holds for theorem 13, namely given a rational $\frac{p}{q} \in \text{int}(\rho_V(\tilde{f}))$ such that the integers p, q have no common factors, the chosen periodic orbit Q which realizes this vertical rotation number must also have period q . The existence of such an orbit follows from the main theorems in [9] and [3].

Using the notation from theorem 11, our next result tells us that if ϕ is pseudo-Anosov, then the complicated dynamics of ϕ is in some sense shared by

f . To be more precise, we will present the following analog of Handel's global shadowing [17], see theorem 3.3.1 of [22], which is more or less taken from theorem 3.2 of [6], stated in a version adequate to our needs:

Theorem 14 : *If $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a diffeomorphism homotopic to the identity or to a Dehn twist, A is a finite f -invariant set and f is isotopic rel A to some map $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ which is pseudo-Anosov rel A , then there exists a compact f -invariant set $W \subset \mathbb{T}^2$ and a continuous surjection $s : W \rightarrow \mathbb{T}^2$ that is homotopic to the inclusion map $i : W \rightarrow \mathbb{T}^2$ and semi-conjugates $f|_W$ to ϕ , that is, $s \circ f|_W = \phi \circ s$.*

Remark: The set W , which is the domain of s , is the closure of another set denoted W' that satisfies the following. The map s restricted to W' is one to one and $s(W') = \{\text{periodic points of } \phi\}$, which is a dense subset of $\mathbb{T}^2$, see the proof of theorem 3.2 of [6]. In this way, given a compact subset M of W , if $s(M) = \mathbb{T}^2$, then $M = W$.

As $s : W \rightarrow \mathbb{T}^2$ is homotopic to the inclusion map $i : W \rightarrow \mathbb{T}^2$, we get that s has a lift $\tilde{s} : p^{-1}(W) \rightarrow \mathbb{R}^2$ such that

$$\tilde{s} \circ \tilde{f}|_{p^{-1}(W)} = \tilde{\phi} \circ \tilde{s} \text{ and } \sup_{\tilde{z} \in p^{-1}(W)} \|\tilde{s}(\tilde{z}) - \tilde{z}\| < \infty,$$

where $\tilde{f}$ and $\tilde{\phi}$ are lifts that are equivariantly homotopic rel $p^{-1}(A)$.

The next results we present are related to theorems 12 and 13 in the following sense: Assume $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a homeomorphism of the torus, either homotopic to the identity or to a Dehn twist, which is pseudo-Anosov relative to some finite ϕ -invariant set Q . Also suppose that its rotation set, or vertical rotation set (depending on whether ϕ is homotopic to the identity or to a Dehn twists) has 0 as an interior point. The first result appears in [12] and [22]:

Theorem 15 : *There exists a Markov partition $\mathcal{R} = \{R_1, \dots, R_N\}$ for ϕ . If G is a graph with the set of vertices $\{1, \dots, N\}$ that has an edge from i to j whenever $\phi(R_i) \cap \text{int}(R_j) \neq \emptyset$, then the subshift of finite type (Λ, σ) associated to G is mixing and factors onto ϕ . More precisely, for any $x = (x_i)_{i \in \text{integers}} \in \Lambda$ the*

intersection $\bigcap_{i \in \mathbb{Z}} \phi^{-i}(R_{x_i})$ consists of a single point, denoted $h(x)$ and the map $h : \Lambda \rightarrow \mathbb{T}^2$ has the following properties:

1. $h \circ \sigma = \phi \circ h$;
2. h is continuous and surjective;
3. h is finite to one;
4. h is one to one on a topologically residual subset consisting of all points whose full orbits never hit the boundary of the Markov partition;

The subshift of finite type that appears in the above theorem can be defined as follows: Let A be a $N \times N$ matrix whose entries satisfy $a_{ij} = 1$ if and only if $\phi(R_i) \cap \text{int}(R_j) \neq \emptyset$ and $a_{ij} = 0$ otherwise. Following the theorem notation, Λ is the space of bi-infinite sequences $x = (\dots, x_{-1}, x_0, x_1, \dots) \in \{1, \dots, N\}^{\mathbb{Z}}$ such that $a_{x_l x_{l+1}} = 1$ for all integers l and $\sigma : \Lambda \rightarrow \Lambda$ is just the shift transformation $\sigma(x)_n = (x)_{n+1}$.

Now, assume $\psi : \Lambda \rightarrow \mathbb{R}^2$ is a function depending only on x_0 and x_1 for which there exists an Euclidean triangle $\Delta \subset \mathbb{R}^2$ such that

$$\Delta \subset \left\{ \int_{\Lambda} \psi d\nu : \nu \text{ is a } \sigma\text{-invariant Borel probability measure} \right\}$$

and for all $\rho \in \text{int}(\Delta)$ there exists a σ -invariant ergodic Borel probability measure ν that satisfies $\int_{\Lambda} \psi d\nu = \rho$. Now we state a theorem which is contained in theorems 2.2.1 and 2.2.2 of [22]:

Theorem 16 : *Under the above assumptions, for every $\rho \in \text{int}(\Delta)$, there exists a compact σ -invariant set $K_{\rho} \subset \Lambda$ such that $h_{\text{top}}(\sigma|_{K_{\rho}}) > 0$ (the topological entropy of σ restricted to K_{ρ} is non zero) and for some constant $\text{Const} > 0$, all $x \in K_{\rho}$ and all integers $n > 0$ we have:*

$$\left\| \sum_{j=0}^{n-1} \psi \circ \sigma^j(x) - n \cdot \rho \right\| \leq \text{Const}$$

2.2 Pesin-Katok theory

This is a series of amazing results, which show the existence of a certain type of hyperbolicity everytime a diffeomorphism is sufficiently smooth (at least $C^{1+\epsilon}$, for some $\epsilon > 0$) and satisfies certain conditions on invariant measures. Unfortunately results on this subject are quite technical and the definitions needed are really complicated. For this reason, we decided to present a very naive description of the theory in the particular case of surfaces and then state a theorem.

In case of surface diffeomorphisms, positive topological entropy is the crucial hypothesis, because of the following. By the entropy variational principle, the so called Ruelle-Pesin inequality and the fact that ergodic measures are extreme points of the set of Borel probability invariant measures, when topological entropy is positive, there always exist ergodic invariant measures μ with non-zero Lyapunov exponents, one positive and one negative and positive metric entropy, $h_\mu(f) > 0$ (see for instance [30] and [19]). These measures are called hyperbolic measures. Below we state a theorem adapted from Katok's work on the subject. For proofs, see [20] and the supplement of [19] by Katok-Mendoza.

Theorem 17 : *Let f be a $C^{1+\epsilon}$ (for some $\epsilon > 0$) diffeomorphism of a surface M and suppose μ is an ergodic hyperbolic Borel probability f -invariant measure with $h_\mu(f) > 0$ and compact support. Then, for any $\alpha > 0$ and any $x \in \text{supp}(\mu)$, there exists a hyperbolic periodic point $Q \in B_\alpha(x)$ which has a transversal homoclinic intersection and the whole orbit of Q is contained in the α -neighborhood of $\text{supp}(\mu)$.*

3 Proofs

In this section we will prove our main results. With this purpose we will prove several auxiliary propositions and lemmas.

3.1 Some preliminary results

The first one is a by product of some results in Kwapisz thesis [22] and Katok's work on Pesin theory, see theorem 17.

Lemma 18 : *Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$. Then for any rational vector $\left(\frac{p}{q}, \frac{r}{q}\right) \in \text{int}(\rho(\tilde{f}))$ (with $q > 0$), $\tilde{f}^q(\bullet) - (p, r)$ has a hyperbolic periodic point $\tilde{Q} \in \mathbb{R}^2$ and $W^u(\tilde{Q})$ has a transverse intersection with $W^s(\tilde{Q})$.*

Proof:

Given $\left(\frac{p}{q}, \frac{r}{q}\right) \in \text{int}(\rho(\tilde{f}))$, let $\tilde{g} \stackrel{\text{def}}{=} \tilde{f}^q(\bullet) - (p, r)$. It is easy to see that $(0, 0) \in \text{int}(\rho(\tilde{g}))$ and a periodic point for $\tilde{g}$ corresponds to a periodic point for f with rotation vector (for $\tilde{f}$) equal $\left(\frac{p}{q}, \frac{r}{q}\right)$. So without loss of generality, we can suppose that $\left(\frac{p}{q}, \frac{r}{q}\right) = (0, 0) \in \text{int}(\rho(\tilde{f}))$.

Choose three periodic orbits Q_1, Q_2 and Q_3 as in theorem 12, such that their rotation vectors form a triangle Δ that contains $(0, 0)$ in its interior. Theorem 12 tells us that the isotopy class of f relative to $Q_1 \cup Q_2 \cup Q_3$ contains a pseudo-Anosov map $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \text{ rel } Q_1 \cup Q_2 \cup Q_3$. As we already said, pseudo-Anosov maps have very rich dynamics, for instance theorem 15 holds.

Moreover, as $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a homeomorphism of the torus homotopic to the identity, there exists a lift of ϕ to the plane, denoted $\tilde{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is equivariantly homotopic to $\tilde{f} \text{ rel } p^{-1}(Q_1 \cup Q_2 \cup Q_3)$ and its rotation set $\rho(\tilde{\phi}) \supset \Delta$. The rotation set of $\tilde{\phi}$ can also be obtained in the following way: Let $D_\phi : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ be the displacement function, given by $D_\phi(x, y) = \tilde{\phi}(\tilde{x}, \tilde{y}) - (\tilde{x}, \tilde{y})$, for any $(\tilde{x}, \tilde{y}) \in p^{-1}(x, y)$. Then

$$\rho(\tilde{\phi}) = \left\{ \int_{\mathbb{T}^2} D_\phi d\mu : \mu \text{ is a } \phi\text{-invariant Borel probability measure} \right\}. \quad (5)$$

As we already said when we presented the definition of rotation set, in [13] and [28] it is proved that every interior point ω of the rotation set is realized by a compact f -invariant subset of the torus and so by Krylov-Bogoliubov's theorem (see [30]) there exists an ergodic f -invariant measure μ which realizes this rotation vector, that is $\omega = \int_{\mathbb{T}^2} D_\phi d\mu$. In [28] it is also shown that if ω is an extreme point of the rotation set, which is a convex subset of the plane,

then there also exists an ergodic f -invariant measure μ which realizes ω . This justifies why the equality in (5) is true.

Using the notation introduced in theorem 15, it is not very difficult to see (see claim 3.2.1 of [22]) that we can define a function $\psi : \Lambda \rightarrow \mathbb{R}^2$ (depending only on x_0 and x_1) such that for any $x = (x_i)_{i \in \text{integers}} \in \Lambda$ and any natural n ,

$$\left\| \sum_{j=0}^{n-1} \psi \circ \sigma^j(x) - \sum_{j=0}^{n-1} D_\phi \circ \phi^j(h(x)) \right\| \leq 2 \max_{1 \leq i \leq N} \{\text{diam}(R_i)\}. \quad (6)$$

As we explained above, for all $\rho \in \text{int}(\Delta)$ there exists a ϕ -invariant ergodic Borel probability measure μ such that $\int_{T^2} D_\phi d\mu = \rho$. So if we consider the measure $\nu(\bullet) \stackrel{\text{def.}}{=} \mu(h(\bullet))$, it is a σ -invariant ergodic Borel probability measure which satisfies $\int_\Lambda \psi d\nu = \rho$. Thus if we fix $\rho = (0, 0)$, we get from theorems 15, 16 and expression (6) that there exists a compact $\tilde{\phi}$ -invariant set $K_\phi \subset \mathbb{R}^2$ such that $h_{\text{top}}(\tilde{\phi}|_{K_\phi}) > 0$. We want to have a similar statement for the map $\tilde{f}$. With this purpose, we use theorem 14 and the remark after it.

In this way, we obtain a compact $\tilde{f}$ -invariant set $K_f = \tilde{s}^{-1}(K_\phi) \subset \mathbb{R}^2$ such that $h_{\text{top}}(\tilde{f}|_{K_f}) > 0$. As $h_{\text{top}}(\tilde{f}|_{K_f}) > 0$, there exists a hyperbolic ergodic Borel probability $\tilde{f}$ -invariant measure μ , with positive metric entropy, whose support is contained in K_f . So, given $\epsilon > 0$, theorem 17 implies that $\tilde{f}$ has a hyperbolic periodic point $\tilde{Q}$ with a transversal homoclinic intersection and the whole orbit of $\tilde{Q}$ is contained in $V_\epsilon(\text{supp}(\mu)) \subset V_\epsilon(K_f)$. $\square$

The next result is an analog of lemma 18 for maps homotopic to Dehn twists.

Lemma 19 : *Suppose f belongs to $\text{Diff}_k^{1+\epsilon}(T^2)$. Then for any rational $\frac{p}{q} \in \text{int}(\rho_V(\tilde{f}))$ (with $q > 0$) and any integer s , $\tilde{f}^q(\bullet) - (s, p)$ has a hyperbolic periodic point with a transversal homoclinic intersection.*

Proof:

As in the previous lemma, without loss of generality, we can suppose that $(q, p, s) = (1, 0, 0)$. Here we will present a different argument, that can also be used to prove lemma 18. The reason for this is the following. Using the ideas from the proof of lemma 18 in this context, we could not have any control on

the integer s that appears in the statement of the present lemma. We just could prove that if $f \in \text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $\frac{p}{q} \in \text{int}(\rho_V(\tilde{f}))$, then f has a hyperbolic periodic point with a transversal homoclinic intersection, whose vertical rotation number is p/q . The control on s is necessary, for instance, to prove corollary 5. So let us start the argument.

Choose two periodic orbits Q_1 and Q_2 as in theorem 13, such that their vertical rotation numbers, ρ_1 and ρ_2 are, one positive and one negative. Theorem 13 tells us that f is isotopic relative to $Q_1 \cup Q_2$ to a pseudo-Anosov map $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ rel $Q_1 \cup Q_2$. Clearly $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is an area preserving homeomorphism of the torus homotopic to a Dehn twist. So, there exists a lift of ϕ to the plane, denoted $\tilde{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is equivariantly homotopic to $\tilde{f}$ rel $p^{-1}(Q_1 \cup Q_2)$ and its vertical rotation set satisfies $0 \in]\rho_2, \rho_1[\subset \text{int}(\rho_V(\tilde{\phi}))$. This implies by theorem 6 of [3] that $\tilde{\phi}$ has fixed points. As ϕ is a pseudo-Anosov homeomorphism of the torus relative to some finite set, the foliations $\mathcal{F}^u, \mathcal{F}^s$ may have a finite number of singularities. Some of these singularities are p -prong singularities, for some $p \geq 3$ and in $Q_1 \cup Q_2$ the foliations may have 1-prong singularities, see [12] and [18]. So if $P \in \mathbb{T}^2$ is a ϕ -periodic point whose vertical rotation number belongs to $] \rho_2, \rho_1 [$, the dynamics of some adequate iterate of ϕ near P is generated by finitely many invariant hyperbolic sectors glued together. In each sector the dynamics is locally like the dynamics in the first quadrant of the map $(\tilde{x}, \tilde{y}) \rightarrow (\alpha \tilde{x}, \beta \tilde{y})$, for some real numbers $0 < \beta < 1 < \alpha$. The main difference from the dynamics in a neighborhood of a hyperbolic periodic point of a two-dimensional diffeomorphism is the fact that there may be more than four hyperbolic sectors (when P coincides with a p -prong singularity, for some $p \geq 3$), but never less because the vertical rotation number of points in $Q_1 \cup Q_2$ belongs to $\{\rho_2, \rho_1\}$.

Proposition 20 : *For any rational $\frac{r}{n} \in]\rho_2, \rho_1[$ (with $n > 0$) and any integer s , $\tilde{\phi}^n(\bullet) - (s, r)$ has a hyperbolic periodic point with a transversal homoclinic intersection when projected to the torus.*

Remarks:

1. When we say hyperbolic in the statement of this proposition, we mean that the local dynamics is obtained by gluing exactly four sectors, or equivalently the point is a regular point of the foliations.
2. An analogous result holds in the homotopic to the identity case, namely suppose g belongs to $Diff_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0, 0) \in \text{int}(\rho(\tilde{g}))$. Then, as we did in lemma 18 we can choose three periodic orbits P_1, P_2 and P_3 such that their rotation vectors form a triangle Δ that contains $(0, 0)$ in its interior and the isotopy class of g relative to $P_1 \cup P_2 \cup P_3$ contains a pseudo-Anosov map $\varphi : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \text{ rel } P_1 \cup P_2 \cup P_3$. The analogous version of proposition 20 to this situation is the following. *For any rational $(\frac{p}{n}, \frac{r}{n}) \in \text{int}(\Delta)$, $\tilde{\varphi}^n(\bullet) - (p, r)$ has a hyperbolic periodic point with a transversal homoclinic intersection when projected to the torus.* The same proof works in both cases.

Proof of proposition 20:

Without loss of generality, we can suppose that $(n, r, s) = (1, 0, 0)$. As we already said, from theorem 6 of [3] $\tilde{\phi}$ has fixed points. As ϕ is pseudo-Anosov relative to some finite set, the fixed points of $\tilde{\phi}$ project to the torus into a finite set K_1 . By the generalized Lefschetz index formula and the fact that ϕ is homotopic to a Dehn twist, the sum of the topological indexes of ϕ on these fixed points is zero. This follows from the fact that

$$\det \left(Id - \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \right) = 0,$$

so all Nielsen classes are non-essential for ϕ , see for instance [19], more precisely the comments after theorem 8.7.1.

But for some appropriate sufficiently large integer $m_2 > 0$, the local dynamics at points in K_1 implies that

$$\sum_{z \in K_1} \text{ind}(\phi^{m_2}, z) < 0.$$

This happens because all points in K_1 with negative index are saddles (maybe with more than four sectors) and points with positive index are rotating saddles.

By this we mean that the hyperbolic sectors around the point rotate under iterations of ϕ until they fall on themselves. The orientation reversing saddle $(\tilde{x}, \tilde{y}) \rightarrow (-2\tilde{x}, -0.3\tilde{y})$ is an example such that for it, $m_2 = 2$. So, for an adequate sufficiently large iterate of ϕ , all points in K_1 are saddles with each separatrix fixed, and thus they all have negative index.

Let us look at all the fixed points of $\tilde{\phi}^{m_2}$. Clearly, when projected to the torus, this set, denoted $K_2 \supset K_1$, is also finite and not equal to K_1 because, in the same way as above, the sum of the topological indexes of ϕ^{m_2} on points belonging to K_2 is zero (because ϕ^{m_2} is also homotopic to a Dehn twist). Again, in the same way as above, for some adequate sufficiently large integer $m_3 > 0$, the local dynamics at points in K_2 implies that

$$\sum_{z \in K_2} \text{ind}(\phi^{m_3 \cdot m_2}, z) < 0.$$

So, if we project the fixed points of $\tilde{\phi}^{m_3 \cdot m_2}$ to the torus, we get a finite set $K_3 \supset K_2$, not equal to K_2 by the same reason as above. In this way, we get a strictly increasing sequence of finite sets K_i . So at some i_0 , the cardinality of K_{i_0} is larger than the number of singularities of the foliations $\mathcal{F}^u, \mathcal{F}^s$. This implies that some $\tilde{\phi}$ -periodic point $\tilde{Q}$ does not fall into a singularity of the foliations $\mathcal{F}^u, \mathcal{F}^s$ when projected to the torus. So $Q \stackrel{\text{def.}}{=} p(\tilde{Q})$ is hyperbolic and its stable and unstable manifolds have a transverse intersection in the torus. This follows from the fact that ϕ is pseudo-Anosov relative to some finite set. More precisely, for any two ϕ -periodic points (which may coincide), the stable manifold of one of them intersects the unstable manifold of the other, see [12]. $\square$

Now we continue the proof of lemma 19. Note that ϕ is transitive, preserves area and by proposition 20, ϕ has hyperbolic periodic points with transversal homoclinic intersections in the torus for all rotation numbers in $]\rho_2, \rho_1[$. The following proposition can be extracted from the proof of lemma 24:

Proposition 21 : *Assume that $\varphi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a transitive area preserving homeomorphism of the torus, either homotopic to the identity or to a Dehn twist, and either $(0, 0)$ or 0 belongs to the interior of the rotation set of a (fixed) lift*

of φ to the plane, denoted $\tilde{\varphi}$. Assume also that φ has hyperbolic periodic points with transversal homoclinic intersections for all rotation vectors or numbers in a small neighborhood of the origin. Then, $\tilde{\varphi}$ has a hyperbolic periodic point denoted $\tilde{Q}$, such that $W^u(\tilde{Q})$ has a topologically transverse intersection with $W^s(\tilde{Q} + (a, b))$ for some pair of coprime integers a, b with $(a, b) \neq (0, 0)$.

Remark: When we say that $\tilde{\varphi}$ has some hyperbolic periodic point, we mean that it is differentiable at the whole orbit of this point and satisfies the usual definition of hyperbolicity on the orbit.

In the same way, the following statement can be extracted from the proofs of theorems 3 and 4:

Proposition 22 : *Under the same hypothesis as in proposition 21, the following holds: $\tilde{\varphi}$ has some periodic point denoted $\tilde{Q}$, which is hyperbolic and $W^u(\tilde{Q})$ has a topologically transverse intersection with $W^s(\tilde{Q} + (a, b))$ for all pairs of integers (a, b) .*

So $\tilde{\varphi}$ has a hyperbolic periodic point $\tilde{Q}$ such that $W^u(\tilde{Q})$ has a transverse intersection with $W^s(\tilde{Q})$. The intersection is really transverse, not only topologically transverse because ϕ is pseudo-Anosov rel to some finite ϕ -invariant set. Thus all intersections between the unstable and stable foliations $\mathcal{F}^u, \mathcal{F}^s$ (at regular points, which is the case) are transverse, see [12].

And this gives a compact $\tilde{\varphi}$ -invariant set $K_\phi \subset \mathbb{R}^2$ which carries topological entropy, that is $h_{top}(\tilde{\varphi}|_{K_\phi}) > 0$ (K_ϕ is simply an invariant set of the horseshoe $\tilde{\varphi}$ has at the point $\tilde{Q}$). The only place in the proofs of lemma 24 and theorems 3 and 4 where $C^{1+\epsilon}$ differentiability is necessary is in order to guarantee the existence of hyperbolic periodic points with transversal homoclinic intersections in the torus for rotation numbers in the interior of the vertical rotation set. As we already said, here this is achieved by proposition 20.

To conclude, we need a similar statement for $\tilde{f}$, namely we have to show that there exists a compact $\tilde{f}$ -invariant set $K_f \subset \mathbb{R}^2$ such that $h_{top}(\tilde{f}|_{K_f}) > 0$. With this purpose, we use theorem 14 and the remark after it. And finally we

apply theorem 17. This part of the proof is exactly as in end of the proof of lemma 18. $\square$

Lemma 23 : *Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho(\tilde{f}))$ is not empty or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho_V(\tilde{f}))$ is not empty . If f is transitive, then f can not have a periodic open disk.*

Proof:

The proof in both cases is analogous, so let us do it in the homotopic to the identity case. By contradiction, suppose that for some open disk $D \subset \mathbb{T}^2$ there exists $n > 0$ such that $f^n(D) = D$. Then, as f is transitive,

$$\text{Orb}(D) = D \cup f(D) \cup \dots \cup f^{n-1}(D)$$

is a f -invariant open set, dense in the torus. Moreover, as $f^n(D) = D$, we get that there exists an integer vector (k_1, k_2) such that for any connected component $\tilde{D}$ of $p^{-1}(D)$, we have:

$$\tilde{f}^n(\tilde{f}^i(\tilde{D})) = \tilde{f}^i(\tilde{D}) + (k_1, k_2), \text{ for all } 0 \leq i \leq n-1$$

As $\text{int}(\rho(\tilde{f})) \neq \emptyset$, choose a rational vector $\left(\frac{p}{q}, \frac{r}{q}\right) \in \text{int}(\rho(\tilde{f})) \setminus \left(\frac{k_1}{n}, \frac{k_2}{n}\right)$. From lemma 18, f has a hyperbolic $m.q$ -periodic point z (for some integer $m > 0$) which realizes this rotation vector, with a transversal homoclinic intersection.

Now we claim that $(W^u(z) \cup z \cup W^s(z)) \cap \text{Orb}(D) = \emptyset$. Clearly, $z \notin \text{Orb}(D)$, because $\rho(\tilde{z}) = \left(\frac{p}{q}, \frac{r}{q}\right) \neq \left(\frac{k_1}{n}, \frac{k_2}{n}\right)$. Suppose by contradiction, that a branch Γ of $W^u(z)$ or of $W^s(z)$ intersects some $f^i(D)$ (without loss of generality, we can suppose that $i = 0$). Let $w \in \Gamma \cap D$. As $f^{n.m.q}(\Gamma) = \Gamma$, first suppose that Γ is contained in D . In this case, given $\tilde{D}$ a connected component of $p^{-1}(D)$, there exists a connected component $\tilde{\Gamma} \in p^{-1}(\Gamma)$ such that $\tilde{\Gamma} \subset \tilde{D}$. As $\tilde{f}^{n.m.q}(\tilde{\Gamma}) = \tilde{\Gamma} + (n.m.p, n.m.r)$, $\tilde{f}^{n.m.q}(\tilde{D}) = \tilde{D} + (m.q.k_1, m.q.k_2)$ and $(n.m.p, n.m.r) \neq (m.q.k_1, m.q.k_2)$, we get a contradiction. So, there are points $w', w'' \in \Gamma \cap \partial D$ such that w belongs to the arc in Γ between w' and w'' and apart from its end points, this arc, denoted γ is contained in D . Clearly γ divides D into two open disks, D_1 and D_2 . So, as $f^{n.m.q} \times f^{n.m.q} : D \times D \rightarrow D \times D$ preserves

volume, we get from the Poincaré recurrence theorem that there exists an integer $N > 0$ such that $f^{N.n.m.q} \times f^{N.n.m.q}(D_1 \times D_2)$ intersects $D_1 \times D_2$. But this is a contradiction with the preservation of area of $f^{n.m.q}$ and the fact that γ is an arc contained in Γ , because Γ is a branch of $W^u(z)$ or of $W^s(z)$. So γ can not be f -periodic neither can some f -iterate of γ have a topologically transverse intersection with γ . This argument appears in the proof of lemma 6.1 of [14].

So Γ does not intersect $Orb(D)$, that is, $(W^u(z) \cup z \cup W^s(z)) \subset \partial Orb(D)$ because $Orb(D)$ is dense in the torus. As $Orb(D)$ is an open f -invariant set, if K is a connected component of $Orb(D)$, as f is transitive, the first natural n_K such that $f^{n_K}(K) = K$ (the existence of n_K follows from the fact that f is area preserving) satisfies the following

$$Orb(D) = K \cup f(K) \cup \dots \cup f^{n_K-1}(K) \text{ and the union is disjoint.}$$

Thus, $Orb(D)$ has exactly n_K connected components. But as $(W^u(z) \cup z \cup W^s(z)) \subset \partial Orb(D)$ and $W^u(z)$ has a transversal intersection with $W^s(z)$, we get that $Orb(D)$ has infinitely many connected components, a contradiction. $\square$

Lemma 24 : *Suppose f belongs to $Diff_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $Diff_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If f is transitive and $\tilde{Q} \in \mathbb{R}^2$ is a hyperbolic periodic point for $\tilde{f}$ with transversal homoclinic intersections when projected to the torus, then for some pair of integers $(a,b) \neq (0,0)$, a,b coprimes, $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (a,b))$.*

Proof:

Again, the proof in both situations is analogous, so let us present the arguments in the homotopic to the identity case. Let $\tilde{Q} \in \mathbb{R}^2$ be a hyperbolic periodic point for $\tilde{f}$ with a transversal homoclinic intersection, which exists by lemma 18. First note that there exists arbitrarily small topological rectangles $D_Q \subset \mathbb{T}^2$ such that $Q = p(\tilde{Q})$ is a vertex of D_Q and the sides of D_Q , denoted $\alpha_Q, \beta_Q, \gamma_Q$ and δ_Q are contained in $W^s(Q), W^u(Q), W^s(Q)$ and $W^u(Q)$ respectively. Let us choose another hyperbolic periodic point $P \in \mathbb{T}^2$ such that its rotation vector is not $(0,0)$, which also has a transversal homoclinic intersection

(the existence of this point P also follows from lemma 18). Let n be a natural number such that $f^n(Q) = Q$ and $f^n(P) = P$ and Df^n has positive eigenvalues at both points. Clearly the orbit of Q is disjoint from the orbit of P . So we can also choose another arbitrarily small topological rectangle $D_P \subset T^2$ such that P is a vertex of D_P and the sides of D_P , denoted $\alpha_P, \beta_P, \gamma_P$ and δ_P are contained in $W^s(P), W^u(P), W^s(P)$ and $W^u(P)$ respectively, in such a way that

$$(D_Q \cup f^{-1}(D_Q) \cup \dots \cup f^{-n+1}(D_Q)) \cap (D_P \cup f^{-1}(D_P) \cup \dots \cup f^{-n+1}(D_P)) = \emptyset$$

and for all $0 \leq i \leq n-1$, each set $f^{-i}(D_Q), f^{-i}(D_P)$ is a small rectangle in T^2 .

Also note that there exists an integer $m_0 > 0$ such that for all $0 \leq i \leq n-1$, if $m \geq m_0$, then

$$\begin{aligned} \partial(f^{n,m}(f^{-i}(D_Q))) &\subset W^u(f^{-i}(Q)) \cup f^{-i}(\alpha_Q) \\ &\text{and} \\ \partial(f^{n,m}(f^{-i}(D_P))) &\subset W^u(f^{-i}(P)) \cup f^{-i}(\alpha_P), \end{aligned}$$

because

$$f^{n,m}(f^{-i}(\alpha_Q)) \subset f^{-i}(\alpha_Q) \text{ and } f^{n,m}(f^{-i}(\alpha_P)) \subset f^{-i}(\alpha_P) \text{ for all integers } m > 0$$

and for $m \geq m_0$, $f^{n,m}(f^{-i}(\gamma_Q)) \subset f^{-i}(\alpha_Q)$ and $f^{n,m}(f^{-i}(\gamma_P)) \subset f^{-i}(\alpha_P)$.

As f is transitive, for all $0 \leq i \leq n-1$, there exist integers $l_P(i), l_Q(i) \geq m_0.n$ such that

$$f^{l_Q(i)}(f^{-i}(D_Q)) \cap D_P \neq \emptyset \text{ and } f^{l_P(i)}(f^{-i}(D_P)) \cap D_Q \neq \emptyset.$$

So for any $0 \leq i \leq n-1$, there exists integers $m_P(i), m_Q(i) \geq m_0$ and other integers $0 \leq r_P(i), r_Q(i) \leq n-1$ such that

$$f^{m_Q(i).n}(f^{-i}(D_Q)) \cap f^{-r_P(i)}(D_P) \neq \emptyset \text{ and } f^{m_P(i).n}(f^{-i}(D_P)) \cap f^{-r_Q(i)}(D_Q) \neq \emptyset.$$

This means that for all $0 \leq i \leq n-1$, $W^u(f^{-i}(Q)) \pitchfork W^s(f^{-r_P(i)}(P))$ and $W^u(f^{-i}(P)) \pitchfork W^s(f^{-r_Q(i)}(Q))$. Then a simple combinatorial argument implies that there exists $0 \leq i, j \leq n-1$ such that $W^u(f^{-i}(Q)) \pitchfork W^s(f^{-j}(P))$ and $W^u(f^{-j}(P)) \pitchfork W^s(f^{-i}(Q))$.

So maybe after renaming the points in the orbit of P , we can assume that $W^u(Q) \pitchfork W^s(P)$ and $W^u(P) \pitchfork W^s(Q)$. If we go to the plane, we get that, fixed

some $\tilde{Q}_0 \in p^{-1}(Q)$, there exists $\tilde{P}_0 \in p^{-1}(P)$ such that $W^u(\tilde{Q}_0) \pitchfork W^s(\tilde{P}_0)$. The choice of Q and P implies that $\tilde{f}^n(\tilde{Q}_0) = \tilde{Q}_0$ and there exists some pair of integers $(a_1, b_1) \neq (0, 0)$ such that $\tilde{f}^n(\tilde{P}_0) = \tilde{P}_0 + (a_1, b_1)$. So $W^u(\tilde{Q}_0) \pitchfork \left(W^s(\tilde{P}_0 + m \cdot (a_1, b_1)) \right)$ for all integers $m > 0$ (clearly $W^s(\tilde{P}_0 + m \cdot (a_1, b_1)) = W^s(\tilde{P}_0) + m \cdot (a_1, b_1)$). From the fact that $W^u(P) \pitchfork W^s(Q)$ we get that for any $\tilde{P} \in p^{-1}(P)$, there exists a certain $\tilde{Q} = \text{function}(\tilde{P})$, such that $W^u(\tilde{P}) \pitchfork W^s(\tilde{Q})$ and $\|\tilde{P} - \tilde{Q}\| < \text{Const}$, which does not depend on the choice of $\tilde{P} \in p^{-1}(P)$. From the topological transversality, we get that a piece of $W^u(Q)$ gets arbitrarily close in the Hausdorff topology to any given compact part of $W^u(P)$ (see definition 9), so if we choose an integer $m > 0$ sufficiently large, we get that a piece of $W^u(\tilde{Q}_0)$ is sufficiently close to part of $W^u(\tilde{P}_0 + m \cdot (a_1, b_1))$, something that forces $W^u(\tilde{Q}_0)$ to have a topological transverse intersection with $W^s(\tilde{Q}')$ for some $\tilde{Q}' \in p^{-1}(Q)$ such that

$$\|(\tilde{P}_0 + m \cdot (a_1, b_1)) - \tilde{Q}'\| < \text{Const}.$$

Thus if $m > 0$ is sufficiently large, $\tilde{Q}' \neq \tilde{Q}_0$. To conclude the proof we just have to note that $W^u(\tilde{Q}_0) \cup \tilde{Q}_0 \cup W^s(\tilde{Q}_0)$ is an arc connected subset of $\mathbb{R}^2$, which transversely intersects $(W^u(\tilde{Q}_0) \cup \tilde{Q}_0 \cup W^s(\tilde{Q}_0)) + (c', d')$, for some pair of integers $(c', d') \neq (0, 0)$. If (c', d') equals some integer j times (c, d) , with c and d coprimes, we need the following lemma, which is a consequence of the so called Brouwer lemma on translation arcs, see corollary 3.3 of [7].

Lemma 25 : *Let K be an arc connected subset of $\mathbb{R}^2$ and let $\tilde{h} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an orientation preserving homeomorphism of the plane with no fixed points. If $\tilde{h}(K) \cap K = \emptyset$, then $\tilde{h}^n(K) \cap K = \emptyset$ for all integers $n \neq 0$.*

The above lemma applied to $K = W^u(\tilde{Q}_0) \cup \tilde{Q}_0 \cup W^s(\tilde{Q}_0)$ and $\tilde{h} = Id + (c, d)$ implies that $W^u(\tilde{Q}_0) \cup \tilde{Q}_0 \cup W^s(\tilde{Q}_0)$ must intersect $(W^u(\tilde{Q}_0) \cup \tilde{Q}_0 \cup W^s(\tilde{Q}_0)) + (c, d)$, also in a topologically transverse way, and so the proof of lemma 24 is over. $\square$

Remarks:

1. note that the above argument also implies that there exists a pair of

integers $(p, q) \neq (0, 0)$, p and q coprimes, such that for any $\tilde{P} \in p^{-1}(P)$, $W^u(\tilde{P}) \cap W^s(\tilde{P} + (p, q))$. As the only hypothesis on $\tilde{P}$ was that its rotation vector is not $(0, 0)$, we get that this is a general property, true for all rational vectors in the interior of $\rho(\tilde{f})$;

2. As $W^u(\tilde{Q}) \cap W^s(\tilde{Q} + (a, b))$, it is easy to prove that $\overline{W^u(\tilde{Q})} \supset W^u(\tilde{Q}) + (a, b)$;

Lemma 26 : *In case $f \in \text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$, let P be a hyperbolic m -periodic point for f with a transversal homoclinic intersection, whose rotation vector lies in the interior of $\rho(\tilde{f})$ and in case $f \in \text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$, let P be a hyperbolic m -periodic point for f with a transversal homoclinic intersection, whose vertical rotation number lies in the interior of $\rho_V(\tilde{f})$ (in both cases, $m > 0$ is an integer). If f is transitive, then $\overline{W^u(P)} = \overline{W^s(P)} = \mathbb{T}^2$.*

Proof:

The proof is analogous for $W^u(P)$ and $W^s(P)$, so let us present it for $W^u(P)$. As $f^m(\overline{W^u(P)}) = \overline{W^u(P)}$, if we suppose by contradiction that $\overline{W^u(P)} \neq \mathbb{T}^2$, then the open set $(\overline{W^u(P)})^c$ is non-empty and invariant under f^m . First, suppose that some of its connected components contains a homotopically non trivial simple closed curve γ . Then

$$(\gamma \cup f^m(\gamma)) \cap (B_\epsilon(P) \cup W^u(P)) = \emptyset, \quad (7)$$

for a sufficiently small $\epsilon > 0$.

From remark 1 right after the proof of lemma 24, we get that

$$\begin{aligned} &\text{there exists a pair of integers } (p, q) \neq (0, 0), \text{ } p \text{ and } q \text{ coprimes,} \\ &\text{such that for any } \tilde{P} \in p^{-1}(P), W^u(\tilde{P}) \cap (W^s(\tilde{P}) + (p, q)). \end{aligned} \quad (8)$$

In the following we will analyze separately the homotopic to the identity case and the Dehn twist case and in both we will arrive at a contradiction.

- First suppose that $f \in \text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$.

Note that there exists a pair of integers (a_1, a_2) such that for any $\tilde{P} \in p^{-1}(P)$, $\tilde{f}^m(\tilde{P}) - (a_1, a_2) = \tilde{P}$ and $(0, 0) \in \text{int}(\rho(\tilde{f}^m(\bullet) - (a_1, a_2)))$. If we set

$\tilde{g}(\bullet) \stackrel{\text{def.}}{=} \tilde{f}^m(\bullet) - (a_1, a_2)$, then $\tilde{g}(\tilde{P}) = \tilde{P}$ for any $\tilde{P} \in p^{-1}(P)$ and so we can construct a simple curve η in the plane which consists of a piece of $W^u(\tilde{P})$ starting at $\tilde{P}$ until it continues through the piece of $W^s(\tilde{P}) + (p, q)$ which is inside $B_\epsilon(\tilde{P}) + (p, q)$ and contains $\tilde{P} + (p, q)$, see figure 2. Let θ be equal to $\bigcup_{i \in \text{integers}} \eta + i(p, q)$, which is a closed connected set. Then, it is easy to see that

$$\theta \cap p^{-1}(\gamma) = \emptyset.$$

So, each connected component $\tilde{\gamma}$ of $p^{-1}(\gamma)$ is parallel to θ , namely $\tilde{\gamma} = \tilde{\gamma} + (p, q)$. Moreover, the following holds:

1. $\tilde{g}^l(\theta) \subset \bigcup_{i \in \text{integers}} \left(\left(B_\epsilon(\tilde{P}) \cup W^u(\tilde{P}) \right) + i.(p, q) \right)$, for all integers $l > 0$;
2. $\tilde{g}^l(\theta)$ intersects $p^{-1}(\gamma)$ if $l > 0$ is sufficiently large (this happens because $\tilde{g}^l(\theta) \cap \theta \neq \emptyset$ and g has periodic points in the torus with rotation vector parallel to $(-q, p)$);

So $B_\epsilon(P) \cup W^u(P)$ intersects γ , a contradiction with expression (7).

- Now suppose $f \in Dif_k^{1+\epsilon}(\mathbb{T}^2)$.

If γ is not homotopic to $(1, 0)$, then as in this case $f^m(\gamma)$ and γ are in different homotopic classes and $B_\epsilon(P) \cup W^u(P)$ contains a homotopically non-trivial simple closed curve, we get that $B_\epsilon(P) \cup W^u(P)$ either intersects γ or $f^m(\gamma)$, a contradiction. So we are left to consider the case when γ is an horizontal curve, that is, γ is homotopic to $(1, 0)$. In this case, concerning the value of q in expression (8), there are 2 possibilities:

i) $q \neq 0$. This implies that there exists a homotopically non-trivial closed curve contained in $B_\epsilon(P) \cup W^u(P)$ which is homotopic to (m, q) , for some integer m . Note that m may not be equal to p , which appears in (8). The reason for this is that we may have to iterate the (p, q) curve contained in $W^u(P) \cup P \cup W^s(P)$ many times under f^m so that its piece contained in $W^s(P)$ becomes very small, more precisely, contained in $B_\epsilon(P)$. Nevertheless, as $q \neq 0$, this curve must intersect γ , a contradiction;

ii) $q = 0$. In this case, if we lift everything to the cylinder $S^1 \times \mathbb{R}$, we obtain a homotopically non trivial closed curve $\hat{\theta} \subset W^u(\hat{P}) \cup \hat{P} \cup W^s(\hat{P})$, for some $\hat{P} \in S^1 \times \mathbb{R}$ such that $p(\pi^{-1}(\hat{P})) = P \in T^2$. Moreover, $\hat{\theta}$ is obtained as the union of two simple connected arcs, one starting at $\hat{P}$, contained in $W^u(\hat{P})$ and the other contained in $W^s(\hat{P})$, ending at $\hat{P}$. As P is f -periodic with period m , there exists an integer r such that

$$\hat{f}^m(\hat{P}) = \hat{P} + (0, r).$$

So if we define $\hat{g}(\bullet) \stackrel{def}{=} \hat{f}^m(\bullet) - (0, r)$, we get that $\hat{g}^l(\hat{\theta}) \cap \hat{\theta} \neq \emptyset$ for all integers l and if $l > 0$ is sufficiently large, then $\hat{g}^l(\hat{\theta}) \subset W^u(\hat{P}) \cup B_\epsilon(\hat{P})$ and it intersects $\pi(p^{-1}(\gamma))$, which again contradicts (7). This happens because of two facts: a) if l is sufficiently large, then $\hat{g}^l(\hat{\theta} \cap W^s(\hat{P})) \subset B_\epsilon(\hat{P})$; b) if $\tilde{g}$ is a lift of $\hat{g}$ to the plane, then $0 \in \text{int}(\rho_V(\tilde{g}))$, so $\tilde{g}^l(\hat{\theta})$ is connected and satisfies $p_2(\tilde{g}^l(\hat{\theta})) \xrightarrow{l \rightarrow \infty} \mathbb{R}$.

Thus, in both cases, all connected components of the open set $\left(\overline{W^u(P)}\right)^c$ do not contain homotopically non-trivial (considered as torus curves) closed curves.

To conclude, suppose for some connected component U of $\left(\overline{W^u(P)}\right)^c$, there exists a simple closed curve α contained in U which is not contractible in U . The following argument works in both the homotopic to the identity case and the Dehn twist case.

From what we did above, α is homotopically trivial as a curve in the torus. So in the open disk bounded by α there are points that belong to $\overline{W^u(P)}$, otherwise α would be contractible in U . But this means that the whole $\overline{W^u(P)}$ is contained in this disk, because $\overline{W^u(P)}$ is connected. And this is a contradiction with the first thing we did, the fact that all connected components of the open set $\left(\overline{W^u(P)}\right)^c$ do not contain homotopically non-trivial simple closed curves. So all simple closed curves contained in U are contractible in U . This means U is an open disk, which is periodic under f (because f is area-preserving). And this contradicts lemma 23 and proves the present lemma. $\square$

Now we are ready to prove our main results. Before presenting the proofs, we will restate each one of them.

3.2 Proofs of theorems 3 and 4

Proof of theorem 3:

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$, it is transitive and $\left(\frac{p}{q}, \frac{r}{q}\right) \in \text{int}(\rho(\tilde{f}))$. Then, $\tilde{f}^q(\bullet) - (p, r)$ has a hyperbolic periodic point $\tilde{Q}$ such that $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \mathbb{R}^2$.

We know from lemmas 18 and 24 that there exists a hyperbolic periodic point $Q \in \mathbb{T}^2$ such that its rotation vector is $\left(\frac{p}{q}, \frac{r}{q}\right)$, $W^u(Q)$ has a transverse intersection with $W^s(Q)$ and for some pair of integers $(a, b) \neq (0, 0)$, a, b coprimes, for all $\tilde{Q}' \in p^{-1}(Q)$, $W^u(\tilde{Q}') \cap W^s(\tilde{Q}' + (a, b))$. As a and b are coprimes, there are integers c, d such that

$$a.d - c.b = 1. \quad (9)$$

In this way, for any given pair of integers (i_*, j_*) , there exists another pair of integers (i, j) such that

$$i.(a, b) + j.(c, d) = (i_*, j_*). \quad (10)$$

So fixed some $\tilde{Q} \in p^{-1}(Q)$, $W^u(\tilde{Q} + j(c, d) + i(a, b)) \cap W^s(\tilde{Q} + j(c, d) + (i+1)(a, b))$, for all integers i, j . Now let $\tilde{g}(\bullet) \stackrel{\text{def.}}{=} \tilde{f}^q(\bullet) - (p, r)$ and suppose $m > 0$ is an integer such that $\tilde{g}^m(\tilde{Q}) = \tilde{Q}$. As in the proof of lemma 26, we can construct a path connected set $\theta \subset \mathbb{R}^2$ such that:

1. $\theta = \theta + (a, b)$;
2. θ contains $\tilde{Q} + i(a, b)$, for all integers i ;
3. θ contains a simple arc η from $\tilde{Q}$ to $\tilde{Q} + (a, b)$ of the following form: η starts at $\tilde{Q}$, contains a piece λ of a branch of $W^u(\tilde{Q})$ until it reaches $W^s(\tilde{Q}) + (a, b)$ and then η contains a piece μ of a branch of $W^s(\tilde{Q}) + (a, b)$ until it reaches $\tilde{Q} + (a, b)$, see figure 3. Clearly $\mu \subset W^s(\tilde{Q}) + (a, b)$ can be chosen arbitrarily small. The set θ is obtained from η in the following natural way: $\theta = \bigcup_{i \in \text{integers}} \eta + i(a, b)$;
4. θ is bounded in the $(-b, a)$ direction, that is, θ is contained between two straight lines, both parallel to (a, b) ;

For any given integer l , let us choose two straight lines, $L_1(l)$ and $L_2(l)$ both parallel to (a, b) such that $\theta + l(c, d)$ is contained between them and $(L_1(l) \cup L_2(l)) \cap \theta = \emptyset$. Now we remember that as $(0, 0) \in \text{int}(\rho(\tilde{g}))$, there exist two periodic points for g , one with rotation vector of the form $(\frac{-b}{N}, \frac{a}{N})$ and the other with rotation vector of the form $(\frac{b}{N}, \frac{-a}{N})$, for some sufficiently large integer $N > 0$, see [13]. So as $\tilde{g}^{m.t}(\theta) \cap \theta \neq \emptyset$ for all integers t , there exists an integer $t_* > 0$ such that $\tilde{g}^{m.t}(\theta)$ intersects both $L_1(l)$ and $L_2(l)$ for all $t \geq t_*$. The way θ is constructed implies that there exists an integer i_l such that

$$W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + l(c, d) + i(a, b)), \text{ for all } i \geq i_l. \quad (11)$$

To prove that for any given pair of integers (i_*, j_*) , $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (i_*, j_*))$, from expressions (9), (10) and (11) it is enough to show that $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} - (a, b))$. And this is achieved by an argument similar to the one in the proof of lemma 24. To be precise, in that proof we can choose the hyperbolic periodic point $P \in T^2$ such that its rotation vector (for the map $\tilde{g}$) is $(\frac{-a}{n'}, \frac{-b}{n'})$, for some integer $n' > 0$ such that $(\frac{-a}{n'}, \frac{-b}{n'}) \in \text{int}(\rho(\tilde{g}))$. Clearly, the rotation vector of P with respect to $\tilde{f}$ is

$$\left(\frac{n'p - a}{n'q}, \frac{n'r - b}{n'q} \right).$$

So, maybe after renaming the points in the orbit of P , we can assume that $W^u(Q) \pitchfork W^s(P)$ and $W^u(P) \pitchfork W^s(Q)$. If we go back to the plane we get that there exists $\tilde{P} \in p^{-1}(P)$ such that $W^u(\tilde{Q}) \pitchfork W^s(\tilde{P})$.

Let $n > 0$ be an integer such that $g^n(Q) = Q$, $g^n(P) = P$ and Dg^n has positive eigenvalues at both points. The choice of Q and P implies that $\tilde{g}^n(\tilde{Q}) = \tilde{Q}$ and there exists an integer $c > 0$ such that $\tilde{g}^n(\tilde{P}) = \tilde{P} + c(-a, -b)$. So $W^u(\tilde{Q}) \pitchfork W^s(\tilde{P} + t.c(-a, -b))$ for all integers $t > 0$. From the fact that $W^u(P) \pitchfork W^s(Q)$ we get that for any $\tilde{P}' \in p^{-1}(P)$, there exists a certain $\tilde{Q}' = \text{function}(\tilde{P}')$ such that $W^u(\tilde{P}') \pitchfork W^s(\tilde{Q}')$ and $\|\tilde{P}' - \tilde{Q}'\| < \text{Const}$. Moreover, $\text{Const} > 0$ does not depend on the choice of $\tilde{P}' \in p^{-1}(P)$. From the topological transversality, we get that $\overline{W^u(Q)}$ contains $W^u(P)$, so if we choose an integer $t > 0$ sufficiently large, we get that a connected piece of $W^u(\tilde{Q})$ is sufficiently close in the Hausdorff topology to part of $W^u(\tilde{P} + t.c(-a, -b))$,

something that forces $W^u(\tilde{Q})$ to have a topological transverse intersection with $W^s(\tilde{Q}')$ for some $\tilde{Q}' \in p^{-1}(Q)$ such that $\|(\tilde{P} + t.c(-a, -b)) - \tilde{Q}'\| < \text{Const.}$ Thus $\tilde{Q}' = \tilde{Q} + t.c(-a, -b) + e_1(a, b) + e_2(c, d)$, for some integer pair (e_1, e_2) such that $\max\{|e_1|, |e_2|\}$ is bounded independently of $t > 0$. Now using expression (11) we get that if $t > 0$ is sufficiently large, then $W^u(\tilde{Q}') \pitchfork W^s(\tilde{Q} - i(a, b))$, for some $i > 0$. Thus $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} - (a, b))$.

To conclude our proof, note that lemma 26 implies that given an open ball $U \subset \mathbb{R}^2$, there exists $\tilde{Q}' \in p^{-1}(Q)$ such that $W^u(\tilde{Q}') \cap U \neq \emptyset$. As $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q}')$, we get that $W^u(\tilde{Q}) \cap U \neq \emptyset$ and we are done. The proof for $W^s(\tilde{Q})$ is analogous. $\square$

Proof of theorem 4:

Statement: Suppose f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$, it is transitive, $\frac{p}{q} \in \text{int}(\rho_V(\tilde{f}))$ and s is any integer number. Then, $\tilde{f}^q(\bullet) - (s, p)$ has a hyperbolic periodic point $\tilde{Q}$ such that $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \mathbb{R}^2$.

Let $f \in \text{Diff}_{k_{Dehn}}^{1+\epsilon}(\mathbb{T}^2)$ for some integer $k_{Dehn} \neq 0$. Without loss of generality, let us suppose that $k_{Dehn} > 0$. We know from lemma 24 that there exists a hyperbolic periodic point $\tilde{Q} \in \mathbb{R}^2$ for $\tilde{g}(\bullet) \stackrel{\text{def.}}{=} \tilde{f}^q(\bullet) - (s, p)$, of period $m > 0$, such that for some pair of integers $(a, b) \neq (0, 0)$, a, b coprimes, $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (a, b))$.

First, let us suppose that $b > 0$. As $\tilde{g}^m(\tilde{Q}) = \tilde{Q}$ and $\tilde{g}$ is homotopic to a Dehn twist, we get that for any integer l , $\tilde{g}^m(\tilde{Q} + (0, l)) = \tilde{Q} + (k_{Dehn}.q.m.l, l)$. So, $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (a + k_{Dehn}.m.q.b, b))$ and there exists a compact path connected set $\gamma \subset \mathbb{R}^2$ connecting $\tilde{Q}$ to $\tilde{Q} + (k_{Dehn}.m.q.b, 0)$ of the following form. The set γ is contained in

$$W^u(\tilde{Q}) \cup W^s(\tilde{Q} + (a + k_{Dehn}.m.q.b, b)) \cup W^u(\tilde{Q} + (k_{Dehn}.m.q.b, 0))$$

and it contains a connected arc of $W^s(\tilde{Q} + (a + k_{Dehn}.m.q.b, b))$ that has one endpoint in $\tilde{Q} + (a + k_{Dehn}.m.q.b, b)$.

Thus $\pi(\gamma) \stackrel{\text{def.}}{=} \hat{\gamma} \subset S^1 \times \mathbb{R}$ contains a homotopically non trivial simple closed curve and $\hat{\gamma} \subset W^u(\hat{Q}) \cup W^s(\hat{Q} + (0, b))$, where $\hat{Q} = \pi(\tilde{Q})$. As $\hat{\gamma}$ is compact, for every integer n there exist numbers $M_-(n) < M_+(n)$ such that:

1. $\widehat{\gamma} + (0, n) \subset S^1 \times]M_-(n), M_+(n)[$;
2. $S^1 \times \{M_-(n), M_+(n)\}$ does not intersect $\widehat{\gamma}$;

As $\widehat{g}$ has points with positive and negative vertical rotation number and $\widehat{g}^{i.m}(\widehat{\gamma}) \cap \widehat{\gamma} \neq \emptyset$ for all integers i , we get that for every integer n , there exists an integer $i(n) > 0$ such that

$$\widehat{g}^{i.m}(\widehat{\gamma}) \text{ intersects } S^1 \times \{M_-(n)\} \text{ and } S^1 \times \{M_+(n)\} \text{ for all } i > i(n).$$

This means that $W^u(\widehat{Q}) \pitchfork W^s(\widehat{Q} + (0, n + b))$ for all integers n . So, $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c(l), l))$ for all integers l and for some function $c(l)$.

If we remember that $\widetilde{g}^{m.i}(\widetilde{Q} + (0, l)) = \widetilde{Q} + (k_{Dehn}.q.m.i.l, l)$ for any integers l and $i > 0$, we get that $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c(l) + k_{Dehn}.q.m.i.l, l))$ for all integers l and $i > 0$. Using this it is easy to see that for some integer constants $c_- < 0 < c_+$, the following intersections hold: $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c_+, 0))$ and $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c_-, 0))$. And finally, the same argument applied in the end of the proof of lemma 24 using lemma 25 implies that $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (1, 0))$ and $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} - (1, 0))$.

If $b = 0$, there exists a continuous arc $\gamma \subset \mathbb{R}^2$ connecting $\widetilde{Q}$ to $\widetilde{Q} + (a, 0)$ contained in $W^u(\widetilde{Q}) \cup W^s(\widetilde{Q} + (a, 0))$. Thus $\pi(\gamma) = \widehat{\gamma} \subset S^1 \times \mathbb{R}$ contains a homotopically non trivial simple closed curve and $\pi(\gamma) \subset W^u(\widehat{Q}) \cup W^s(\widehat{Q})$, where $\widehat{Q} = \pi(\widetilde{Q})$. Arguing exactly as in the $b > 0$ case we get that $W^u(\widehat{Q}) \pitchfork W^s(\widehat{Q} + (0, n))$ for all integers n . So, $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c(n), n))$ for all integers n and for some function $c(n)$. And finally we obtain, again exactly as in the $b > 0$ case, that $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (1, 0))$ and $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} - (1, 0))$.

The case $b < 0$ is analogous to the case $b > 0$.

In this way, given a pair of integers (c, d) , we know that for some integer s , $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (s, d))$. But this implies that $W^u(\widetilde{Q} + (c - s, 0)) \pitchfork W^s(\widetilde{Q} + (c, d))$. As $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c - s, 0))$, we finally get that $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q} + (c, d))$. As in the previous theorem, note that lemma 26 implies that given an open ball $U \subset \mathbb{R}^2$, there exists $\widetilde{Q}' \in p^{-1}(Q)$ such that $W^u(\widetilde{Q}') \cap U \neq \emptyset$. As $W^u(\widetilde{Q}) \pitchfork W^s(\widetilde{Q}' + (a, 0))$, for all integers a , we get that $W^u(\widetilde{Q}) \cap U \neq \emptyset$ and the proof is over. The proof for $W^s(\widetilde{Q})$ is analogous. $\square$

3.3 Proof of corollary 5

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If f is transitive, then $\tilde{f}$ is topologically mixing.

Remember that topologically mixing means the following: given U, V be arbitrarily small open balls in $\mathbb{R}^2$, there exists an integer $N = N(U, V) > 0$ such that if $n \geq N$, then $\tilde{f}^n(U) \cap V \neq \emptyset$.

First note that, from the previous theorems, $\tilde{f}$ has a hyperbolic m -periodic point $\tilde{Q}$ such that $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \mathbb{R}^2$ (this is all we need to get the topological mixing). So, $W^u(\tilde{f}^i(\tilde{Q}))$ intersects V and $W^s(\tilde{f}^i(\tilde{Q}))$ intersects U , for all $0 \leq i \leq m-1$. Let λ_i^s be a compact connected piece of a branch of $W^s(\tilde{f}^i(\tilde{Q}))$ such that λ_i^s starts at $\tilde{f}^i(\tilde{Q})$ and intersects U , for all $0 \leq i \leq m-1$. Clearly, there exists $N > 0$, a large integer, such that if $n \geq N$, then a connected piece of $\tilde{f}^{-n}(V)$ is sufficiently close in the Hausdorff topology to $\lambda_{i(n)}^s$, for some $0 \leq i(n) \leq m-1$, in a way that this forces $\tilde{f}^{-n}(V)$ to intersect U . And so $V \cap \tilde{f}^n(U) \neq \emptyset$. $\square$

3.4 Proof of lemma 1

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. Then there exists $\tilde{Q} \in \mathbb{R}^2$, which is a hyperbolic periodic point for $\tilde{f}$ such that for some pair of integers $(a, b) \neq (0, 0)$, a, b coprimes, $W^u(\tilde{Q}) \cap W^s(\tilde{Q} + (a, b))$ (note that $W^s(\tilde{Q} + (a, b)) = W^s(\tilde{Q}) + (a, b)$). In particular, if $Q = p(\tilde{Q})$ then $W^s(Q) \cup Q \cup W^u(Q)$ contains a homotopically non-trivial simple closed curve in $\mathbb{T}^2$.

This proof is analogous in both cases, so let us suppose that f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$. As we did in lemma 18, choose three periodic orbits Q_1, Q_2 and Q_3 as in theorem 12, such that their rotation vectors form a triangle Δ that contains $(0,0)$ in its interior. Theorem 12 tells us that the isotopy class of f relative to $Q_1 \cup Q_2 \cup Q_3$ contains a pseudo-Anosov map $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2 \text{ rel } Q_1 \cup Q_2 \cup Q_3$.

As $\phi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a homeomorphism of the torus homotopic to the identity, there exists a lift of ϕ to the plane, denoted $\tilde{\phi} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is equivariantly homotopic to $\tilde{f}$ rel to $p^{-1}(Q_1 \cup Q_2 \cup Q_3)$ and its rotation set $\rho(\tilde{\phi}) \supset \Delta \supset \text{int}(\Delta) \ni (0,0)$.

Proposition 27 : *The map $\tilde{\phi}$ is topologically mixing in $\mathbb{R}^2$, which implies that for every integer $N > 0$, $\tilde{\phi}^N$ is a transitive map of the plane.*

Proof:

In the proof of lemma 19 we presented propositions 21 and 22, which can be directly extracted from the proofs of lemma 24 and theorems 3 and 4. Analogously, the next proposition can be extracted from the proof of lemma 26:

Proposition 28 : *Assume that $\varphi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ is a transitive area preserving homeomorphism of the torus, either homotopic to the identity or to a Dehn twist, such that for some (fixed) lift of φ to the plane, denoted $\tilde{\varphi}$, an open neighborhood of zero, denoted V , belongs to the interior of its rotation set. Also assume that φ has hyperbolic periodic points with transversal homoclinic intersections for all rotation vectors (or numbers) in V . If $Q \in \mathbb{T}^2$ is a hyperbolic φ -periodic point with transversal homoclinic intersections, whose rotation vector (or number) belongs to V , then $\overline{W^u(Q)} = \overline{W^s(Q)} = \mathbb{T}^2$.*

As ϕ is transitive, preserves area and for all rationals $\rho \in \text{int}(\Delta)$, ϕ has a hyperbolic (remember that in this context, hyperbolic means four sectors) periodic point with transverse homoclinic intersections in the torus whose rotation vector is ρ (see proposition 20 and remark 2 right after it), we get from propositions 21, 22 and 28 that there exists a hyperbolic $\tilde{\phi}$ -periodic point $\tilde{Q} \in \mathbb{R}^2$ such that $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \mathbb{R}^2$ and so corollary 5 finally proves the present proposition. $\square$

Remark: The above proof also works in case f is homotopic to a Dehn twist.

From theorem 14 and the remark right after it, for any fixed integer $N > 0$, we get that

$$s \circ f^N|_W = \phi^N \circ s \text{ and } \tilde{s} \circ \tilde{f}^N|_{p^{-1}(W)} = \tilde{\phi}^N \circ \tilde{s} \quad (12)$$

for a certain lift $\tilde{s} : p^{-1}(W) \rightarrow \mathbb{R}^2$ such that $\sup_{\tilde{z} \in p^{-1}(W)} \|\tilde{s}(\tilde{z}) - \tilde{z}\| < \infty$. As $\tilde{\phi}^N$ is a transitive map of the plane (by proposition 27), there exists a point $\tilde{z}^* \in p^{-1}(W)$ such that $\tilde{s}(\tilde{z}^*)$ has a dense orbit under iterates of $\tilde{\phi}^N$. This is equivalent to saying that the ω -limit set of $\tilde{s}(\tilde{z}^*)$ under $\tilde{\phi}^N$ is the whole plane. Expression (12) then implies that

$$\tilde{s}(\omega\text{-limit set of } \tilde{z}^* \text{ under } \tilde{f}^N) = (\omega\text{-limit set of } \tilde{s}(\tilde{z}^*) \text{ under } \tilde{\phi}^N) = \mathbb{R}^2.$$

So as $\tilde{s}|_{p^{-1}(W')}$ is one to one, $\overline{p^{-1}(W')} = p^{-1}(W)$ and $\tilde{s}(p^{-1}(W')) = p^{-1}(\{\text{periodic points of } \phi\})$, which is a dense subset of the plane (again, see the remark right after theorem 14), we get that the ω -limit set of $\tilde{z}^*$ under $\tilde{f}^N$ is the whole $p^{-1}(W)$. In other words, we have proved the following proposition:

Proposition 29 : *For every integer $N > 0$, there exists a point $\tilde{z}^* \in p^{-1}(W)$ such that its orbit under $\tilde{f}^N$ is dense in $p^{-1}(W)$.*

Clearly, $p^{-1}(W) \supset K_f$, where $K_f \subset \mathbb{R}^2$ is a $\tilde{f}$ -invariant compact set such that $h_{top}(\tilde{f}|_{K_f}) > 0$, see the proof of lemma 18. Let us fix some $\epsilon > 0$. As we already explained in subsection 2.2, there exists a hyperbolic ergodic $\tilde{f}$ -invariant Borel probability measure μ^* with positive entropy, whose support is contained in K_f . Now we describe two results from [8] on Pesin theory.

For every $\delta > 0$, there exists a compact subset $\Lambda_\delta \subset K_f$ with $\mu^*(\Lambda_\delta) > 1 - \delta$, which is the so called δ -Pesin set. Each of its points satisfies some local hyperbolicity assumptions, see for instance section 3 of the supplement by Katok-Mendoza in [19]. Given any point $\tilde{w} \in \Lambda_\delta$, there exists a sufficiently small compact neighborhood V of $\tilde{w}$ such that the local stable manifolds of points $\tilde{w}' \in V \cap \Lambda_\delta$ are either disjoint or equal and they depend continuously on the point $\tilde{w}' \in V \cap \Lambda_\delta$. The same holds for the unstable manifolds. There is also a continuous product structure: given two points in $V \cap \Lambda_\delta$ the intersection between the stable manifold of one of them with the unstable manifold of the other is transverse and thus consists of exactly one point, which varies continuously with the first two points and may not be in Λ_δ . There are also some new technical definitions in [8], called inaccessibility and accessibility of a point in

$V \cap \Lambda_\delta \cap \text{Recurrent}^\pm$, where $\text{Recurrent}^\pm$ is the subset of all points in K_f which are both forward and backward recurrent, see definition 6 of [8]. With this said, we can state lemmas 5 and 6 of [8]:

Lemma 5 of [8]: *Let $\tilde{w}' \in V \cap \Lambda_\delta \cap \text{Recurrent}^\pm$ be an inaccessible point. Then there exist rectangles enclosing $\tilde{w}'$ having sides along the invariant manifolds of two hyperbolic saddles in V and having arbitrarily small diameter.*

Lemma 6 of [8]: *The set of accessible points in $V \cap \Lambda_\delta \cap \text{Recurrent}^\pm$ has zero μ^* measure.*

As $\mu^*(\text{Recurrent}^\pm) = 1$ and we can start with some point $\tilde{w} \in \Lambda_\delta \cap \text{supp}(\mu^*)$, the following holds: $\mu^*(V \cap \Lambda_\delta \cap \text{Recurrent}^\pm) > 0$, so lemmas 5 and 6 of [8] then say that there exists a point $\tilde{z} \in K_f$ such that arbitrarily small rectangles enclosing $\tilde{z}$ can be obtained, having sides along the invariant manifolds of two hyperbolic $\tilde{f}$ -periodic points $\tilde{Q}_{V1}, \tilde{Q}_{V2}$, whose orbits are contained in $V_\epsilon(K_f)$, see figure 4 and theorem 17. Clearly, if we want smaller rectangles, the points $\tilde{Q}_{V1}, \tilde{Q}_{V2}$ change, getting closer and closer to $\tilde{z}$. Denote these rectangles by $\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2})$. Standard arguments in Pesin theory imply that the invariant manifolds of these periodic points have transverse intersections. So,

$$\overline{W^s(\tilde{Q}_{V1})} = \overline{W^s(\tilde{Q}_{V2})} \text{ and } \overline{W^u(\tilde{Q}_{V1})} = \overline{W^u(\tilde{Q}_{V2})}. \quad (13)$$

Suppose $\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2})$ is sufficiently small in a way that for all pairs of integers $(a, b) \neq (0, 0)$,

$$\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2}) \cap (\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2}) + (a, b)) = \emptyset. \quad (14)$$

Let us fix an integer $N > 0$ which is a common period for $\tilde{Q}_{V1}$ and $\tilde{Q}_{V2}$. For every pair of integers (a, b) , as the interiors of both rectangles, $\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2})$ and $\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2}) + (a, b)$, intersect $p^{-1}(W)$, we get from proposition 29 that there exists an integer $k(a, b) > 0$ such that

$$\tilde{f}^{k(a,b) \cdot N}(\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2})) \cap (\widetilde{\text{Ret}}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2}) + (a, b)) \neq \emptyset. \quad (15)$$

So from expressions (13), (14) and (15) we get, that $W^u(\tilde{Q}_{V1}) \pitchfork W^s(\tilde{Q}_{V1} + (a, b))$. $\square$

3.5 Proof of theorem 6

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. Then, $\tilde{f}$ has a hyperbolic periodic point $\tilde{Q}$ such that for any pair of integers (a,b) , $W^u(\tilde{Q}) \cap W^s(\tilde{Q} + (a,b))$, so $\overline{W^u(\tilde{Q})} = \overline{W^u(\tilde{Q})} + (a,b)$ and $\overline{W^s(\tilde{Q})} = \overline{W^s(\tilde{Q})} + (a,b)$. Moreover, $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})}$, this set is $\tilde{f}$ -invariant ($\tilde{f}(\overline{W^u(\tilde{Q})}) = \overline{W^u(\tilde{Q})}$) and all connected components of the complement of $\overline{W^u(\tilde{Q})}$ are open disks with diameter uniformly bounded from above, f -periodic when projected to the torus.

This proof is analogous in both cases, so let us suppose that f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$.

The first part of the theorem follows from lemma 1 which says that there exists a hyperbolic $\tilde{f}$ -periodic point $\tilde{Q}_0$ of period $m > 0$, such that for all pairs of integers (a,b) , $W^u(\tilde{Q}_0) \cap W^s(\tilde{Q}_0 + (a,b))$.

So $\overline{W^u(\tilde{Q}_0)}$ is invariant under integer translations and the same holds for $\overline{W^s(\tilde{Q}_0)}$. In particular, this means that if $Q_0 = p(\tilde{Q}_0)$, then

$$\overline{W^u(\tilde{Q}_0)} = p^{-1}(\overline{W^u(Q_0)}) \text{ and } \overline{W^s(\tilde{Q}_0)} = p^{-1}(\overline{W^s(Q_0)}). \quad (16)$$

Now let us prove that $\overline{W^u(\tilde{Q}_0)} = \overline{W^s(\tilde{Q}_0)}$ and each connected component of $\left(\overline{W^u(\tilde{Q}_0)}\right)^c$ is a disk with diameter uniformly bounded from above. We need another lemma.

Lemma 30 : Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho(\tilde{f}))$ is not empty or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho_V(\tilde{f}))$ is not empty. Then, f can not have a periodic unbounded open disk. Moreover, all periodic open disks have diameter uniformly bounded from above.

Proof:

This proof will be based on the proof of lemma 23 and it will be presented in case f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho(\tilde{f}))$ is not empty because a similar argument works in case f is homotopic to a Dehn twist.

Suppose that for some open disk $D \subset \mathbb{T}^2$ there exists $n > 0$ such that $f^n(D) = D$. Then there exists a integer vector (k_1, k_2) such that for any con-

nected component $\tilde{D}$ of $p^{-1}(D)$, we have:

$$\tilde{f}^n(\tilde{f}^i(\tilde{D})) = \tilde{f}^i(\tilde{D}) + (k_1, k_2), \text{ for all } 0 \leq i \leq n-1$$

Choose a rational vector $\left(\frac{p}{q}, \frac{r}{q}\right) \in \text{int}(\rho(\tilde{f})) \setminus \left(\frac{k_1}{n}, \frac{k_2}{n}\right)$. From what we already proved in theorem 6, $\tilde{f}^q(\bullet) - (p, r)$ has a hyperbolic m -periodic point $\tilde{Q}$ (for some integer $m > 0$) such that $W^u(\tilde{Q}) \cap W^s(\tilde{Q} + (1, 0))$ and $W^u(\tilde{Q}) \cap W^s(\tilde{Q} + (0, 1))$. Now let us consider the following curves in the plane:

1. let α_H be a simple curve connecting $\tilde{Q}$ to $\tilde{Q} + (1, 0)$, contained in $W^u(\tilde{Q}) \cup W^s(\tilde{Q} + (1, 0))$ in such a way that its intersection with $W^u(\tilde{Q})$ and also with $W^s(\tilde{Q} + (1, 0))$ is connected. Let $\theta_H = \bigcup_{i \in \text{integers}} \alpha_H + (i, 0)$;
2. let α_V be a simple curve connecting $\tilde{Q}$ to $\tilde{Q} + (0, 1)$, contained in $W^u(\tilde{Q}) \cup W^s(\tilde{Q} + (0, 1))$ in such a way that its intersection with $W^u(\tilde{Q})$ and also with $W^s(\tilde{Q} + (0, 1))$ is connected. Let $\theta_V = \bigcup_{i \in \text{integers}} \alpha_V + (0, i)$;

Finally, let

$$K = \left(\bigcup_{i \in \text{integers}} \theta_H + (0, i) \right) \cup \left(\bigcup_{i \in \text{integers}} \theta_V + (i, 0) \right).$$

If $\tilde{D}$ is a connected component of $p^{-1}(D)$, then there exists a constant $Max > 0$ such that if $\text{diam}(\tilde{D}) > Max$, then $p(K) \cap D \neq \emptyset$. So suppose $\text{diam}(\tilde{D}) > Max$. As $\left(\frac{p}{q}, \frac{r}{q}\right) \neq \left(\frac{k_1}{n}, \frac{k_2}{n}\right)$, $Q = p(\tilde{Q}) \notin D$. So $W^u(Q) \cup W^s(Q)$ intersects D . Suppose some point $w \in p(K) \cap W^u(Q)$ belongs to D . Let Γ be the branch of $W^u(Q)$ that contains w . As we did in the proof of lemma 23, there are points $w', w'' \in \Gamma \cap \partial D$ such that w belongs to the arc in Γ between w' and w'' and apart from its end points, this arc is contained in D . And this is a contradiction with the Poincaré recurrence theorem, see the end of the proof of lemma 23. $\square$

Back to the proof of theorem 6, as $\tilde{f}^m(\overline{W^u(\tilde{Q}_0)}) = \overline{W^u(\tilde{Q}_0)}$ and this set is closed and connected, any connected component $\tilde{M}$ of its complement is an open disk such that $\tilde{M} \cap (\tilde{M} + (a, b)) = \emptyset$ for all integers $(a, b) \neq (0, 0)$. To see this, suppose by contradiction that for some $(a', b') \neq (0, 0)$, $\tilde{M}$ intersects

$\widetilde{M} + (a', b')$. As all integer translates of $\widetilde{M}$ are in the complement of $\overline{W^u(\widetilde{Q}_0)}$, this contradicts the fact that $\overline{W^u(\widetilde{Q}_0)}$ is invariant under integer translations and connected. So $M = p(\widetilde{M})$ is a open disk in the torus and there exists an integer $k > 0$ such that

$$f^{k.m}(M) = M.$$

This follows from the fact that M is a connected component of the complement of $\overline{W^u(Q_0)}$, which is invariant under f^m . By lemma 30, $\text{diam}(\widetilde{M})$ is uniformly bounded from above. An analogous argument applied to $\overline{W^s(\widetilde{Q}_0)}$ implies that any connected component of the complement of $\overline{W^s(\widetilde{Q}_0)}$ is also a connected component of the lift of an open disk of the torus, with diameter uniformly bounded from above.

In order to prove that $\overline{W^u(\widetilde{Q}_0)} = \overline{W^s(\widetilde{Q}_0)}$, from expression (16) it is enough to show that $\overline{W^u(Q_0)} = \overline{W^s(Q_0)}$. If M is a connected component of the complement of $\overline{W^s(Q_0)}$, from what we did above, there exists an integer $k > 0$ such that $f^{k.m}(M) = M$ and M is an open disk in the torus, with diameter bounded from above.

Claim 1 : $W^u(Q_0)$ does not intersect M .

Proof:

If $W^u(Q_0)$ intersects M , as any connected component of $p^{-1}(M)$ has the same diameter, which is bounded from above, and for all pair of integers (i, j) , $W^u(\widetilde{Q}_0) \cap W^s(\widetilde{Q}_0 + (i, j))$, we get that some branch Γ of $W^u(Q_0)$ intersects M at some point w and there are points $w', w'' \in \Gamma \cap \partial M$ such that w belongs to the arc in Γ between w' and w'' and apart from its end points, this arc is contained in M . As in the end of the proof of lemma 30, this is a contradiction with the Poincaré recurrence theorem. $\square$

So $\overline{W^u(Q_0)} \subset \overline{W^s(Q_0)}$ and an analogous argument gives $\overline{W^s(Q_0)} \subset \overline{W^u(Q_0)}$, which finishes this part of the argument.

Finally, let us prove that

$$\widetilde{f}(\overline{W^u(\widetilde{Q}_0)}) = \overline{W^u(\widetilde{Q}_0)}. \quad (17)$$

The equalities in expression (16) imply that expression (17) is true if and only if,

$$f(\overline{W^u(Q_0)}) = \overline{W^u(Q_0)}. \quad (18)$$

If expression (18) does not hold, then there exists a connected component M of $\left(\overline{W^u(Q_0)}\right)^c$ such that either:

1. $W^u(f(Q_0))$ intersects M ;
2. $W^u(Q_0)$ intersects $f(M)$;

As M is a bounded f -periodic open disk, claim 1 implies that both cases above can not happen. $\square$

3.6 Proof of lemma 2

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho(\tilde{f}))$ is not empty or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $\text{int}(\rho_V(\tilde{f}))$ is not empty. If f is transitive, then f can not have a periodic open disk. In the general case, there exists $M = M(f) > 0$ such that if $D \subset \mathbb{T}^2$ is a f -periodic open disk, then for any connected component $\tilde{D}$ of $p^{-1}(D)$, $\text{diam}(\tilde{D}) < M$.

This is contained in lemmas 23 and 30. $\square$

3.7 Proof of corollary 7

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If $\tilde{D} \subset \mathbb{R}^2$ is a wandering open disk, then for all integers n , $\text{diam}(\tilde{f}^n(\tilde{D}))$ is uniformly bounded from above and there exists a f -periodic open disk $D_+ \subset \mathbb{T}^2$ such that $D_+ \supset D = p(\tilde{D})$.

From theorem 6, $\tilde{f}$ has a hyperbolic m -periodic point $\tilde{Q}$ such that for all pair of integers (a,b) , $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})} = \overline{W^u(\tilde{Q})} + (a,b)$ and all connected components of $\left(\overline{W^u(\tilde{Q})}\right)^c$ are open disks, with uniformly bounded diameter.

So, if $\text{diam}(\tilde{f}^n(\tilde{D}))$ is not uniformly bounded, then for some integer n_0 , $\tilde{f}^{n_0}(\tilde{D})$ intersects both $W^s(\tilde{Q})$ and $W^u(\tilde{Q})$. But this means that there exists a subset of $\tilde{f}^{n_0-n.m}(\tilde{D})$ that gets closer and closer (in the Hausdorff topology)

as $n \rightarrow \infty$, to a piece of $W^s(\tilde{Q})$ that contains $\tilde{Q}$ and in a similar way, there exists a subset of $\tilde{f}^{n_0+n.m}(\tilde{D})$ that gets closer and closer (also in the Hausdorff topology) as $n \rightarrow \infty$, to a piece of $W^u(\tilde{Q})$ that also contains $\tilde{Q}$. Thus if $n > 0$ is sufficiently large, $\tilde{f}^{n_0-n.m}(\tilde{D})$ intersects $\tilde{f}^{n_0+n.m}(\tilde{D})$, which implies that

$$\tilde{f}^{2n.m}(\tilde{D}) \text{ intersects } \tilde{D},$$

a contradiction with the assumptions on $\tilde{D}$. In a similar way, if $\tilde{D}$ intersects $W^u(\tilde{Q})$, then theorem 6 implies that $\tilde{D}$ intersects $W^s(\tilde{Q})$ and so it can not be wandering. Thus $\tilde{D} \subset \left(\overline{W^u(\tilde{Q})}\right)^c$ and the corollary follows by noticing that $\tilde{D}_+$ is the connected component of $\left(\overline{W^u(\tilde{Q})}\right)^c$ that contains $\tilde{D}$ and $D_+ = p(\tilde{D}_+)$. $\square$

3.8 Proof of corollary 8

Statement: *If f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $\{(\frac{p}{q}, \frac{r}{q}), (0, 0)\} \in \text{int}(\rho(\tilde{f}))$ then $\tilde{f}^q(\bullet) - (p, r)$ has a hyperbolic periodic point $\tilde{P}$ such that for any pair of integers (a, b) , $W^u(\tilde{P}) \cap W^s(\tilde{P} + (a, b))$ and $R.I.(f) = \overline{W^u(p(\tilde{P}))}$. In case f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $\{(\frac{p}{q}, 0)\} \in \text{int}(\rho_V(\tilde{f}))$, then, for all integers s , $\tilde{f}^q(\bullet) - (s, p)$ has a hyperbolic periodic point $\tilde{P}$ such that for any pair of integers (a, b) , $W^u(\tilde{P}) \cap W^s(\tilde{P} + (a, b))$ and $R.I.(f) = \overline{W^u(p(\tilde{P}))}$. Moreover, if $\tilde{Q}$ is the $\tilde{f}$ -periodic point that appears in the statement of theorem 6, then $W^u(p(\tilde{P})) \cap W^s(p(\tilde{Q}))$ and $W^u(p(\tilde{Q})) \cap W^s(p(\tilde{P}))$.*

The complement of $R.I.(f)$ is a union of f -periodic disks (with diameter bounded from above when lifted to the plane) and by theorem 6, there exists a point $\tilde{P} \in \mathbb{R}^2$ such that $p(\tilde{P})$ is a hyperbolic f -periodic point with adequate rotation vector or number and for any pair of integers (a, b) , $W^u(\tilde{P}) \cap W^s(\tilde{P} + (a, b))$. Moreover, the complement of $\overline{W^u(p(\tilde{P}))}$ is also a union of f -periodic disks, with diameter uniformly bounded from above when lifted to the plane. So if $(R.I.(f))^c \neq \left(\overline{W^u(p(\tilde{P}))}\right)^c$, then there exists a f -periodic disk D which intersects the unstable manifold of a hyperbolic f -periodic point R (R can be equal to $p(\tilde{Q})$ or $p(\tilde{P})$) such that the union of its stable and unstable manifolds contains closed curves in the torus of all homotopy types. This implies that some branch Γ of $W^s(R)$ or of $W^u(R)$ intersects D at some point w and there

are points $w', w'' \in \Gamma \cap \partial D$ such that w belongs to the arc in Γ between w' and w'' and apart from its end points, this arc is contained in D . As in the end of the proof of lemma 30, this is a contradiction with the Poincaré recurrence theorem. So $(R.I.(f))^c = \left(\overline{W^u(p(\tilde{P}))}\right)^c$, which implies that $R.I.(f) = \overline{W^u(p(\tilde{P}))}$. Now we prove the last part of the corollary.

Let us choose a closed curve $\gamma \subset T^2$, homotopic to $(1, 0)$, such that γ is the projection under p of a simple connected arc $\tilde{\gamma} \subset \mathbb{R}^2$ made of two connected pieces: one starts at $\tilde{Q}$, contained in $W^u(\tilde{Q})$ until it reaches $W^s(\tilde{Q}) + (1, 0)$ and the other piece is contained in $W^s(\tilde{Q}) + (1, 0)$ until it reaches $\tilde{Q} + (1, 0)$. As $p(\tilde{P}) \neq p(\tilde{Q})$, we can choose a closed curve $\alpha \subset W^s(p(\tilde{P})) \cup p(\tilde{P}) \cup W^u(p(\tilde{P}))$ whose homotopy class is $(0, 1)$, such that $\alpha \cap W^s(p(\tilde{P}))$ is a connected arc, sufficiently close to $p(\tilde{P})$, in a way that $\gamma \cap (\alpha \cap W^s(p(\tilde{P}))) = \emptyset$. This is possible because $p(\tilde{P}) \cap \gamma = \emptyset$.

As α and γ must intersect, the previous choice made on α implies that $W^u(p(\tilde{P})) \pitchfork W^s(p(\tilde{Q}))$. Interchanging $\tilde{Q}$ and $\tilde{P}$ in the above argument we also obtain that $W^u(p(\tilde{Q})) \pitchfork W^s(p(\tilde{P}))$ and the proof is over. $\square$

3.9 Proof of theorem 9

Statement: Suppose f belongs to $Diff_0^{1+\epsilon}(T^2)$ and $(0, 0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $Diff_k^{1+\epsilon}(T^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. Then, for all $\theta \in [0, 2\pi]$, $\omega(B_\theta^\infty) = \emptyset$ and $\omega(B_S^\infty) = \omega(B_N^\infty) = \emptyset$.

The case when θ is a rational multiple of 2π , $f \in Diff_0^{1+\epsilon}(T^2)$ and $(0, 0) \in \text{int}(\rho(\tilde{f}))$ is analogous to the case when $f \in Diff_k^{1+\epsilon}(T^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. In this case, if $f \in Diff_0^{1+\epsilon}(T^2)$, by conjugating f with a suitable map, we can assume that $\theta = 0$. So, suppose by contradiction that $\omega(B_0^\infty) \neq \emptyset$. Let Γ be a connected component of $\omega(B_0^\infty)$. Then $\Gamma \subset [0, \infty[\times \mathbb{R}$ and Γ is closed and unbounded. Now, we remember theorem 6 and let $\tilde{Q}$ be a hyperbolic m -periodic point for $\tilde{f}$ such that for any pair of integers (a, b) , $W^u(\tilde{Q}) \pitchfork W^s(\tilde{Q} + (a, b))$ and $\overline{W^u(\tilde{Q})} = \overline{W^s(\tilde{Q})}$. So we can pick two simple arcs, $\alpha_{(1,0)}$ and $\alpha_{(0,1)}$ such that $\alpha_{(1,0)} \subset W^u(\tilde{Q}) \cup W^s(\tilde{Q} + (1, 0))$ and $\alpha_{(1,0)}$ connects $\tilde{Q}$ to $\tilde{Q} + (1, 0)$ and similarly, $\alpha_{(0,1)} \subset W^u(\tilde{Q}) \cup W^s(\tilde{Q} + (0, 1))$ and $\alpha_{(0,1)}$ connects $\tilde{Q}$ to $\tilde{Q} + (0, 1)$.

Let us define two path connected sets $\theta_{(1,0)}, \theta_{(0,1)} \subset \mathbb{R}^2$ in the following way:
 $\theta_{(1,0)} = \bigcup_{i \in \text{integers}} \alpha_{(1,0)} + i(1,0)$ and $\theta_{(0,1)} = \bigcup_{i \in \text{integers}} \alpha_{(0,1)} + i(0,1)$. Then, the next consequences hold:

1. $\theta_{(1,0)} = \theta_{(1,0)} + (1,0)$ and $\theta_{(0,1)} = \theta_{(0,1)} + (0,1)$;
2. $\theta_{(1,0)}$ contains $\tilde{Q} + i(1,0)$ and $\theta_{(0,1)}$ contains $\tilde{Q} + i(0,1)$, for all integers i ;

It is easy to see that if $\theta_{(0,1)} + i(1,0)$ does not intersect Γ for all integers i , then as Γ is connected and unbounded, $\theta_{(1,0)} + j(0,1)$ must intersect Γ for some integer j . From this we get that either $W^u(\tilde{Q} + (c, d))$ or $W^s(\tilde{Q} + (c, d))$, for some pair of integers (c, d) , has a topologically transverse intersection with Γ . Suppose it is $W^u(\tilde{Q} + (c, d))$. By theorem 6 we can suppose that $(c, d) = (0, 0)$. This implies that $\tilde{f}^{-n.m}(\Gamma)$ contains a compact connected set that converges in the Hausdorff topology to $W^s(\tilde{Q})$ as $n \rightarrow \infty$, a contradiction because $\tilde{f}(\omega(B_0^\infty)) = \omega(B_0^\infty) \subset [0, \infty[\times \mathbb{R}$ and $W^s(\tilde{Q})$ has points outside $[0, \infty[\times \mathbb{R}$.

In case $\frac{\theta}{2\pi}$ is irrational, we proceed as follows: The theory of continued fractions (see for instance [26]) implies that using integer translates of the arcs $\alpha_{(1,0)}$ and $\alpha_{(0,1)}$ defined above, we can obtain a connected closed set γ in the plane such that γ is contained between two straight lines of slope θ and γ intersects all straight lines of slope $\theta + \pi/2$. We can do the same procedure for $\theta + \pi/2$, that is, also using integer translates of $\alpha_{(1,0)}$ and $\alpha_{(0,1)}$ we can obtain a connected closed set γ^* in the plane such that γ^* is contained between two straight lines of slope $\theta + \pi/2$ and γ^* intersects all straight lines of slope θ . So if $\omega(B_\theta^\infty) \neq \emptyset$, then some integer translate of $\gamma \cup \gamma^*$ intersects $\omega(B_\theta^\infty)$. From this and theorem 6, we get that either $W^u(\tilde{Q})$ or $W^s(\tilde{Q})$ has a topologically transverse intersection with some connected component Γ of $\omega(B_\theta^\infty)$. Suppose it is $W^u(\tilde{Q})$. As in the rational case, this implies that $\tilde{f}^{-n.m}(\Gamma)$ contains a compact connected set that converges in the Hausdorff topology to $W^s(\tilde{Q})$ as $n \rightarrow \infty$, a contradiction because $\tilde{f}(\omega(B_\theta^\infty)) = \omega(B_\theta^\infty) \subset \{\tilde{z} \in \mathbb{R}^2 : \langle \tilde{z}, (\cos(\theta), \sin(\theta)) \rangle \geq 0\}$ and $W^s(\tilde{Q})$ has points outside this set. $\square$

3.10 Proof of corollary 10

Statement: Suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$ or f belongs to $\text{Diff}_k^{1+\epsilon}(\mathbb{T}^2)$ and $0 \in \text{int}(\rho_V(\tilde{f}))$. If f is transitive, then $\overline{p(B_S^\infty)} = \overline{p(B_N^\infty)} = \overline{p(B_\theta^\infty)} = \mathbb{T}^2$, for all $\theta \in [0, 2\pi]$. And for a general f , any of the following sets $\left(\overline{p(B_S^\infty)}\right)^c$, $\left(\overline{p(B_N^\infty)}\right)^c$ and $\left(\overline{p(B_\theta^\infty)}\right)^c$ is the union of f -periodic open disks, with diameters uniformly bounded from above when lifted to the plane.

Without loss of generality, suppose f belongs to $\text{Diff}_0^{1+\epsilon}(\mathbb{T}^2)$ and $(0,0) \in \text{int}(\rho(\tilde{f}))$. Given $\theta \in [0, 2\pi]$, let us look at a connected component U of $\left(\overline{p(B_\theta^\infty)}\right)^c$. Note that U is periodic, because $f(\overline{p(B_\theta^\infty)}) = \overline{p(B_\theta^\infty)}$. So there are two possibilities:

1. U contains a homotopically non-trivial simple closed curve γ in the torus;
2. U does not contain such a curve;

In the first case, without loss of generality, suppose γ is a vertical curve, that is γ is homotopic to $(0, 1)$. As $(0,0) \in \text{int}(\rho(\tilde{f}))$, if $\tilde{\gamma}$ is a connected component of $p^{-1}(\gamma)$, then for all integers n , $\tilde{f}^n(\tilde{\gamma})$ intersects $\tilde{\gamma}$, otherwise $(0,0)$ would not be an interior point of $\rho(\tilde{f})$. And given any integer $i \neq 0$, we get that for some integer $n(i) > 0$, $\tilde{f}^{n(i)}(\tilde{\gamma})$ intersects $\tilde{\gamma} + (i, 0)$. So some vertical translate of B_θ^∞ intersects $\tilde{f}^{n(2)}(p^{-1}(\gamma))$, which means that $p(B_\theta^\infty)$ intersects $f^{n(2)}(\gamma)$, a contradiction. Thus case 1 does not happen.

In the second case let us prove that U is an open disk. If it is not, then there exists a simple closed curve $\alpha \subset U$, which is contractible as a curve in the torus, such that in the disk bounded by α there are points of $\overline{p(B_\theta^\infty)}$. But this is impossible, because each connected component Γ of B_θ^∞ is an unbounded closed subset of the plane. So U is an open disk.

If f is transitive, lemma 23 tells us that case 2 also does not happen, which means that $\overline{p(B_\theta^\infty)} = \mathbb{T}^2$. In the general case, $\left(\overline{p(B_\theta^\infty)}\right)^c$ is equal the union of f -periodic open disks, which by lemma 30, all have diameters uniformly bounded from above when lifted to the plane. $\square$

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Figure captions.

Figure 1. Diagram showing a closed connected set K which satisfies $K \cap W^u(P)$.

Figure 2. Diagram showing the simple arc η .

Figure 3. Diagram showing the simple arc η and how the set θ is obtained.

Figure 4. Diagram showing the points $\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2}$ and the rectangle $\widetilde{Ret}(\tilde{z}, \tilde{Q}_{V1}, \tilde{Q}_{V2})$.







