Ergod. Th. & Dynam. Sys. (First published online 2018), page 1 of 18*
doi:10.1017/etds.2018.120
*Provisional—final page numbers to be inserted when paper edition is published

A consequence of the growth of rotation sets for families of diffeomorphisms of the torus

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(*Received* 16 *March* 2018 *and accepted in revised form* 10 *September* 2018)

Abstract. In this paper we consider C^∞-generic families of area-preserving diffeomorphisms of the torus homotopic to the identity and their rotation sets. Let $f_t : \mathrm{T}^2 \to \mathrm{T}^2$ be such a family, $\widetilde{f}_t : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed family of lifts and $\rho(\widetilde{f}_t)$ be their rotation sets, which we assume to have interior for t in a certain open interval I. We also assume that some rational point $(p/q, l/q) \in \partial\rho(\widetilde{f}_{\bar{t}})$ for a certain parameter $\bar{t} \in I$, and we want to understand the consequences of the following hypothesis: for all $t > \bar{t}$, $t \in I$, $(p/q, l/q) \in \mathrm{int}(\rho(\widetilde{f}_t))$. Under these very natural assumptions, we prove that there exists a $f_{\bar{t}}^q$-fixed hyperbolic saddle $P_{\bar{t}}$ such that its rotation vector is $(p/q, l/q)$. We also prove that there exists a sequence $t_i > \bar{t}$, $t_i \to \bar{t}$, such that if P_t is the continuation of $P_{\bar{t}}$ with the parameter, then $W^u(\widetilde{P}_{t_i})$ (the unstable manifold) has quadratic tangencies with $W^s(\widetilde{P}_{t_i}) + (c, d)$ (the stable manifold translated by (c, d)), where $\widetilde{P}_{t_i}$ is any lift of P_{t_i} to the plane. In other words, $\widetilde{P}_{t_i}$ is a fixed point for $(\widetilde{f}_{t_i})^q - (p, l)$, and $(c, d) \neq (0, 0)$ are certain integer vectors such that $W^u(\widetilde{P}_{\bar{t}})$ do not intersect $W^s(\widetilde{P}_{\bar{t}}) + (c, d)$, and these tangencies become transverse as t increases. We also prove that, for $t > \bar{t}$, $W^u(\widetilde{P}_t)$ has transverse intersections with $W^s(\widetilde{P}_t) + (a, b)$, for all integer vectors (a, b), and thus one may consider that the tangencies above are associated to the birth of the heteroclinic intersections in the plane that do not exist for $t \leq \bar{t}$.

Key words: bifurcation theory, smooth dynamics, topological dynamics, low-dimensional dynamics
2010 Mathematics Subject Classification: 37E30, 37E45 (Primary); 37C20, 37C25, 37C29 (Secondary)

1. *Introduction*

1.1. *General explanations.* In this paper, in a certain sense, we continue the study initiated in [**3**]. There, we looked at the following problem. Suppose $f : \mathrm{T}^2 \to \mathrm{T}^2$ is a homeomorphism homotopic to the identity, and its rotation set—which is supposed to have interior—has a point ρ in its boundary with both coordinates rational. The question studied was the following. Is it possible to find two different arbitrarily small C^0-perturbations of f, denoted f_1 and f_2 in such a way that ρ does not belong to the rotation set of f_1, and that ρ is contained in the interior of the rotation set of f_2? In other words, is the rational mode locking found by de Carvalho, Boyland and Hall [**4**] in their particular family of homeomorphisms, in a certain sense, a general phenomenon or not? Our main theorems and examples from the earlier work showed that the answer to this question depends on the set of hypotheses assumed. For instance, regarding C^2-generic families, we proved that if $\rho \in \partial\rho(\widetilde{f_{\bar{t}}})$ for some $\bar{t}$, and if for $t < \bar{t}$, close to $\bar{t}$, $\rho \notin \rho(\widetilde{f_t})$, then for all sufficiently small $t - \bar{t} > 0$, $\rho \notin \operatorname{int}(\rho(\widetilde{f_t}))$. Now, in the current paper, we are interested in the dynamical consequences of a situation that may be obtained as a continuation of the previous situation. Suppose that $f_t : \mathrm{T}^2 \to \mathrm{T}^2$ is a one-parameter family of diffeomorphisms of the torus homotopic to the identity, for which the rotation set $\rho(\widetilde{f_t})$ at a certain parameter $t = \bar{t}$, has interior, some rational vector $\rho \in \partial\rho(\widetilde{f_{\bar{t}}})$ and, for all sufficiently small $t > \bar{t}$, $\rho \in \operatorname{int}(\rho(\widetilde{f_t}))$. We want to understand what happens for the family f_t, $t > \bar{t}$. In other words, we are assuming that at $t = \bar{t}$ the rotation set is ready to grow locally in a neighborhood of ρ.

The usual situation for (generic) families is as follows. As the parameter changes, the rotation set hits a rational vector; this vector stays for a while in the boundary of the rotation set until, finally, it is consumed by the rotation set, i.e. it becomes an interior point.

Here we consider C^∞-generic area-preserving families in the sense of Meyer [**19**], and that also satisfy other generic conditions. Suppose that, for instance, $f_t : \mathrm{T}^2 \to \mathrm{T}^2$ is a family satisfying the above generic conditions and such that the rotation set at $t = \bar{t}$ has interior, $(0, 0) \in \partial\rho(\widetilde{f_{\bar{t}}})$ and for all sufficiently small $t > \bar{t}$, $(0, 0) \in \operatorname{int}(\rho(\widetilde{f_t}))$. Then our main theorem implies that $\widetilde{f_{\bar{t}}} : \mathbb{R}^2 \to \mathbb{R}^2$ has a hyperbolic fixed saddle $\widetilde{P_{\bar{t}}}$, and if $\widetilde{P_t}$ is the continuation of $\widetilde{P_{\bar{t}}}$ with the parameter t, then there exists a sequence $t_i > \bar{t}$, converging to $\bar{t}$, such that $W^u(\widetilde{P_{t_i}})$ has heteroclinic tangencies with certain special integer translates of the stable manifold, $W^s(\widetilde{P_{t_i}}) + (a, b)$, $(a, b) \in \mathbb{Z}^2$, which unfold (i.e. become transversal) as t increases. The integer vectors (a, b) mentioned above belong to a set $K_{\mathbb{Z}^2} \subset \mathbb{Z}^2$ and satisfy the following. For $t \leq \bar{t}$, $W^u(\widetilde{P_t})$ cannot have intersections with $W^s(\widetilde{P_t}) + (a, b)$ whenever $(a, b) \in K_{\mathbb{Z}^2}$. Moreover, the sequence $t_i \to \bar{t}$ depends on the choice of $(a, b) \in K_{\mathbb{Z}^2}$.

So, increasing the parameter until the critical value at $t = \bar{t}$ is reached (the moment when the rotation set is ready to grow locally), this critical parameter is accumulated from the other side by parameters at which there are heteroclinic tangencies in the plane (homoclinic in the torus) not allowed to exist when $t \leq \bar{t}$. In fact, as we will prove, for any $t > \bar{t}$,

$$W^u(\widetilde{P_t}) \text{ has transverse intersections with } W^s(\widetilde{P_t}) + (a, b) \quad \text{for all } (a, b) \in \mathbb{Z}^2. \tag{1}$$

In this way, the creation of the heteroclinic intersections for integers (a, b) in (1) that do not exist for $t \leq \bar{t}$ also produces tangencies.

In order to state things clearly and to present our main result precisely, a few definitions are necessary.

1.2. *Basic notation and some definitions.*

(1) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $p : \mathbb{R}^2 \longrightarrow T^2$ be the associated covering map. Coordinates are denoted as $\widetilde{z} \in \mathbb{R}^2$ and $z \in T^2$.

(2) Let $\mathrm{Diff}_0^r(T^2)$ be the set of C^r diffeomorphisms ($r = 0, 1, \ldots, \infty$) of the torus homotopic to the identity and let $\mathrm{Diff}_0^r(\mathbb{R}^2)$ be the set of lifts of elements from $\mathrm{Diff}_0^r(T^2)$ to the plane. Maps from $\mathrm{Diff}_0^r(T^2)$ are denoted f and their lifts to the plane are denoted $\widetilde{f}$.

(3) Let $p_{1,2} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the standard projections, respectively in the horizontal and vertical components.

(4) Given $f \in \mathrm{Diff}_0^0(T^2)$ (a homeomorphism) and a lift $\widetilde{f} \in \mathrm{Diff}_0^0(\mathbb{R}^2)$, the so-called rotation set of $\widetilde{f}$, $\rho(\widetilde{f})$, can be defined following Misiurewicz and Ziemian [**20**] as

$$\rho(\widetilde{f}) = \bigcap_{i \geq 1} \overline{\bigcup_{n \geq i} \left\{ \frac{\widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} : \widetilde{z} \in \mathbb{R}^2 \right\}}. \tag{2}$$

This set is a compact convex subset of $\mathbb{R}^2$ (see [**20**]), and it was proved in [**12**] and [**20**] that all points in its interior are realized by compact f-invariant subsets of T^2, which can be chosen as periodic orbits in the rational case. By saying that some vector $\rho \in \rho(\widetilde{f})$ is realized by a compact f-invariant set, we mean that there exists a compact f-invariant subset $K \subset T^2$ such that, for all $z \in K$ and any $\widetilde{z} \in p^{-1}(z)$,

$$\lim_{n \to \infty} \frac{\widetilde{f}^n(\widetilde{z}) - \widetilde{z}}{n} = \rho. \tag{3}$$

Moreover, the above limit, whenever it exists, is called the rotation vector of the point z, denoted $\rho(z)$.

1.3. *Some background and the main theorem.*

1.3.1. *Prime ends compactification of open disks.* If $U \subset \mathbb{R}^2$ is an open topological disk whose boundary is a Jordan curve and $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving homeomorphism such that $\widetilde{f}(U) = U$, it is easy to see that $\widetilde{f} : \partial U \to \partial U$ is conjugate to a homeomorphism of the circle, and so a real number $\rho(U) =$ rotation number of $\widetilde{f}|_{\partial U}$ can be associated to this problem. Clearly, if $\rho(U)$ is rational there exists a periodic point in ∂U. If it is not, then there are no such points. This has been known since Poincaré. The difficulties arise when we do not assume ∂U to be a Jordan curve.

Prime ends compactification is a way to attach to U a circle called the circle of prime ends of U, obtaining a space $U \sqcup S^1$ with a topology that makes it homeomorphic to the closed unit disk. If, as above, we assume the existence of a planar orientation-preserving homeomorphism $\widetilde{f}$, such that $\widetilde{f}(U) = U$, then $\widetilde{f}|_U$ extends to $U \sqcup S^1$. The prime ends rotation number of $\widetilde{f}|_U$, still denoted $\rho(U)$, is the usual rotation number of the orientation-preserving homeomorphism induced on S^1 by the extension of $\widetilde{f}|_U$. However, things may be quite different in this setting. In full generality, it is not true that when $\rho(U)$ is rational, there are periodic points in ∂U and for some examples, $\rho(U)$ is irrational and ∂U is not periodic-point free. Nevertheless, in the area-preserving case which is the case considered in this paper, many interesting results have been obtained. We refer the reader to [**13**, **14**, **18**] and [**15**]. To conclude, we present some results extracted from these works and adapted to our hypotheses.

Assume $h : \mathrm{T}^2 \to \mathrm{T}^2$ is an area-preserving diffeomorphism of the torus, homotopic to the identity such that, for each integer $n > 0$, h has finitely many n-periodic points. Moreover, we also assume more technical conditions on h: for each $n > 0$, at all n-periodic points a Lojasiewicz condition is satisfied, see [**11**]. In addition, if the eigenvalues of Dh^n at such a periodic point are both equal to 1, then the point is topologically degenerate; it has a zero topological index. As explained in [**3**, §2], the dynamics near such a point is similar to the one in Figure 2 (see later). In particular, h has one stable separatrix (like a branch of a hyperbolic saddle) and an unstable one, both h-invariant. Topologically, the local dynamics in a neighborhood of the periodic point is obtained by gluing exactly two hyperbolic sectors.

Fix some $\widetilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$, a lift of h to the plane. Given a $\widetilde{h}$-invariant continuum $K \subset \mathbb{R}^2$, if O is a connected component of K^c, then O is a topological open disk in the sphere $S^2 \overset{\text{def.}}{=} \mathbb{R}^2 \sqcup \infty$, the one point compactification of the plane. Assume that O is $\widetilde{h}$-invariant (it could be periodic with period larger than 1) and let α be the rotation number of the prime ends compactification of O. From the hypothesis on h, we have the following.

THEOREM A. *If α is rational, then ∂O has accessible $\widetilde{h}$-periodic points. In addition, if such a point has period n, the eigenvalues of $D\widetilde{h}^n$ at this periodic point must be real and cannot be equal to* -1. *So, from the above properties assumed on h, in ∂O we either have accessible hyperbolic periodic saddles or periodic points with both eigenvalues equal to* 1, *with local dynamics as shown in Figure 2 (see later). Then, there exist connections between separatrices of the periodic points, these separatrices being either stable or unstable branches of hyperbolic saddles, or the unstable or stable separatrix of a point as shown in Figure 2.*

The existence of accessible $\widetilde{h}$-periodic points can be found in [**7**]. The information about the eigenvalues is a new result from [**15**] and the existence of connections in the above situation can be found in [**13**, **18**] and also [**15**].

THEOREM B. *If α is irrational and O is bounded, then there is no periodic point in ∂O.*

This is a result from [**14**].

1.3.2. *Some results on diffeomorphisms of the torus homotopic to the identity.* As the rotation set of a homeomorphism of the torus homotopic to the identity is a compact convex subset of the plane, there are three possibilities for its shape:

(1) it is a point;
(2) it is a linear segment;
(3) it has interior.

We consider the situation when the rotation set has interior.

Whenever a rational vector $(p/q, l/q) \in \mathrm{int}(\rho(\widetilde{f}))$ for some $\widetilde{f} \in \mathrm{Diff}_0^2(\mathbb{R}^2)$,

$$\begin{aligned}&\widetilde{f}^q(\bullet) - (p, l) \text{ has a hyperbolic periodic saddle } \widetilde{P} \in \mathbb{R}^2\\ &\quad \text{such that } W^u(\widetilde{P}), \text{ its unstable manifold, has a topologically transverse}\\ &\quad \text{intersection with } W^s(\widetilde{P}) + (a, b) \quad \text{for all integer vectors } (a, b).\end{aligned} \tag{4}$$

That is, the unstable manifold of $\widetilde{P}$ intersects all integer translations of its stable manifold. This result is proved in [**1**]. Now it is time to precisely define what we mean by a topologically transverse intersection.

Definition. (Topologically transverse intersections) If $f : M \to M$ is a C^1 diffeomorphism of an orientable boundaryless surface M and $p, q \in M$ are f-periodic saddle points, then we say that $W^u(p)$ has a topologically transverse intersection with $W^s(q)$, whenever the following happens. There exists a point $z \in W^s(q) \cap W^u(p)$ (z clearly can be chosen arbitrarily close to q or to p) and an open topological disk B centered at z, such that $B\backslash\alpha = B_1 \cup B_2$, where α is the connected component of $W^s(q) \cap B$ which contains z, with the following property: there exists a closed connected piece of $W^u(p)$ denoted β such that $\beta \subset B$, $z \in \beta$, and $\beta\backslash z$ has two connected components, one contained in $B_1 \cup \alpha$ and the other contained in $B_2 \cup \alpha$, such that $\beta \cap B_1 \neq \emptyset$ and $\beta \cap B_2 \neq \emptyset$. Clearly a C^1 transverse intersection is topologically transverse. See Figure 1 for a sketch of some possibilities. Note that as $\beta \cap \alpha$ may contain a connected arc containing z, the disk B may not be chosen arbitrarily small.

In order to have a picture in mind, consider z close to q, so that z belongs to a connected arc in $W^s(q)$ containing q, which is almost a linear segment. In this way, it is easy to find B as stated above; it could be chosen as an open Euclidean ball. Clearly, this is a symmetric definition: we can consider a negative iterate of z, for some $n < 0$ such that $f^n(z)$ belongs to a connected piece of $W^u(p)$ containing p, which is also almost a linear segment. Then, a completely analogous construction can be made, switching a stable manifold with an unstable one. Choose an open Euclidean ball B' centered at $f^n(z)$, such that $B'\backslash\beta' = B'_1 \cup B'_2$, where β' is the connected component of $W^u(p) \cap B'$ which contains z, with the following property. There exists a closed connected piece of $W^s(q)$ denoted α' (where $\alpha' \subset B'$, $f^n(z) \in \alpha'$, and $\alpha'\backslash f^n(z)$ has two connected components, one contained in $B'_1 \cup \beta'$ and the other contained in $B'_2 \cup \beta'$) such that $\alpha' \cap B'_1 \neq \emptyset$ and $\alpha' \cap B'_2 \neq \emptyset$. So, $f^{-n}(B')$, $f^{-n}(\beta')$ and $f^{-n}(\alpha')$ are the corresponding sets at z.

The most important consequence of a topologically transverse intersection for us is a C^0 λ-lemma: if $W^u(p)$ has a topologically transverse intersection with $W^s(q)$, then $W^u(p)$ C^0-accumulates on $W^u(q)$.

As pointed out in [**2**], the following converse of (4) is true. If $\widetilde{g} \overset{\text{def.}}{=} \widetilde{f}^q(\bullet) - (p, l)$ has a hyperbolic periodic saddle $\widetilde{P} \in \mathbb{R}^2$ such that $W^u(\widetilde{P})$ has a topologically transverse intersection with $W^s(\widetilde{P}) + (a_i, b_i)$, for integer vectors (a_i, b_i), $i = 1, 2, \ldots, k$, such that

$$(0, 0) \in \mathrm{ConvexHull}\{(a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\},$$

then $(0, 0) \in \mathrm{int}(\rho(\widetilde{g})) \Leftrightarrow (p/q, l/q) \in \mathrm{int}(\rho(\widetilde{f}))$. This follows from the lemma shown below.

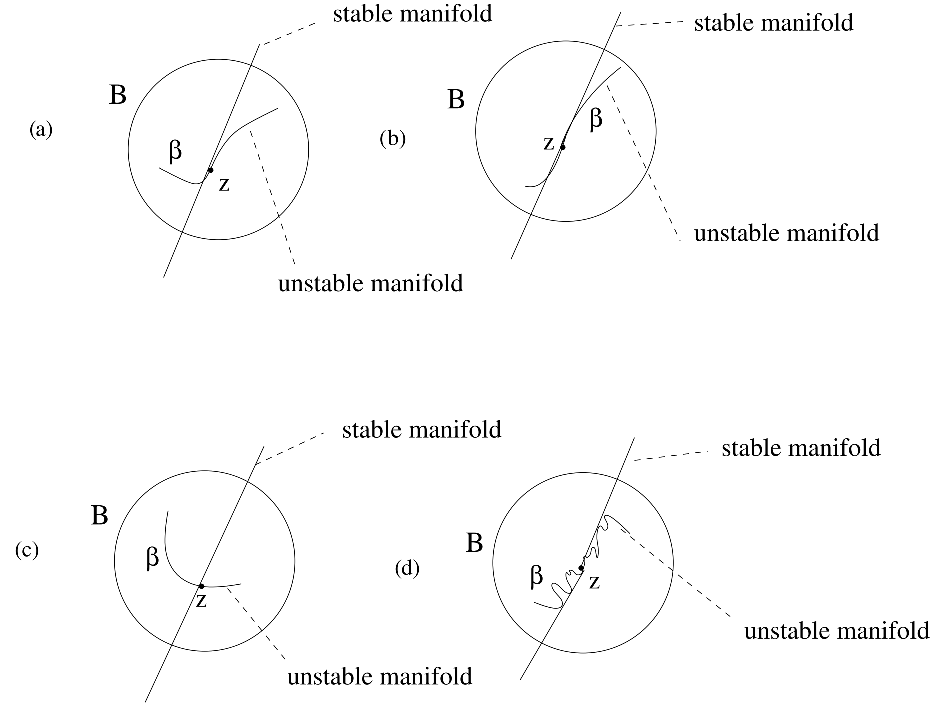


FIGURE 1. Four cases of topologically transverse intersections: (a) z is an odd-order tangency; (b) there is a segment in the intersection of the manifolds; (c) a C^1-transverse crossing; (d) z is accumulated on both sides by even-order tangencies.

LEMMA 0. *Let* $g \in \mathrm{Diff}_0^1(\mathrm{T}^2)$ *and* $\widetilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a lift of* g *which has a hyperbolic periodic saddle point* $\widetilde{P}$ *such that* $W^u(\widetilde{P})$ *has a topologically transverse intersection with* $W^s(\widetilde{P}) + (a, b)$*, for some integer vector* $(a, b) \neq (0, 0)$*. Then* $\rho(\widetilde{g})$ *contains* $(0, 0)$ *and a rational vector parallel to* (a, b) *with the same orientation as* (a, b)*.*

In order to prove this lemma, one just has to note that if $W^u(\widetilde{P})$ has a topologically transverse intersection with $W^s(\widetilde{P}) + (a, b)$, then we can produce a topological horseshoe for $\widetilde{g}$ (see [**2**]), for which a certain periodic sequence will correspond to points with a rotation vector parallel to, and with the same orientation as, (a, b). So, when $(p/q, l/q) \in \partial\rho(\widetilde{f})$ for some $\widetilde{f} \in \mathrm{Diff}_0^2(\mathbb{R}^2)$, it may be the case that $\widetilde{f}^q(\bullet) - (p, l)$ has a hyperbolic periodic saddle $\widetilde{P}$ such that $W^u(\widetilde{P})$ has a topologically transverse intersection with $W^s(\widetilde{P}) + (a, b)$, for some integer vectors (a, b), but not for all.

Moreover, if r is a supporting line at $(p/q, l/q) \in \partial\rho(\widetilde{f})$ (which means that r is a straight line that contains $(p/q, l/q)$ and does not intersect $\mathrm{int}(\rho(\widetilde{f}))$) and if $\vec{v}$ is a vector orthogonal to r, such that $-\vec{v}$ points towards the rotation set, then $W^u(\widetilde{P})$ has a topologically transverse intersection with $W^s(\widetilde{P}) + (a, b)$, for some integer vector $(a, b) \Rightarrow (a, b).\vec{v} \leq 0$. If $\rho(\widetilde{f})$ intersects r only at $(p/q, r/q)$, then $(a, b).\vec{v} \geq 0 \Rightarrow (a, b) = (0, 0)$.

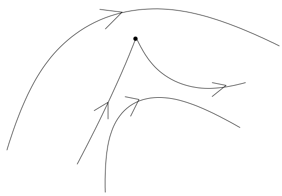


FIGURE 2. Dynamics in a neighborhood of the degenerate periodic points.

Regarding the family f_t and the rational $\rho = (p/q, l/q)$, which is in the boundary of the rotation set $\rho(\widetilde{f_t})$ at the critical parameter $t = \bar{t}$, without loss of generality, we can assume ρ to be (0, 0). Instead of considering f_t and its lift $\widetilde{f_t}$, we simply have to consider f_t^q and the lift $\widetilde{f_t}^q - (p, l)$. This is a standard procedure for this type of problem. We just have to be careful because we will assume some hypotheses for the family f_t, $\widetilde{f_t}$ and we must ensure that they also hold for f_t^q, $\widetilde{f_t}^q - (p, l)$. We look at this next.

1.3.3. *Hypotheses of the main theorem.* Assume $f_t \in \mathrm{Diff}_0^\infty(\mathrm{T}^2)$ is a generic C^∞-family of area-preserving diffeomorphisms ($t \in]\bar{t} - \epsilon, \bar{t} + \epsilon[$ for some parameter $\bar{t}$ and $\epsilon > 0$), among other things, in the sense of Meyer [**19**] such that:

(1) $\rho(\widetilde{f_t})$ has interior for all $t \in]\bar{t} - \epsilon, \bar{t} + \epsilon[$;

(2) $(p/q, l/q) \in \partial\rho(\widetilde{f_{\bar{t}}})$, r is a supporting line for $\rho(\widetilde{f_{\bar{t}}})$ at $(p/q, l/q)$, $\vec{v}$ is a unitary vector orthogonal to r, such that $-\vec{v}$ points towards the rotation set;

(3) $(p/q, l/q) \in \mathrm{int}(\rho(\widetilde{f_t}))$ for all $t \in]\bar{t}, \bar{t} + \epsilon[$;

(4) the genericity in the sense of Meyer implies that if, for some parameter $t \in]\bar{t} - \epsilon, \bar{t} + \epsilon[$, a f_t-periodic point has 1 as an eigenvalue, then it is a saddle-elliptic type of point—one which will give birth to a saddle and an elliptic point when the parameter moves in one direction and will disappear if the parameter moves in the other direction. As the family is generic, for each period there are only finitely many periodic points. Moreover, as is explained on p. 3 of the summary of [**11**], we can assume that at, all periodic points a Lojasiewicz condition is satisfied. In particular, this holds at each n-periodic point (for all integers $n > 0$) which has 1, 1 as eigenvalues (the point is isolated among the n-periodic points and must have zero topological index). So, as explained in [**3**, §2], the dynamics near such a point is as shown in Figure 2.

(5) Saddle-connections are a phenomenon of infinity codimension (see [**10**]). Therefore, as we are considering C^∞-generic 1-parameter families, we can also assume that, for all $t \in]\bar{t} - \epsilon, \bar{t} + \epsilon[$, f_t does not have connections between invariant branches of periodic points, which can be either hyperbolic saddles or degenerate as in Figure 2. This is not explicitly described in the literature on this subject, but a proof for this more general situation can be obtained in exactly the same way as the proof for the case when only hyperbolic saddles are considered.

(6) Moreover, as explained in [**21**, §6 of Ch. II], a much stronger statement holds: for C^∞ 1-parameter generic families f_t, if a point $z \in \mathrm{T}^2$ belongs to the intersection of a stable and an unstable manifold of some f_t-periodic hyperbolic saddles and

the intersection is not C^1-transverse, then it is a quadratic tangency, that is, it is not topologically transverse. This implies that every time an unstable manifold has a topologically transverse intersection with a stable manifold of some hyperbolic periodic points, this intersection is actually C^1-transverse. In addition, when a tangency appears, it unfolds generically with the parameter (with positive speed, see [**21**, Remark 6.2]). This means that if some hyperbolic periodic saddles q_t and p_t, such that $W^u(q_{t'})$ and $W^s(p_{t'})$ have a quadratic tangency at a point $z_{t'} \in \mathrm{T}^2$ for some parameter t', then for t close to t', in suitable coordinates near $z_{t'} = (0, 0)$, we can write $W^u(q_t) = (x, f(x) + (t - t'))$ and $W^s(p_t) = (x, 0)$, where f is a C^∞ function defined in a neighborhood of 0 such that $f(0) = 0$, $f'(0) = 0$ and $f''(0) > 0$. This implies that if the parameter varies in a neighborhood of the tangency parameter, then to one side a C^1-transverse intersection is created and to the other side the intersection disappears.

This is stated for families of general diffeomorphisms in [**21**], but the same result holds for families of area-preserving diffeomorphisms. It is not hard to see that in order to avoid degenerate tangencies (of order ≥ 3), the kind of perturbations needed can preserve area.

If f_t, $\widetilde{f_t}$ satisfy the above hypotheses, then f_t^q, $\widetilde{f_t}^q - (p, l)$ clearly satisfy the same set of hypotheses with respect to $\rho = (0, 0)$. Note that the supporting line at $(0, 0)$ for $\rho(\widetilde{f_{\bar{t}}}^q - (p, l))$ is parallel to r, the supporting line for $\rho(\widetilde{f_{\bar{t}}})$ at $(p/q, l/q)$. So there is no restriction in assuming $(p/q, l/q)$ to be $(0, 0)$.

1.3.4. *Statement of the main theorem.*

THEOREM 1. *Under the six hypotheses assumed in §1.3.3 for the family f_t with $(p/q, l/q) = (0, 0)$, the following holds: $\widetilde{f_{\bar{t}}}$ has a hyperbolic fixed saddle $\widetilde{P_{\bar{t}}}$ such that $W^u(\widetilde{P_{\bar{t}}})$ has a topologically transverse (and therefore a C^1-transverse) intersection with $W^s(\widetilde{P_{\bar{t}}}) + (a, b)$ for some integer vector $(a, b) \neq (0, 0)$. In addition, there exists $K(f) > 0$ such that, for any $(c, d) \in \mathbb{Z}^2$ for which $(c, d).\vec{v} > K(f)$, if $\widetilde{P_t}$ is the continuation of $\widetilde{P_{\bar{t}}}$ for $t > \bar{t}$, then there exists a sequence $t_i > \bar{t}$, $t_i \overset{i\to\infty}{\to} \bar{t}$, such that $W^u(\widetilde{P_{t_i}})$ has a quadratic tangency with $W^s(\widetilde{P_{t_i}}) + (c', d')$, for some $(c', d') \in \mathbb{Z}^2$ that satisfies $|(c' - c, d' - d).\vec{v}| \leq K(f)/4$, and the tangency generically unfolds for $t > t_i$. The vector (c', d') is within a bounded distance from (c, d) in the direction of $\vec{v}$, but may be far in the direction of $\vec{v}^{\perp}$. These tangencies are heteroclinic intersections for $\widetilde{f_{t_i}}$ that cannot exist at $t = \bar{t}$. Finally, we point out that, for all $t > \bar{t}$ and for all integer vectors (a, b), $W^u(\widetilde{P_t})$ has a C^1 transverse intersection with $W^s(\widetilde{P_t}) + (a, b)$.*

Remarks

- As mentioned above, the tangencies given in the previous theorem are precisely for some integer vectors (c', d') which could not exist at $t = \bar{t}$. This will become clear in the proof.
- We were not able to produce tangencies at $t = \bar{t}$, even when $(c', d').\vec{v} > 0$ was small. In fact, ongoing work by Jager, Koropecki and Tal suggests that, for the particular family they are studying, these tangencies may not exist at the bifurcating parameter.

Actually, it is easy to see this for an area-preserving diffeomorphism of the sphere. If it has a hyperbolic saddle fixed point p and $W^u(p)$ intersects $W^s(p)$, then there is a topologically transverse intersection between $W^u(p)$ and $W^s(p)$. So, for a generic family of such maps, a horseshoe is not preceded by a tangency: the existence of a tangency already implies a horseshoe. This is clearly not true out of the the area-preserving world and shows how subtle is the problem of the birth of a horseshoe in the conservative case.

- Intuitively, as the rotation set becomes larger, one would expect the topological entropy to grow, at least for a tight model. In fact, in Kwapisz [**17**], some lower bounds for the topological entropy related to the two-dimensional size of the rotation set are presented (it is conjectured there that the area of the rotation set could be used, but what is actually used is a more technical computation on the size of the rotation set). Our main theorem says that every time the rotation set locally grows near a rational point, then nearby maps must have tangencies, which generically unfold as the parameter changes. This is a phenomenon which is associated to the growth of topological entropy, see [**5**] and [**23**]. More precisely, in these two papers it is proved that, generically, if a surface diffeomorphism f has arbitrarily close neighbors with larger topological entropy, then f has a periodic saddle point with a homoclinic tangency. Both these results were not stated in the area-preserving case. In fact they may not be true in the area-preserving case, but they indicate that whenever topological entropy is ready to grow, it is expected to find tangencies nearby.
- The unfolding of the above tangencies create generic elliptic periodic points, see [**9**].
- An analytic version of the above theorem can also be proved. We have to assume that:
  (1) the family has no connections between separatrices of periodic points;
  (2) for each period, there are only finitely many periodic points;
  (3) if a periodic point has a negative topological index, then it is a hyperbolic saddle.
  These conditions are generic among C^∞-1-parameter families, but this author does not know the situation for analytic families. The tangencies obtained in this case have finite order, but are not necessarily quadratic. To prove such a result, first remember that all isolated periodic points for analytic area-preserving diffeomorphisms satisfy a Lojasiewicz condition, see [**3**, §2]. Moreover, if an isolated periodic point has a characteristic curve (see [**11**] and again [**3**, §2]), then from the preservation of area, the dynamics in a neighborhood of such a point is obtained, at least in a topological sense, by gluing a finite number of saddle sectors.

Another important ingredient is the main result of [**15**] quoted here as Theorem A, which, among other things, says that for an area-preserving diffeomorphism f of the plane, which for every $n > 0$ has only isolated n-periodic points, if it has an invariant topological open disk U with compact boundary, whose prime ends rotation number is rational, then ∂U contains periodic points, all of the same period $k > 0$ and the eigenvalues of (Df^k) at these periodic points contained in ∂U are real and positive. So, if such a map f is analytic, the k-periodic points in ∂U satisfy a Lojasiewicz condition. If a certain periodic point, for instance denoted P, has topological index 1, then the eigenvalues of $(Df^k\mid_P)$ are both equal to 1. Under these conditions, the main result of [**24**] implies the existence of periodic orbits rotating around P with many different velocities with respect to an isotopy I_t from the Id to f. Additionally, a technical result in [**15**] says that if a periodic point belongs to ∂U, then this cannot happen. So, the topological indexes of all periodic points in ∂U are less than or equal to zero and thus, from [**3**, §2], they all have characteristic curves. Therefore, locally, all periodic points in ∂U are saddle-like. They may have two sectors (index zero) or four sectors (index -1). From [**18**], if ∂U is bounded, connections must exist. Thus, the hypothesis that there are no connections between separatrices of periodic points implies the irrationality of the prime ends rotation number for all open invariant disks whose boundaries are compact.

Finally, the last result we need is due to Churchill and Rod [**8**]. They show that for analytic area-preserving diffeomorphisms, the existence of a topologically transverse homoclinic point for a certain saddle implies the existence of a C^1-transverse homoclinic point for that saddle. Using these results in the appropriate places of the proof in the next section, an analytic version of the main theorem can be obtained.

In the next section of this paper we prove our main result.

2. *Proof of the main theorem*

The proof will be divided in two steps.

2.1. *Step 1.*

Here we prove that $\widetilde{f_{\bar{t}}}$ has a hyperbolic fixed saddle such that its unstable manifold has a topologically transverse intersection (therefore, C^1-transverse) with its stable manifold translated by a non-zero integer vector (a, b). Clearly, from Lemma 0, $(a, b).\vec{v} \leq 0$.

First of all, note that since $(0, 0) \in \partial\rho(\widetilde{f_{\bar{t}}})$ and for all $t > \bar{t}$, $(0, 0) \in \mathrm{int}(\rho(\widetilde{f_t}))$, then $\widetilde{f_{\bar{t}}}$ has (finitely many) fixed points up to $\mathbb{Z}^2$ translations, and it cannot be fixed-point free. The finiteness comes from the generic assumptions. If all of these fixed points had a zero topological index, as explained before the statement of Theorem 1, the dynamics near each of them would be as in Figure 2. In this situation, [**3**, Theorem 1] implies that $(0, 0) \notin \mathrm{int}(\rho(\widetilde{f_t}))$ for any t close to $\bar{t}$.

So there must be $\widetilde{f_{\bar{t}}}$-fixed points with a non-zero topological index. From the Nielsen–Lefschetz index theorem, we obtain a fixed point for $\widetilde{f_{\bar{t}}}$ with a negative index. From the genericity of our family, the only negative index allowed is -1 and fixed points with topological indices equal to -1 are hyperbolic saddles. Assume there are $k > 0$ such points in the fundamental domain $[0, 1[^2$, denoted $\{\widetilde{P}^1_{\bar{t}}, \ldots, \widetilde{P}^k_{\bar{t}}\}$. So, in $[0, 1[^2$ $\widetilde{f_{\bar{t}}}$ has k hyperbolic fixed saddle points.

Now let us choose a rational vector in $\mathrm{int}(\rho(\widetilde{f_{\bar{t}}}))$. Without loss of generality, conjugating f with some adequate integer matrix if necessary, we can suppose that this rational vector is of the form $(0, -1/n)$ for some $n > 0$.

By some results from [**1**], let $\widetilde{Q} \in \mathbb{R}^2$ be a periodic hyperbolic saddle point for $(\widetilde{f_{\bar{t}}})^n + (0, 1)$ such that $W^u(\widetilde{Q})$ has a topologically transverse intersection with $W^s(\widetilde{Q}) + (a, b)$ for all integer vectors (a, b). In this case, $\overline{W^u(\widetilde{Q})} = \overline{W^s(\widetilde{Q})} = \mathrm{RI}(\widetilde{f_{\bar{t}}})$, the region of instability of $\widetilde{f_{\bar{t}}}$, a $\widetilde{f_{\bar{t}}}$-invariant equivariant set such that, if $\widetilde{D}$ is a connected component of its complement, then $\widetilde{D}$ is a connected component of the lift of a $f_{\bar{t}}$-periodic open disk in the torus and for every $f_{\bar{t}}$-periodic open disk $D \subset \mathrm{T}^2$, $p^{-1}(D) \subset (\mathrm{RI}(\widetilde{f_{\bar{t}}}))^c$. Also, for any rational vector $(p/q, l/q) \in \mathrm{int}(\rho(\widetilde{f_{\bar{t}}}))$, there exists a hyperbolic periodic saddle point for $(\widetilde{f_{\bar{t}}})^q - (p, l)$, such that its unstable manifold also has topologically transverse intersections with all integer translates of its stable manifold, and so the closure of its stable manifold is equal the closure of its unstable manifold and they are both equal $\mathrm{RI}(\widetilde{f_{\bar{t}}})$. As we said, these results were proved in [**1**] and similar statements hold for homeomorphisms [**16**].

If, for some $1 \leq i^* \leq k$, $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ and $W^s(\widetilde{P_{\bar{t}}^{i^*}})$ are both unbounded subsets of the plane, then it can be proved that $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ must have a topologically transverse intersection with $W^s(\widetilde{Q})$, and $W^s(\widetilde{P_{\bar{t}}^{i^*}})$ must have a topologically transverse intersection with $W^u(\widetilde{Q})$. But, as the rotation vector of $\widetilde{Q}$ is not zero, this gives what we want in Step 1. More precisely, the following fact holds.

Fact. If $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ *and* $W^s(\widetilde{P_{\bar{t}}^{i^*}})$ *are both unbounded subsets of the plane, then* $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ *has a topologically transverse intersection with* $W^s(\widetilde{P_{\bar{t}}^{i^*}}) - (0, 1)$.

Proof. As $W^s(\widetilde{Q})$ has a topologically transverse intersection with $W^u(\widetilde{Q}) + (a, b)$ for all integer vectors (a, b), this implies that if $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ is unbounded, then $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ has a topologically transverse intersection with $W^s(\widetilde{Q})$. This follows from the following idea: there is a compact arc λ_u in $W^u(\widetilde{Q})$ that contains $\widetilde{Q}$ and a compact arc λ_s in $W^s(\widetilde{Q})$ that also contains $\widetilde{Q}$, such that λ_u has topologically transverse intersections with $\lambda_s + (0, 1)$ and $\lambda_s + (1, 0)$. This implies that the connected components of the complement of

$$\bigcup_{(a,b)\in\mathbb{Z}^2} \lambda_u \cup \lambda_s + (a, b)$$

are all open topological disks, with diameter uniformly bounded from above. So, if $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ is unbounded, it must have a topologically transverse intersection with some translate of λ_s. As $W^s(\widetilde{Q})$ C^0-accumulates on all its integer translates, we finally get that $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ has a topologically transverse intersection with $W^s(\widetilde{Q})$.

So, as for some integer $m > 0$,

$$(\widetilde{f_{\bar{t}}})^{m.n}(\widetilde{Q}) = \widetilde{Q} - (0, m),$$

$W^u(\widetilde{P_{\bar{t}}^{i^*}})$ C^0-accumulates on compact pieces of $W^u(\widetilde{Q}) - (0, k_j)$ for a certain sequence $k_j \to \infty$. That is, given a compact arc $\widetilde{\theta}$ contained in $W^u(\widetilde{Q})$, there exists a sequence $k_j \to \infty$ such that, for some arcs $\widetilde{\theta}_j \subset W^u(\widetilde{P_{\bar{t}}^{i^*}})$, $\widetilde{\theta}_j + (0, k_j) \to \widetilde{\theta}$ in the Hausdorff topology as $j \to \infty$. An analogous argument implies that if $W^s(\widetilde{P_{\bar{t}}^{i^*}})$ is unbounded, then $W^s(\widetilde{P_{\bar{t}}^{i^*}})$ has a topologically transverse intersection with $W^u(\widetilde{Q})$. So if we choose a compact arc κ_u contained in $W^u(\widetilde{Q})$ which has a topologically transverse intersection with $W^s(\widetilde{P_{\bar{t}}^{i^*}})$, we get that $W^u(\widetilde{P_{\bar{t}}^{i^*}})$ accumulates on $\kappa_u - (0, k_j)$ and thus it has a topologically transverse intersection with $W^s(\widetilde{P_{\bar{t}}^{i^*}}) - (0, k_j)$ for some $k_j > 0$ sufficiently large. As we pointed out after the definition of topologically transverse intersections, all the above follows from a C^0-version of the λ-lemma that holds for topologically transverse intersections. Now consider a compact subarc of a branch of $W^u(\widetilde{P_{\bar{t}}^{i^*}})$, denoted α^u, starting at $\widetilde{P}_{\tilde{t}}^{i^*}$ and a compact subarc of a branch of $W^s(\widetilde{P}_{\tilde{t}}^{i^*})$, denoted α^s, starting at $\widetilde{P}_{\tilde{t}}^{i^*}$ such that

$$\alpha^u \text{ has a topologically transverse intersection with } \alpha^s - (0, k_j) \quad \text{for some } k_j > 0. \tag{5}$$

Using Brouwer's lemma on translation arcs exactly as in [**1**, Lemma 24], we get that either α^u has an intersection with $\alpha^s - (0, 1)$ or $\alpha^u - (0, 1)$ has an intersection with α^s. If α^u had only non-topologically transverse intersections with $\alpha^s - (0, 1)$ and $\alpha^u - (0, 1)$ had only non-topologically transverse intersections with α^s, then we could C^0-perturb α^u and α^s in an arbitrarily small way such that $(\alpha^u_{\text{per}} \cup \alpha^s_{\text{per}}) \cap ((\alpha^u_{\text{per}} \cup \alpha^s_{\text{per}}) - (0, 1)) = \emptyset$ and $\alpha^u_{\text{per}} \cap (\alpha^s_{\text{per}} - (0, k_j)) \neq \emptyset$ (because of the topologically transverse assumption (5)). However, this contradicts Brouwer's lemma [**6**]. So, either α^u has a topologically transverse intersection with $\alpha^s - (0, 1)$ or $\alpha^u - (0, 1)$ has a topologically transverse intersection with α^s. The first possibility is what we want and the second is not possible because Lemma 0 would imply that $\rho(\widetilde{f_{\tilde{t}}})$ contains a point of the form $(0, a)$ for some $a > 0$. As $(0, -1/n) \in \text{int}(\rho(\widetilde{f_{\tilde{t}}}))$ these two facts contradict the assumption that $(0, 0) \in \partial\rho(\widetilde{f_{\tilde{t}}})$. □

In order to conclude this step, we need the following lemma.

LEMMA 2. *There exists $1 \leq i^* \leq k$ such that for any choice of $\lambda_u^{i^*}$ and $\lambda_s^{i^*}$, one unstable and one stable branch at $\widetilde{P}_{\tilde{t}}^{i^*}$, they are both unbounded.*

Proof. For each $1 \leq i \leq k$, as $\widetilde{P}_{\tilde{t}}^{i}$ has a topological index equal to -1, the two stable and the two unstable branches are each $\widetilde{f_{\tilde{t}}}$-invariant. Fix some unstable branch λ_u^i and some stable branch λ_s^i and let

$$K_u^i = \overline{\lambda_u^i} \quad \text{and} \quad K_s^i = \overline{\lambda_s^i}.$$

Both are connected $\widetilde{f_{\tilde{t}}}$-invariant sets. Let K^i be equal to either K_u^i or K_s^i and assume it is bounded. Without loss of generality, suppose that $K^i = K_u^i$. First, we collect some properties about K^i.

- If K^i intersects a connected component $\widetilde{D}$ of the complement of $\text{RI}(\widetilde{f_{\tilde{t}}})$, then, from [**13**, Lemma 6.1], $\lambda_u^i \backslash \widetilde{P}_{\tilde{t}}^{i}$ is contained in $\widetilde{D}$, which then is a $\widetilde{f_{\tilde{t}}}$-invariant bounded open disk (see [**1**] and [**16**]). As the family of diffeomorphisms considered is generic (in particular, it does not have connections between stable and unstable separatrices of periodic points), the rotation number of the prime ends compactification of $\widetilde{D}$, denoted β, must be irrational by Theorem A. So, if $\widetilde{P}_{\tilde{t}}^{i} \in \partial\widetilde{D}$, as $\lambda_u^i \backslash \widetilde{P}_{\tilde{t}}^{i} \subset \widetilde{D}$, this would be a contradiction with the irrationality of β, because $\widetilde{f_{\tilde{t}}}(\lambda_u^i) = \lambda_u^i$. Thus, $\widetilde{P}_{\tilde{t}}^{i}$ is contained in $\widetilde{D}$ and the topological index of $\widetilde{f_{\tilde{t}}}$ with respect to $\widetilde{D}$ is $+1$ (because β is irrational, therefore not zero). By the topological index of $\widetilde{f_{\tilde{t}}}$ with respect to $\widetilde{D}$ we mean the sum of the indices at all the $\widetilde{f_{\tilde{t}}}$-fixed points contained in $\widetilde{D}$. This information will be used in the end of the proof.
- Suppose now that $K^i = K_u^i$ is contained in $\text{RI}(\widetilde{f_{\tilde{t}}})$. It is not possible that $K_u^i \cap \lambda_s^i = \widetilde{P}_{\tilde{t}}^{i}$ because that would imply that the connected component M of $(K^i)^c$ which contains $\lambda_s^i \backslash \widetilde{P}_{\tilde{t}}^{i}$ has a rational prime ends rotation number and, as we have already said, this does not happen under our generic conditions. So, from Oliveira [**22**, Lemma 2], we get that either $K_u^i \supset \lambda_s^i$ or λ_u^i intersects λ_s^i. If λ_u^i intersects λ_s^i, then

λ_s^i intersects $\mathrm{RI}(\widetilde{f_{\bar{t}}})$, so it is contained in $\mathrm{RI}(\widetilde{f_{\bar{t}}})$ and we can find a Jordan curve τ contained in $\lambda_u^i \cup \lambda_s^i$, $\widetilde{P_{\bar{t}}^i} \in \tau$. Theorem B implies that there is no periodic point in the boundary of a connected component of the complement of $\mathrm{RI}(\widetilde{f_{\bar{t}}})$, because such a component is $f_{\bar{t}}$-periodic when projected to the torus and it has an irrational prime ends rotation number, by Theorem A. So, interior(τ) intersects $\mathrm{RI}(\widetilde{f_{\bar{t}}})$, and thus interior($\tau$) intersects both $W^u(\widetilde{Q})$ and $W^s(\widetilde{Q})$, something that contradicts the assumption that K_u^i is bounded because λ_u^i must intersect $W^s(\widetilde{Q})$, which implies that it is unbounded. Moreover, if $K_u^i \supset \lambda_s^i$, then from our assumption that K_u^i is bounded, we get that $K_s^i = \overline{\lambda_s^i}$ is also bounded. Arguing as above, we obtain that $K_s^i \cap \lambda_u^i \neq \widetilde{P_{\bar{t}}^i}$.

Otherwise, if M^* is the connected component of $(K_s^i)^c$ that contains $\lambda_u^i \backslash \widetilde{P_{\bar{t}}^i}$, as $\widetilde{f_{\bar{t}}}(\lambda_u^i) = \lambda_u^i$, the rotation number of the prime ends compactification of M^* would be rational. As this implies connections between separatrices of periodic points, which do not exist under our hypotheses, K_s^i must intersect $\lambda_u^i \backslash \widetilde{P_{\bar{t}}^i}$. Again, from Oliveira [**22**, Lemma 2], $K_s^i \supset \lambda_u^i$ or λ_u^i intersects λ_s^i and we are done.

Thus, the almost-final situation we have to deal is when $K_u^i \supset \lambda_s^i$ and $K_s^i \supset \lambda_u^i$. However, it is contained in the proof of the main theorem of [**22**] that these relations imply that λ_u^i intersects λ_s^i. As we explained above, this is a contradiction with the assumption that K_u^i is bounded. If $K^i = K_s^i$ and $K_s^i \subset \mathrm{RI}(\widetilde{f_{\bar{t}}})$, an analogous argument could be applied in order to arrive at similar contradictions.

Thus, if K^i is bounded, it must be contained in the complement of $\mathrm{RI}(\widetilde{f_{\bar{t}}})$. In order to conclude the proof, we are left to consider the case when, for every $1 \leq i \leq k$, we can choose K^i equal to either K_u^i or K_s^i, such that it is bounded and contained in a connected component $\widetilde{D}_i$ of the complement of $\mathrm{RI}(\widetilde{f_{\bar{t}}})$. As we already obtained, the topological index of $\widetilde{f_{\bar{t}}}$ restricted to $\widetilde{D}_i$ is $+1$. This clearly contradicts the Nielsen–Lefschetz index formula because the sum of the indices of $f_{\bar{t}}$ at its fixed points which have (0, 0) rotation vector would be positive. So, for some $1 \leq i^* \leq k$, both unstable and both stable branches at $\widetilde{P_{\bar{t}}^{i^*}}$ are unbounded. □

This concludes Step 1.

2.2. *Step 2.* From the previous step we know that there exists a $\widetilde{f_{\bar{t}}}$-fixed point denoted $\widetilde{P_{\bar{t}}}$ such that $W^u(\widetilde{P_{\bar{t}}})$ has a topologically transverse, and therefore a C^1-transverse, intersection with $W^s(\widetilde{P_{\bar{t}}}) - (0, 1)$.

So, there exists a compact connected piece of a branch of $W^u(\widetilde{P_{\bar{t}}})$, starting at $\widetilde{P_{\bar{t}}}$, denoted $\lambda_u^{\bar{t}}$, and a compact connected piece of a branch of $W^s(\widetilde{P_{\bar{t}}})$, starting at $\widetilde{P_{\bar{t}}}$, denoted $\lambda_s^{\bar{t}}$, such that $\lambda_u^{\bar{t}} \cup (\lambda_s^{\bar{t}} - (0, 1))$ contains a continuous curve connecting $\widetilde{P_{\bar{t}}}$ to $\widetilde{P_{\bar{t}}} - (0, 1)$. The end point of $\lambda_u^{\bar{t}}$, denoted w, belongs to $\lambda_s^{\bar{t}} - (0, 1)$ and it is a C^1-transverse heteroclinic point. The main consequence of the above is the following fact.

Fact. The curve $\widetilde{\gamma}_V^{\bar{t}}$ connecting $\widetilde{P_{\bar{t}}}$ to $\widetilde{P_{\bar{t}}} - (0, 1)$ contained in $\lambda_u^{\bar{t}} \cup (\lambda_s^{\bar{t}} - (0, 1))$ projects to a (not necessarily simple) closed curve in the torus, homotopic to $(0, -1)$ and it has a continuous continuation for $t \geq \bar{t}$ sufficiently small. That is, for $t - \bar{t} \geq 0$ sufficiently small, there exists a curve $\widetilde{\gamma}_V^t$ connecting $\widetilde{P_t}$ to $\widetilde{P_t} - (0, 1)$ made by a piece of an unstable branch of $W^u(\widetilde{P}_t)$ and a piece of a stable branch of $W^s(\widetilde{P}_t) - (0, 1)$ such that $t \to \widetilde{\gamma}_V^t$ is continuous for $t - \bar{t} \geq 0$ sufficiently small.

Proof. The proof is immediate from the fact that w is a C^1-transversal heteroclinic point which has a continuous continuation for all diffeomorphisms C^2-close to $\widetilde{f}_{\bar{t}}$. □

As $(0, -1/n)$ is contained in $\text{int}(\rho(\widetilde{f}_{\bar{t}}))$, there are rational points in $\text{int}(\rho(\widetilde{f}_{\bar{t}}))$ with positive horizontal coordinates. Thus, if we define

$$\Gamma_{V,a}^{\bar{t}} \overset{\text{def.}}{=} \cdots \cup \widetilde{\gamma}_V^{\bar{t}} + (a, 2) \cup \widetilde{\gamma}_V^{\bar{t}} + (a, 1) \cup \widetilde{\gamma}_V^{\bar{t}} + (a, 0) \cup \widetilde{\gamma}_V^{\bar{t}} + (a, -1) \cup \cdots \quad \text{for } a \in \mathbb{Z},$$

we get that, for all sufficiently large integers $n > 0$, $(\widetilde{f}_{\bar{t}})^n(\Gamma_{V,0}^{\bar{t}})$ intersects $\Gamma_{V,1}^{\bar{t}}$ transversely and, moreover, we obtain a curve $\widetilde{\gamma}_H^{\bar{t}}$ connecting $\widetilde{P}_{\bar{t}}$ to $\widetilde{P}_{\bar{t}} + (1, 0)$ of the following form: it starts at $\widetilde{P}_{\bar{t}}$, and goes through the branch of $W^u(\widetilde{P}_{\bar{t}})$ that contains $\lambda_u^{\bar{t}}$ until it hits, in a topologically transverse way (so in a C^1-transverse way), $\lambda_s^{\bar{t}} + (1, b)$ for some integer b. If $b < 0$, we add to this curve the following one:

$$\widetilde{\gamma}_V^{\bar{t}} + (1, 0) \cup \widetilde{\gamma}_V^{\bar{t}} + (1, -1) \cup \cdots \cup \widetilde{\gamma}_V^{\bar{t}} + (1, b + 1) \tag{6}$$

and if $b \geq 0$, we add the following curve:

$$\widetilde{\gamma}_V^{\bar{t}} + (1, 1) \cup \widetilde{\gamma}_V^{\bar{t}} + (1, 2) \cup \cdots \cup \widetilde{\gamma}_V^{\bar{t}} + (1, b) \cup \lambda_s^{\bar{t}} + (1, b). \tag{7}$$

In both cases, we omit a small piece of $\lambda_s^{\bar{t}} + (1, b)$ in order to get a proper curve connecting $\widetilde{P}_{\bar{t}}$ and $\widetilde{P}_{\bar{t}} + (1, 0)$. This follows from the fact that when we consider iterates $(\widetilde{f}_{\bar{t}})^n(\Gamma_{V,0}^{\bar{t}})$ for some large $n > 0$, the arcs contained in stable manifolds are shrinking and arcs contained in unstable manifolds are getting bigger. As there are orbits moving to the right under positive iterates of $\widetilde{f}_{\bar{t}}$, the above holds. So, $\widetilde{\gamma}_H^{\bar{t}}$ is a continuous curve whose end points are $\widetilde{P}_{\bar{t}}$ and $\widetilde{P}_{\bar{t}} + (1, 0)$, made of a connected piece of an unstable branch of $W^u(\widetilde{P}_{\bar{t}})$ added to either one of the two vertical curves above, (6) or (7),with a small piece of $\lambda_s^{\bar{t}} + (1, b)$ deleted. As the intersection between the branch of $W^u(\widetilde{P}_{\bar{t}})$ that contains $\lambda_u^{\bar{t}}$ with $\lambda_s^{\bar{t}} + (1, b)$ is C^1-transverse and $t \to \widetilde{\gamma}_V^t$ is continuous for $t - \bar{t} \geq 0$ sufficiently small, we get that $t \to \widetilde{\gamma}_H^t$ is also continuous for $t - \bar{t} \geq 0$ sufficiently small. See Figure 3(a) and (b) for representations of these two possibilities.

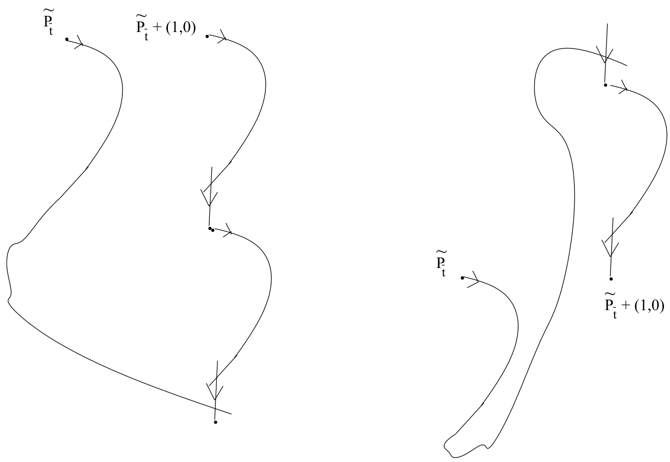
Remember that r is the supporting line at $(0, 0) \in \partial\rho(\widetilde{f}_{\bar{t}})$ and $\vec{v}$ is an unitary vector orthogonal to r such that $-\vec{v}$ points towards the rotation set.

Now we state a more general version of [**2**, Lemma 6]. It does not appear in that paper in this form, but the proof presented there also proves this more abstract version.

LEMMA 6 OF [**2**]. *Let K_H and K_V be two continua in the plane, such that K_H contains* $(0, 0)$ *and* $(1, 0)$ *and K_V contains* $(0, 0)$ *and* $(0, 1)$. *For every vector $\vec{w}$, it is possible to construct a connected closed set $M_{\vec{w}}$ which is equal to the union of well-chosen integer translates of K_H and K_V such that:*

(1) *$M_{\vec{w}}$ intersects every straight line orthogonal to $\vec{w}$;*

(2) *$M_{\vec{w}}$ is bounded in the direction orthogonal to $\vec{w}$, that is, $M_{\vec{w}}$ is contained between two straight lines $r_{M_{\vec{w}}}$ and $s_{M_{\vec{w}}}$, both parallel to $\vec{w}$, and the distance between*



(a) (b)

FIGURE 3. How to construct $\widetilde{\gamma}_H^t$.

these lines is less then $3 + 2.\max\{\text{diameter}(K_H), \text{diameter}(K_V)\}$. *So, in particular,* $(M_{\vec{w}})^c$ *has at least two unbounded connected components, one containing* $r_{M_{\vec{w}}}$*, denoted* $U_r(M_{\vec{w}})$*, and the other containing* $s_{M_{\vec{w}}}$*, denoted* $U_s(M_{\vec{w}})$*.*

So, applying this lemma in the context of this paper gives us a path-connected closed set $\theta^{\bar t}_{\vec{v}^{\perp}} \subset \mathbb{R}^2$ which is obtained as the union of certain integer translates of $\widetilde{\gamma}_V^{\bar t}$ or $\widetilde{\gamma}_H^{\bar t}$ in a way that:

(1) $\theta^{\bar t}_{\vec{v}^{\perp}}$ intersects every straight line parallel to $\vec{v}$;

(2) $\theta^{\bar t}_{\vec{v}^{\perp}}$ is bounded in the direction of $\vec{v}$, that is, $\theta^{\bar t}_{\vec{v}^{\perp}}$ is contained between two straight lines l_- and l_+, both parallel to $\vec{v^{\perp}}$, and the distance between these lines is less then $3 + 2.\max\{\text{diameter}(\widetilde{\gamma}_V^{\bar t}), \text{diameter}(\widetilde{\gamma}_H^{\bar t})\}$. Moreover, $(\theta^{\bar t}_{\vec{v}^{\perp}})^c$ has at least two unbounded connected components, one containing l_-, denoted U_-, and the other containing l_+, denoted U_+.

Assume that l_+ and U_+ were chosen so that if (c, d) is an integer vector such that $\theta^{\bar t}_{\vec{v}^{\perp}} + (c, d)$ belongs to U_+, then

$$(c, d).\vec{v} > 0. \tag{8}$$

It is not hard to see that for integer vectors (c, d) for which $\theta^{\bar t}_{\vec{v}^{\perp}} + (c, d)$ belongs either to U_+ or U_-, an inequality like (8) needs to hold.

For this, note that if $\vec{v}$ is a rational direction for which $(c, d).\vec{v} = 0$, then from the way $\theta^{\bar t}_{\vec{v}^{\perp}}$ is constructed, we get that $\theta^{\bar t}_{\vec{v}^{\perp}} + (c, d)$ intersects $\theta^{\bar t}_{\vec{v}^{\perp}}$, so $\theta^{\bar t}_{\vec{v}^{\perp}} + (c, d)$ does not belong to $(\theta^{\bar t}_{\vec{v}^{\perp}})^c$. And in case $\vec{v}$ is an irrational direction, $(c, d).\vec{v} \neq 0$. In particular, an integer vector (c, d) satisfying the inequality in (8) has the property that any positive multiple of it does not belong to $\rho(\widetilde{f_{\bar t}})$. This will be important in the remainder of the proof.

Moreover, if $(c,d).\vec{v} > 3 + 2.\max\{\text{diameter}(\widetilde{\gamma}_V^{\bar{t}}), \text{diameter}(\widetilde{\gamma}_H^{\bar{t}})\}$, then $\theta_{\vec{v}^\perp}^{\bar{t}} + (c,d)$ belongs to U_+. Now we are ready to finish the proof of the main theorem.

As both $t \to \widetilde{\gamma}_V^t$ and $t \to \widetilde{\gamma}_H^t$ are continuous for $t - \bar{t} \geq 0$ sufficiently small, the same holds for $t \to \theta_{\vec{v}^\perp}^t$. Given any integer vector (c,d) such that

$$(c,d).\vec{v} > K(f) \overset{\text{def.}}{=} 4.(3 + 2.\max\{\text{diameter}(\widetilde{\gamma}_V^{\bar{t}}), \text{diameter}(\widetilde{\gamma}_H^{\bar{t}})\}) + 10,$$

if a $t^* > \bar{t}$ sufficiently close to $\bar{t}$ is fixed, we can assume that

$$\theta_{\vec{v}^\perp}^t + (c,d) \cap \theta_{\vec{v}^\perp}^t = \emptyset \quad \text{for all } t \in [\bar{t}, t^*].$$

In addition, as $(0,0) \in \text{int}(\rho(\widetilde{f}_{t^*}))$, there exists an integer $N > 0$ such that

$$(\widetilde{f}_{t^*})^N(\theta_{\vec{v}^\perp}^{t^*}) \text{ has a topologically transverse intersection with } \theta_{\vec{v}^\perp}^{t^*} + (c,d).$$

However, as $(0,0) \notin \text{int}(\rho(\widetilde{f}_{\bar{t}}))$, and $(c,d).\vec{v}$ is sufficiently large, we get that

$$(\widetilde{f}_{\bar{t}})^N(\theta_{\vec{v}^\perp}^{\bar{t}}) \cap \theta_{\vec{v}^\perp}^{\bar{t}} + (c,d) = \emptyset.$$

The previous property follows from the existence of $(a,b) \in \mathbb{Z}^2$ such that

$$\theta_{\vec{v}^\perp}^{\bar{t}} + (a,b) \text{ is contained between } \theta_{\vec{v}^\perp}^{\bar{t}} \text{ and } \theta_{\vec{v}^\perp}^{\bar{t}} + (c,d)$$

and

$$(a,b).\vec{v} > 2.(3 + 2.\max\{\text{diameter}(\widetilde{\gamma}_V^{\bar{t}}), \text{diameter}(\widetilde{\gamma}_H^{\bar{t}})\}) + 5.$$

If we prove that $(\widetilde{f}_{\bar{t}})^N(\theta_{\vec{v}^\perp}^{\bar{t}})$ cannot have a topologically transverse intersection with $\theta_{\vec{v}^\perp}^{\bar{t}} + (a,b)$, then it clearly cannot intersect $\theta_{\vec{v}^\perp}^{\bar{t}} + (c,d)$. So, if there were such a topologically transverse intersection, as $\theta_{\vec{v}^\perp}^{\bar{t}} + (a,b)$ is disjoint from $\theta_{\vec{v}^\perp}^{\bar{t}}$ and arcs of stable manifolds shrink under positive iterates of $\widetilde{f}_{\bar{t}}$, there would be some $(a',b'), (a'',b'') \in \mathbb{Z}^2$, such that $\widetilde{P}_{\bar{t}} + (a',b')$ belongs to $\theta_{\vec{v}^\perp}^{\bar{t}}$ and its unstable manifold has a transverse intersection with the stable manifold of $\widetilde{P}_{\bar{t}} + (a'',b'')$ which belongs to $\theta_{\vec{v}^\perp}^{t'} + (a,b)$. As $\theta_{\vec{v}^\perp}^{\bar{t}}$ is bounded in the direction of $\vec{v}$ by $3 + 2.\max\{\text{diameter}(\widetilde{\gamma}_V^{\bar{t}}), \text{diameter}(\widetilde{\gamma}_H^{\bar{t}})\}$, we get that $(a''-a', b''-b').\vec{v} > 0$, so Lemma 0 implies the existence of a rotation vector outside $\rho(\widetilde{f}_{\bar{t}})$, a contradiction.

Thus, from the continuity of $t \to \theta_{\vec{v}^\perp}^t$ and $t \to \widetilde{f}_t$, there exists $t' \in]\bar{t}, t^*[$ such that $(\widetilde{f}_{t'})^N(\theta_{\vec{v}^\perp}^{t'})$ has a non-topologically transverse intersection with $\theta_{\vec{v}^\perp}^{t'} + (c,d)$ and, for all $t \in]t', t^*]$, the intersection is topologically transverse. As above, from the fact that stable manifolds shrink under positive iterates of $\widetilde{f}_{t'}$, the intersection that happens for $t = t'$ corresponds to a tangency between the unstable manifold of some translate of $\widetilde{P}_{t'}$ which belongs to $\theta_{\vec{v}^\perp}^{t'}$ with the stable manifold of some translate of $\widetilde{P}_{t'}$ which belongs to $\theta_{\vec{v}^\perp}^{t'} + (c,d)$. In other words, there is a tangency between $W^u(\widetilde{P}_{t'})$ and $W^s(\widetilde{P}_{t'}) + (c^*, d^*)$ for some integer vector (c^*, d^*) such that $|(c^* - c, d^* - d).\vec{v}| \leq 3 + 2.\max\{\text{diameter}(\widetilde{\gamma}_V^{\bar{t}}), \text{diameter}(\widetilde{\gamma}_H^{\bar{t}})\}$. This estimate follows from the fact that if

$\widetilde{P}_t + (e, f)$ belongs to $\theta^t_{\vec{v}^\perp}$, then $|(e, f).\vec{v}| \leq 3 + 2.\max\{\text{diameter}(\widetilde{\gamma}^{\bar{t}}_V), \text{diameter}(\widetilde{\gamma}^{\bar{t}}_H)\}$. Here we are using the fact that $\widetilde{P}_t \in \theta^t_{\vec{v}^\perp}$.

As $t^* > \bar{t}$ is arbitrary, if we remember that for a generic family as we are considering, topologically transverse intersections are C^1-transverse and tangencies are always quadratic, then the proof of the main theorem is almost complete. We are left to deal with the last part of the statement, which says that, for all parameters $t > \bar{t}$,

$$W^u(\widetilde{P}_t) \text{ has transverse intersections with } W^s(\widetilde{P}_t) + (a, b) \quad \text{for all } (a, b) \in \mathbb{Z}^2.$$

As for $t > \bar{t}$, $(0, 0) \in \text{int}(\rho(\widetilde{f}_t))$, the main result of [**1**] implies that (for each $t > \bar{t}$) $\widetilde{f}_t$ has a hyperbolic periodic saddle (not necessarily fixed) $\widetilde{Z}_t \in \mathbb{R}^2$ such that $W^u(\widetilde{Z}_t)$ has transverse intersections with $W^s(\widetilde{Z}_t) + (a, b)$, for all $(a, b) \in \mathbb{Z}^2$. So, as $W^u(\widetilde{P}_t)$ and $W^s(\widetilde{P}_t)$ are both unbounded, exactly as we did in the proof of the fact from Step 1 of this proof, $W^u(\widetilde{P}_t)$ has transverse intersections with $W^s(\widetilde{Z}_t)$ and $W^s(\widetilde{P}_t)$ has transverse intersections with $W^u(\widetilde{Z}_t)$. Thus an application of the λ-lemma concludes the proof.

Acknowledgements. When finishing the writing of this paper, I noticed that if I did not assume C^1-transversal intersections between stable and unstable manifolds of hyperbolic periodic saddles, but only topologically transversal intersections in certain parts of the arguments, the proofs would become much more complicated. So, I started looking for a result saying that for generic 1-parameter families of surface diffeomorphisms, homoclinic and heteroclinic points are either quadratic tangencies or C^1-transverse. When, at first, I could not find what I needed, I contacted several people who have worked on the subject. My thanks go to Carlos Gustavo Moreira, Eduardo Colli, Enrique Pujals, Marcelo Viana and Pedro Duarte whose answers were positive, informing me that this kind of result has been known since [**21**].

I also would like to thank Andres Koropecki for conversations on the results of [**22**] and for explanations on his results in [**14**] and [**15**].

The author is partially supported by CNPq, grant number 306348/2015-2.

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