Stability for the vertical rotation interval of twist mappings

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Abstract

In this paper we consider twist mappings of the torus, $\overline{T}: \mathrm{T}^2 \to \mathrm{T}^2$, and their vertical rotation intervals $\rho_V(T) = [\rho_V^-, \rho_V^+]$, which are closed intervals such that for any $\omega \in]\rho_V^-, \rho_V^+[$ there exists a compact $\overline{T}$-invariant set $\overline{Q}_\omega$ with $\rho_V(\overline{x}) = \omega$ for any $\overline{x} \in \overline{Q}_\omega$, where $\rho_V(\overline{x})$ is the vertical rotation number of $\overline{x}$. In case ω is a rational number, $\overline{Q}_\omega$ is a periodic orbit (this study began in [1] and [2]). Here we analyze how ρ_V^- and ρ_V^+ behave as we perturb $\overline{T}$ when they assume rational values. In particular we prove that for analytic area-preserving mappings these functions are locally constant at rational values.

Key words: twist mappings, vertical rotation set, topological methods, analyticity, bifurcations

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1 Introduction and main result

The dynamics of a homeomorphism of the circle has a well-known and very important invariant, the so called rotation number. Roughly speaking, it measures the average speed the orbit of a point in the circle rotates around it and its rationality or not has strong implications on the dynamics of the homeomorphism (see [14] for a nice didactic exposition). In particular when the rotation number is rational, it is locally constant in the C^0 topology, provided that the mapping is not conjugate to a rigid rotation. Similar results for circle endomorphisms have been proved by Boyland in [8]. As it is well-known, endomorphisms of the circle do not have a single rotation number, they have a closed interval of rotation numbers (the rotation interval), with many interesting properties (see [13] and [20]). The results proved by Boyland in [8] concern the behavior of the extreme points of the rotation interval.

The aim of the present paper is to study a similar problem in the context of twist mappings of the torus. First we remember that twist mappings of the torus have an invariant called vertical rotation interval (see [1], [2], [4] and [5]). To be more precise, given a twist mapping $T: S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ which induces a diffeomorphism $\overline{T}: \mathrm{T}^2 \to \mathrm{T}^2$ homotopic to the Dehn twist $(\phi, I) \to (\phi + I \ mod\ 1, I \ mod\ 1)$, there exists a closed interval $\rho_V(T) = [\rho_V^-, \rho_V^+]$ with the following property:

For $\omega \in int(\rho_V(T))$, there are 2 different situations.

1) $\omega = \frac{p}{q}$ is a rational number. In this case there is a q-periodic point $\overline{x}$ for $\overline{T}$ (in fact there are at least 2 such points) that lifts to a point $x \in S^1 \times \mathbb{R}$ such that $T^q(x) = x + (0, p)$.

2) ω is an irrational number. Then there is a compact, $\overline{T}$-invariant set $\overline{Q} \subset \mathrm{T}^2$ that lifts to a set $Q \subset S^1 \times \mathbb{R}$ such that for any $x \in Q$ we have:

$$\lim_{n \to \infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n} = \omega,$$

where $p_2: S^1 \times \mathbb{R} \to \mathbb{R}$ is the projection in the vertical coordinate.

In [4] we have shown that ρ_V^+ and ρ_V^- are continuous functions of T in the

C^1 topology (in fact a stronger result was proved there). Note that this is in complete similarity with results for circle homeomorphisms and endomorphisms.

As we already said, the main result of this paper is inspired by the one we mentioned for circle homeomorphisms with rational rotation number. If $T: S^1\times\mathbb{R}\rightarrow S^1\times\mathbb{R}$ is an analytic area-preserving twist mapping which induces a diffeomorphism $\overline{T}: \mathrm{T}^2\rightarrow\mathrm{T}^2$ homotopic to the Dehn twist and $\rho_V^-(T) < \rho_V^+(T)\in\mathbb{Q}$, then $\rho_V^+(T)$ is locally constant in the C^1 topology, that is, it can not be increased and decreased by appropriate C^1 small perturbations. A similar result holds for $\rho_V^-(T)$ when it assumes a rational value and the vertical rotation interval is not degenerate to a point.

In [5] we solved the complementary problem: There we proved that for a C^1 twist mapping T, if $\rho_V^-(T) < \rho_V^+(T)\notin\mathbb{Q}$, then $\rho_V^+(T+(0,\alpha)) > \rho_V^+(T)$ for any $\alpha > 0$. Note that this result is also analogous to what happens for a circle homeomorphism with irrational rotation number. For a similar result in the context of torus homeomorphisms homotopic to the identity see [7].

In [6] a partial version of the main result of this paper was proved. The class of mappings studied there was, in a certain sense, a little restrictive, as for instance, it did not contain the standard mapping (in fact, it did not contain any area-preserving mapping):

$$S_M: \begin{cases} \phi' = \phi + I' \ mod\ 1 \\ I' = I + \frac{k}{2\pi}\sin(2\pi\phi)\ mod\ 1 \end{cases} \tag{1}$$

Clearly, as S_M is analytic and area-preserving, the result proved here can be applied to it.

Our main goal with all this analysis of how the vertical rotation interval changes as we perturb T is a better understanding of the bifurcations that happen in a family like S_M.

As the research in this subject has shown, this is a very difficult task. The main problems with our analysis are the following:

1. the particular family may not be "versal" for the bifurcation we are considering. For instance, even when $\rho_V^+(T)$ can be increased by appropriate

perturbations, it may be the case that it remains constant or even decreases if the perturbations are restricted to a particular family.

2. there is not a nice topological characterization (at least not to our knowledge) of all the possible dynamics in the neighborhood of an index one periodic point of an area-preserving mapping. This problem is related to the following one: If $\rho_V^+(T)$ can be increased by appropriate perturbations, then new index one periodic orbits appear. Is the appearance of these orbits associated to invariant sets of positive Lebesgue measure? In other words, is the appearance of these orbits associated to non-ergodicity with respect to Lebesgue measure for the perturbed torus mapping?

3. when $\rho_V^+(T)$ is locally constant at a rational value, are there non-trivial invariant sets with this vertical rotation number? (by non-trivial we mean with positive Lebesgue measure)

This paper is organized as follows. In the first section we present some basic definitions and the precise statement of our main result, in the second section we present a brief summary of the results used in the paper and in the third section we prove our main theorem.

Notation and definitions :

0) Let (ϕ, I) denote the coordinates for the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where ϕ is defined modulo 1. Let $(\widetilde{\phi}, \widetilde{I})$ denote the coordinates for the universal cover of the cylinder, $\mathbb{R}^2$. For all mappings $T : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}$ we define $(\phi', I') = T(\phi, I)$ and $(\widetilde{\phi}', \widetilde{I}') = \widetilde{T}(\widetilde{\phi}, \widetilde{I})$, where $\widetilde{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a lift of T.

1) $\mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2) = \{\widetilde{T} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \ / \ \widetilde{T}$ is a C^1 diffeomorphism of the plane, $\widetilde{I}'(\widetilde{\phi}, \widetilde{I}) \overset{\widetilde{I} \to \pm\infty}{\rightarrow} \pm\infty$, $\partial_{\widetilde{I}} \widetilde{\phi}' > 0$ (twist to the right), $\widetilde{\phi}'(\widetilde{\phi}, \widetilde{I}) \overset{\widetilde{I} \to \pm\infty}{\rightarrow} \pm\infty$ and $\widetilde{T}$ is the lift of a C^1 diffeomorphism $T : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R}\}$.

2) $\mathrm{Diff}^1_{\mathrm{r}}(S^1 \times \mathbb{R}) = \{ T : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R} \ / \ T$ is induced by an element of $\mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2)\}$.

3) Let $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ be the standard projections, respectively in the $\widetilde{\phi}$ and $\widetilde{I}$ coordinates ($p_1(\widetilde{\phi}, \widetilde{I}) = \widetilde{\phi}$ and $p_2(\widetilde{\phi}, \widetilde{I}) = \widetilde{I}$). We also use p_1 and p_2 for the standard projections of the cylinder.

4) Let $\mathrm{D}_{\mathrm{Dehn}} = \{T \in \mathrm{Diff}^1_{\mathrm{r}}(S^1 \times \mathbb{R}) : T$ induces a mapping $\overline{T} : \mathrm{T}^2 \to \mathrm{T}^2$ homotopic to the Dehn twist $(\phi, I) \to (\phi + I \ mod \ 1, I \ mod \ 1)\}$, where $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the 2-torus (coordinates in the torus are denoted by $(\overline{\phi}, \overline{I})$)

5) Let $pro : S^1 \times \mathbb{R} \to \mathrm{T}^2$ and $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ be given by:

$$pro(\phi, I) = (\phi, I \ mod \ 1)$$
$$\pi(\widetilde{\phi}, \widetilde{I}) = (\widetilde{\phi} \ mod \ 1, \widetilde{I})$$

6) Given a point $\overline{x} \in \mathrm{T}^2$, we define its vertical rotation number as (when the limit exists):

$$\rho_V(\overline{x}) = \lim_{n \to \infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n}, \text{ for any } x \in pro^{-1}(\overline{x}) \tag{2}$$

Now we are ready to state our main result:

Theorem 1 *: Let $T \in \mathrm{D}_{\mathrm{Dehn}}$ be an analytic area-preserving mapping such that $\rho_V(T) = [\rho_V^-, \frac{p}{q}]$, with $\frac{p}{q} \in \mathbb{Q}$, $(p,q) = 1$ and $\rho_V^- < \frac{p}{q}$. Then, if $V \subset \mathrm{D}_{\mathrm{Dehn}}$ is any sufficiently small neighborhood of T in the C^1 topology, one and only one of the following possibilities holds:*

1) for all $G \in V$, $\rho_V^+(G) \leq p/q$

2) for all $G \in V$, $\rho_V^+(G) = p/q$

3) for all $G \in V$, $\rho_V^+(G) \geq p/q$

2 Basic tools

In this part of the paper we state some results we use and present references.

2.1 Some results for twist mappings

First we recall some topological results for twist mappings essentially due to Le Calvez (see [15] and [16]). Let $T \in \mathrm{Diff}^1_{\mathrm{r}}(S^1 \times \mathbb{R})$ and $\widetilde{T} \in \mathrm{D}^1_{\mathrm{r}}(\mathbb{R}^2)$ be a lift of

T to the plane. For every pair (s,q), $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we define the following sets:

$$\begin{array}{c} K_{lift}(s,q) = \left\{ (\widetilde{\phi}, \widetilde{I}) \in \mathbb{R}^2 \colon\ p_1 \circ \widetilde{T}^q(\widetilde{\phi}, \widetilde{I}) = \widetilde{\phi} + s \right\} \\ \text{and} \\ K(s,q) = \pi \circ K_{lift}(s,q) \end{array} \tag{3}$$

Then we have the following:

Lemma 1 *: For every $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(s,q) \supset C(s,q)$, a connected compact set that separates the cylinder.*

For all $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we can define the following functions on S^1:

$$\begin{array}{c} \mu^-(\phi) = \min\{p_2(z) \colon\ z \in K(s,q) \text{ and } p_1(z) = \phi\} \\ \mu^+(\phi) = \max\{p_2(z) \colon\ z \in K(s,q) \text{ and } p_1(z) = \phi\} \end{array}$$

And we can define similar functions for $T^q(K(s,q))$:

$$\begin{array}{c} \nu^-(\phi) = \min\{p_2(z) \colon\ z \in T^q \circ K(s,q) \text{ and } p_1(z) = \phi\} \\ \nu^+(\phi) = \max\{p_2(z) \colon\ z \in T^q \circ K(s,q) \text{ and } p_1(z) = \phi\} \end{array}$$

Lemma 2 *: Defining $Graph\{\mu^\pm\}=\{(\phi,\mu^\pm(\phi)) : \phi \in S^1\}$ we have:*

$$Graph\{\mu^-\} \cup Graph\{\mu^+\} \subset C(s,q)$$

So for all $\phi \in S^1$ we have $(\phi, \mu^\pm(\phi)) \in C(s,q)$.

The next lemma is a fundamental result in all this theory:

Lemma 3 *: $T^q(\phi,\mu^-(\phi)) = (\phi,\nu^+(\phi))$ and $T^q(\phi,\mu^+(\phi)) = (\phi,\nu^-(\phi))$.*

For proofs of the previous results see Le Calvez [15] and [16].

2.2 Results previously obtained by the author:

The first theorem we present asserts the existence of the vertical rotation interval and present some of its properties:

Theorem 2 *: To each mapping $T \in \mathrm{D}_{\mathrm{Dehn}}$, we can associate a closed interval $\rho_V(T) = [\rho_V^-, \rho_V^+]$, possibly degenerated to a single point, such that for every $\omega \in]\rho_V^-, \rho_V^+[$ there is a compact $\overline{T}$-invariant set $\overline{Q}_\omega \subset \mathrm{T}^2$ with $\rho_V(\overline{x}) = \omega$, for* *all $\overline{x} \in \overline{Q}_\omega$. If ω is a rational number $\frac{p}{q}$, then $\overline{Q}_\omega$ is a q-periodic orbit. In fact, in this case there are at least 2 such orbits. If they are finite, then at least one has positive index and another one has negative index..*

For a proof, see theorem 6 of the appendix of [3] and theorem 5 of [2].

Theorem 3 *: The functions $\rho_V^+, \rho_V^- : \mathrm{D}_{\mathrm{Dehn}} \to \mathbb{R}$ are continuous in the C^1 topology.*

Lemma 4 *: Given $T \in \mathrm{D}_{\mathrm{Dehn}}$, let $f : \mathbb{R} \to \mathbb{R}$ be given by: $f(\alpha) = \rho_V^+(T + (0, \alpha))$. Then f is a non-decreasing function of α.*

For proofs of the above results, see [4]. To conclude we present a lemma from [5] similar to the main result of [17], which says that we do not need to think of general perturbations in this setting. Vertical translations are enough for most applications. In the following, $T_\alpha(\phi, I) \stackrel{\text{def.}}{=} T(\phi, I) + (0, \alpha)$.

Lemma 5 *: Let $T \in \mathrm{D}_{\mathrm{Dehn}}$ be such that $\rho_V(T) = [\omega^-, \omega^+]$. Suppose that by an arbitrarily C^1-small perturbation applied to T, we can change ω^+, that is, there exists T^* arbitrarily C^1-close to T, such that $\rho_V^+(T^*) \neq \omega^+$. Then for any given $\epsilon > 0$, at least one of the following inequalities must hold:*

1) $\rho_V^+(T_\epsilon) \neq \omega^+$, or

2) $\rho_V^+(T_{-\epsilon}) \neq \omega^+$.

Moreover, given $T \in \mathrm{D}_{\mathrm{Dehn}}$ with $\rho_V(T) = [\omega^-, \omega^+]$, there exists a neighborhood $T \in \mathcal{U} \subset \mathrm{D}_{\mathrm{Dehn}}$ such that for any $T^* \in \mathcal{U}$, $\rho_V^+(T^*) = \omega^+$, if and only if, $\exists \epsilon > 0$ such that $\rho_V^+(T_\alpha) = \omega^+$, for all $\alpha \in [-\epsilon, \epsilon]$.

2.3 On the dynamics near fixed points of analytic diffeomorphisms of the plane

The dynamics near singularities of analytic vector fields of the plane is very well understood (see for instance [10]). It can be proved that, if the singularity is not a focus or a center, then the dynamics can be obtained from a finite number of sectors, glued in an adequate way. Topologically, these sectors can be classified in 4 types: elliptic, hyperbolic, expanding and attracting. Dumortier et al. studied this problem for planar diffeomorphisms near fixed points, see [11].

The situation we want to understand in this section is the following: Is there a topological picture of the dynamics near an index zero isolated fixed point of an analytic area-preserving diffeomorphism of the plane?

It turns out that the area-preservation together with the zero index hypotheses imply that the eigenvalues of the derivative of the diffeomorphism at the fixed point are both equal to 1. The area-preservation implies that in a sufficiently small neighborhood of the fixed point, the diffeomorphism is the time one mapping of a formal vector field (defined by a formal series), see [19]. So we are able to apply results from [11], which say that at least in a topological sense, the dynamics near the fixed point can be obtained as in the vector field setting, gluing a finite number of sectors. As we are supposing that area is preserved by the diffeomorphism, there can not be elliptic, expanding and attracting sectors. As the topological index of the fixed point is zero, it can not be a center and so there must be exactly 2 hyperbolic sectors and the dynamics is topologically as in figure 1.

3 Proof of the main theorem

In this section we are going to prove the main theorem. Our proof is by contradiction. We suppose that there is a mapping T satisfying the theorem hypothesis and such that the following holds:

$$\text{For every } \alpha \neq 0,\ \rho_V^+(T_\alpha) \neq p/q,\ \text{ where } T_\alpha(\phi, I) = T(\phi, I) + (0, \alpha) \tag{4}$$

Using lemma 5, we get that theorem 1 is true if and only if the above assertion is impossible. Thus we are left to show that (4) is impossible. As $\rho_V^- < p/q$ and $\rho_V^+(T_\alpha) > p/q$ for all $\alpha > 0$ (see lemma 4), we get from theorems 2 and 3 that $\overline{T}_0$ has q-periodic points with $\rho_V = p/q$.

Lemma 4 again implies that for any $\alpha < 0$, $\rho_V^+(T_\alpha) < p/q \Rightarrow$ the q-periodic points of $\overline{T}_0$ with $\rho_V = p/q$ are degenerate. Moreover, in the following we are going to prove that they are finite. As T is an analytic mapping, the same argument used in the proof of theorem 3 of [3] implies that if the q-periodic points of $\overline{T}$ with $\rho_V = p/q$ are infinite, then there exists a simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that for all $x \in \gamma$, $T^q(x) = x + (0, p)$. Clearly γ belongs to $K(s, q)$, for a certain $s \in \mathbb{Z}$. And this curve can not be homotopically non-trivial, because $\rho_V^-(T) < p/q$. To see this, note that if γ is homotopically non-trivial, and $T^q(\gamma) = \gamma + (0, p)$, as T induces a mapping on the torus homotopic to the Dehn twist, then the following is true:

$$T^q(\gamma + (0, m)) = \gamma + (0, m) + (0, p), \text{ for all } m \in \mathbb{Z}$$

As $\rho_V^-(T) = \delta < p/q$, there exists a point $z \in S^1 \times \mathbb{R}$ such that $\rho_V(z) < p/q$. Now choose $m \in \mathbb{Z}$ such that z is above $\gamma + (0, m) \Rightarrow T^{n.q}(z)$ is above $\gamma + (0, m) + (0, n.p)$ for all $n > 0$, which contradicts the fact that $\rho_V(z) < p/q$. So γ is homotopically trivial and thus crosses some vertical twice, at say, x and x' $(p_1(x) = p_1(x') = \phi)$. Suppose that $p_2(x) < p_2(x')$. The definition of μ^- and lemma 3, which says that $T^q(\phi, \mu^-(\phi)) = (\phi, \nu^+(\phi))$, imply that $\mu^-(\phi) < p_2(x)$. Thus we get that $\nu^+(\phi) - \mu^-(\phi) > p + (p_2(x') - p_2(x))$, something that contradicts the fact that $\rho_V^+(T_\alpha) < p/q$ for all $\alpha < 0$, because $C(s, q)$ and $T^q(C(s, q))$ are upper semi-continuous functions of T in the Hausdorff topology (see [17]). So the q-periodic points of $\overline{T}$ with $\rho_V = p/q$ are finite.

Let us denote them

$$\{\overline{p}_0^1, \overline{p}_1^1, ..., \overline{p}_{q-1}^1\}, \ \{\overline{p}_0^2, \overline{p}_1^2, ..., \overline{p}_{q-1}^2\}, ..., \{\overline{p}_0^N, \overline{p}_1^N, ..., \overline{p}_{q-1}^N\}, \tag{5}$$

where $\overline{p}_j^i = (\overline{\phi}_j^i, \overline{I}_j^i)$ for $i \in \{1, 2, ..., N\}$ and $j \in \{0, 1, ..., q-1\}$.

Define $F_\alpha(\phi, I) \stackrel{\text{def.}}{=} (T_\alpha)^q(\phi, I) - (0, p)$. Clearly F_α induces a torus diffeomorphism $\overline{F}_\alpha$ with $\rho_V^-(F_\alpha) < 0 < \rho_V^+(F_\alpha)$, for all sufficiently small $\alpha > 0$ and all the points in (5) are fixed points for $\overline{F}_0$ with zero vertical rotation number.

Now for each point of (5) let us choose open neighborhoods $\overline{W}_k$, $k = 1, 2, ..., N.q$, such that:

1. $closure\left(\overline{F}_0(\overline{W}_k) \cup \overline{W}_k \cup \overline{F}_0^{-1}(\overline{W}_k)\right) \cap closure(\overline{W}_l) = \emptyset$, for all $k \neq l$, $k, l \in \{1, 2, ..., N.q\}$.

2. as each $\overline{p}_j^i$ is fixed for $\overline{F}_0$ and has zero topological index, the dynamics inside each $\overline{W}_k$ is as in figure 1. So $\overline{W}_k \supset \overline{E}n_{k,0} \cup \overline{E}x_{k,0}$, where $\overline{E}n_{k,0}$ and $\overline{E}x_{k,0}$ are two open connected sets, free under $\overline{F}_0$ (disjoint from their images), such that every time the orbit of a point enters in $\overline{W}_k$, it must enter through $\overline{E}n_{k,0}$ and leave through $\overline{E}x_{k,0}$. Moreover, if we choose $\overline{W}_k$ properly, we can suppose that $closure(\overline{E}n_{k,0}) \cap closure(\overline{E}x_{k,0}) = \emptyset$ and that there exists a point $\overline{z}_k \in B_r(\overline{z}_k) \subset \overline{E}n_{k,0}$, where $r > 0$ is the same for all $k \in \{1, 2, ..., N.q\}$, such $\overline{F}_0(\overline{z}_k) \in B_s(\overline{F}_0(\overline{z}_k)) \subset \overline{E}x_{k,0}$, also for some fixed $s > 0$, see figure 2.

If we choose sufficiently small neighborhoods $\overline{W}_k$, then there exists $\alpha_1 > 0$, such that for $0 \leq \alpha \leq \alpha_1$, $\overline{F}_\alpha$ has periodic orbits with negative vertical rotation number, disjoint from $\bigcup_{k=1}^{N.q} \overline{W}_k$. This follows from $\rho_V^-(F_0) < 0 = \rho_V^+(F_0)$.

Let $\overline{B}_k \subset \overline{W}_k$ be a open neighborhood of the $\overline{F}_0$-fixed point which is inside $\overline{W}_k$ that satisfies:

$$\overline{W}_k \supset closure(\overline{F}_0^{-2}(\overline{B}_k) \cup \overline{F}_0^{-1}(\overline{B}_k) \cup \overline{B}_k \cup \overline{F}_0(\overline{B}_k) \cup \overline{F}_0^2(\overline{B}_k)) \tag{6}$$

The above condition implies that

$$closure(\overline{B}_k) \cap closure(\overline{F}_0(\overline{E}n_{k,0}) \cup \overline{E}n_{k,0} \cup \overline{E}x_{k,0}) = \emptyset. \tag{7}$$

Following figure 2, let us denote the boundary of $\overline{W}_k$ as follows: $\partial\overline{W}_k = \partial l_k \cup \partial b_k \cup \partial r_k \cup \partial t_k$. Let $\overline{V}_{b,k}, \overline{V}_{t,k} \subset \overline{W}_k$ be 2 small open neighborhoods, respectively of ∂b_k and ∂t_k such that the following conditions hold ($\eta > 0$ is a sufficiently small number and $B_\eta(\overline{V}_{b,k} \cup \overline{V}_{t,k})$ is the η-neighborhood of $\overline{V}_{b,k} \cup \overline{V}_{t,k}$):

1) $\{\overline{z}_k, \overline{F}_0(\overline{z}_k)\} \cap closure(\overline{V}_{b,k} \cup \overline{V}_{t,k}) = \emptyset$

2) there exists $N > 0$ such that for all $\overline{x} \in B_\eta(\overline{V}_{b,k} \cup \overline{V}_{t,k})$,
for some $n = n(\overline{x}) < N$, $\overline{F}_0^n(\overline{x}) \notin \overline{W}_k$ and
for $0 \leq i \leq n$, $\overline{F}_0^i(\overline{x}) \notin closure(\overline{B}_k)$ (8)

3) there exists $M > 0$ such that for all $\overline{x} \in B_\eta(\overline{V}_{b,k} \cup \overline{V}_{t,k})$,
for some $m = m(\overline{x}) < M$, $\overline{F}_0^{-m}(\overline{x}) \notin \overline{W}_k$ and
for $0 \leq i \leq m$, $\overline{F}_0^{-i}(\overline{x}) \notin closure(\overline{B}_k)$

Now choose $\alpha_2 > 0$ such that for $0 \leq \alpha \leq \alpha_2$,

1. all fixed points for $\overline{F}_\alpha$ with zero vertical rotation number are contained in $\overset{N.q}{\underset{k=1}{\cup}} \overline{B}_k$ and there is an isotopy supported in $\overset{N.q}{\underset{k=1}{\cup}} \overline{B}_k$ beetwen $\overline{F}_\alpha$ and a homeomorphism without fixed points of zero vertical rotation number

2. the three conditions in (8) hold when, instead of $\overline{F}_0$, we use $\overline{F}_\alpha$.

3. $\overline{F}_\alpha^{\pm 1}(\partial b_k) \cap \overline{W}_k \subset \overline{V}_{b,k}$ and $\overline{F}_\alpha^{\pm 1}(\partial t_k) \cap \overline{W}_k \subset \overline{V}_{t,k}$

4. condition (6) holds when instead of $\overline{F}_0$ we use $\overline{F}_\alpha$. As before, this implies that $closure(\overline{B}_k) \cap closure(\overline{F}_\alpha(\overline{E}n_{k,\alpha}) \cup \overline{E}n_{k,\alpha} \cup \overline{E}x_{k,\alpha}) = \emptyset$, where $\overline{E}n_{k,\alpha}$ and $\overline{E}x_{k,\alpha}$ are the analogous of $\overline{E}n_{k,0}$ and $\overline{E}x_{k,0}$ for $\alpha > 0$

5. $\overline{z}_k \in \overline{E}n_{k,\alpha}$ and $\overline{F}_\alpha(\overline{z}_k) \in \overline{E}x_{k,\alpha}$

6. if we denote the connected component of $\overline{E}n_{k,\alpha} \backslash closure(\overline{V}_{b,k} \cup \overline{V}_{t,k})$ whose boundary intersects $\partial \overline{W}_k$ by $\overline{E}n'_{k,\alpha}$ and, in the same way, the connected component of $\overline{E}x_{k,\alpha} \backslash closure(\overline{V}_{b,k} \cup \overline{V}_{t,k})$ whose boundary intersects $\partial \overline{W}_k$ is denoted by $\overline{E}x'_{k,\alpha}$, then $\overline{E}n_{k,\alpha} \subset \overline{E}n'_{k,\alpha} \cup B_\eta(\overline{V}_{b,k} \cup \overline{V}_{t,k})$ and $\overline{E}x_{k,\alpha} \subset \overline{E}x'_{k,\alpha} \cup B_\eta(\overline{V}_{b,k} \cup \overline{V}_{t,k})$. Moreover, $closure(\overline{E}n'_{k,\alpha}) \cap closure(\overline{E}x'_{k,\alpha}) = \emptyset$, see figure 3

7. $closure\left(\overline{F}_\alpha(\overline{W}_k) \cup \overline{F}_\alpha^{-1}(\overline{W}_k)\right) \cap closure(\overline{W}_l) = \emptyset$, for all $k \neq l$

8. $\overline{z}_k \in \overline{E}n'_{k,\alpha}$ and $\overline{F}_\alpha(\overline{z}_k) \in \overline{E}x'_{k,\alpha}$

A simple continuity argument implies that all the above conditions are satisfied for sufficiently small $\alpha > 0$.

Now fix $0 < \alpha < \min\{\alpha_1, \alpha_2\}$. Our hypothesis imply that $\rho_V^-(F_\alpha) < 0 < \rho_V^+(F_\alpha)$. From the choice of $\alpha_1 > 0$, there exists $\overline{x}_- \in \mathrm{T}^2$ and $n > 0$ such that $(\overline{F}_\alpha)^n(\overline{x}_-) = \overline{x}_-$, $\rho_V(\overline{x}_-) < 0$ and the orbit of $\overline{x}_-$ is disjoint from $\overset{N.q}{\underset{k=1}{\cup}} \overline{W}_k$. Let $m > 0$ be an integer such that $0 < 1/m < \rho_V^+(F_\alpha)$ and let $\overline{x}_+ \in \mathrm{T}^2$ (the existence of $\overline{x}_+$ follows from theorem 2) be such that $(\overline{F}_\alpha)^m(\overline{x}_+) = \overline{x}_+$ and $\rho_V(\overline{x}_+) = 1/m$. If the orbit of $\overline{x}_+$ is disjoint from $\overset{N.q}{\underset{k=1}{\cup}} \overline{B}_k$, then we are done.

So, suppose that $Orbit(\overline{x}_+) = \{\overline{x}_+, \overline{F}_\alpha(\overline{x}_+), (\overline{F}_\alpha)^2(\overline{x}_+), ..., (\overline{F}_\alpha)^{m-1}(\overline{x}_+)\}$ intersects $\overset{N.q}{\underset{k=1}{\cup}} \overline{B}_k$. From the fact that the three conditions in (8) hold for $\overline{F}_\alpha$, without loss of generality we can suppose that $\overline{x}_+ \in \overline{Ex}'_{1,\alpha} \subset \overline{W}_1$. As $(\overline{F}_\alpha)^m(\overline{x}_+) = \overline{x}_+$, there exists $m' < m$ such that $(\overline{F}_\alpha)^{m'}(\overline{x}_+) \in \overline{En}'_{1,\alpha} \subset \overline{W}_1$. Now let $\{x_+, W_1, W_2, ..., W_{N.q}\} \subset S^1 \times [0,1)$ be lifts of $\overline{x}_+$ and $\overline{W}_1, \overline{W}_2, ..., \overline{W}_{N.q}$ to the cylinder. From the choice of $\overline{x}_+$ we get that $(F_\alpha)^m(x_+) = x_+ + (0,1)$. If we consider all the intermediate pieces of the orbit of x_+, then the following observation is true:

Observation 1 : There exists $r_f = (F_\alpha)^i(x_+)$ and $r_l = (F_\alpha)^j(x_+)$, for some $0 \leq i < j \leq m'$ such that $r_f \in Ex'_{k1,\alpha} + (0,s) \subset W_{k1} + (0,s)$ and $r_l \in En'_{k1,\alpha} + (0, s+l) \subset W_{k1} + (0, s+l)$, for some $1 \leq k1 \leq N.q$ and $s, l \in \mathbb{Z}$, with $l > 0$. And when for 2 points $z = (F_\alpha)^d(x_+)$, $y = (F_\alpha)^e(x_+)$, $i \leq d < e < j$, $pro(z)$ and $pro(y)$ belong to the same $\overline{B}_t$, then $z \in B_t + (0,m)$ and $y \in B_t + (0, m+a)$, for $m, a \in \mathbb{Z}$ with $a \leq 0$. Moreover, $\{\overline{F}_\alpha^i(\overline{x}_+), ..., (\overline{F}_\alpha)^j(\overline{x}_+)\} \cap \overline{B}_{k1} = \emptyset$.

In order to find points r_f and r_l as above, we just have to start with $x_+ \in Ex'_{1,\alpha} \subset W_1$ and $(F_\alpha)^{m'}(x_+) \in En'_{1,\alpha} + (0,1) \subset W_1 + (0,1)$ and check if the conditions in the observation are satisfied for the intermediate points of the orbit. If they are not, then we pick the convenient subset of the orbit from x_+ to $(F_\alpha)^{m'}(x_+)$ and check again. After a finite number of attempts, r_f and r_l are found.

Let us denote the orbit segment from $\overline{r}_f$ to $\overline{r}_l$ by

$$\overline{r}_f = \overline{x}_0, \overline{x}_1, ..., \overline{x}_{j-i} = \overline{r}_l, \text{ where } \overline{x}_n = (\overline{F}_\alpha)^n(\overline{r}_f) \tag{9}$$

and suppose that it intersects $\bigcup_{k=1}^{N.q} \overline{B}_k$ in $\overline{B}_{k2}, \overline{B}_{k3}, ..., \overline{B}_{kL}$. From the fact that expression (8) is true also for $\overline{F}_\alpha$, in each $\overline{W}_{km}$, $2 \leq m \leq L$, there are points $\overline{x}_{km\_first} \in \overline{E}n'_{km,\alpha} \subset \overline{W}_{km}$ and $\overline{x}_{km\_last} \in \overline{E}x'_{km,\alpha} \subset \overline{W}_{km}$, defined as follows:

- $\overline{x}_{km\_first}$ is the first point of the sequence (9) such that $(\overline{F}_\alpha)^n(\overline{x}_{km\_first})$ belongs to $\overline{W}_{km}$ for $n = 0, 1, ..., n_{\max}$, $(\overline{F}_\alpha)^{n_{crit}}(\overline{x}_{km\_first}) \in \overline{B}_{km}$ for some $0 < n_{crit} < n_{\max}$ and of course $(\overline{F}_\alpha)^{n_{\max}}(\overline{x}_{km\_first}) \in \overline{E}x'_{km,\alpha}$
- $\overline{x}_{km\_last}$ is the last point of the sequence (9) such that $(\overline{F}_\alpha)^{-l}(\overline{x}_{km\_last})$ belongs to $\overline{W}_{km}$ for $l = 0, 1, ..., l_{\max}$, $(\overline{F}_\alpha)^{-l_{crit}}(\overline{x}_{km\_last}) \in \overline{B}_{km}$ for some $0 < l_{crit} < l_{\max}$ and of course $(\overline{F}_\alpha)^{-l_{\max}}(\overline{x}_{km\_last}) \in \overline{E}n'_{km,\alpha}$

To understand the above definitions, one just has to remember that every time the orbit of a point falls into some $\overline{B}_k$, then it must have entered $\overline{W}_k$ through $\overline{E}n'_{k,\alpha}$ and it will leave $\overline{W}_k$ through $\overline{E}x'_{k,\alpha}$.

The next step is to perform 2 deformations in each $\overline{W}_{km}$ ($m \geq 1$), one supported in $\overline{E}n'_{km,\alpha}$ and the other supported in $\overline{E}x'_{km,\alpha}$. For $2 \leq m \leq L$, let $\mu_{f\_km} \subset \overline{E}n'_{km,\alpha}$ be a simple arc whose endpoints are $\overline{x}_{km\_first}$ and $\overline{z}_{km}$, such that, apart from $\overline{x}_{km\_first}$, it does not contain any other point of sequence (9). Let $V_{f\_km} \subset \overline{E}n'_{km,\alpha}$ be an open neighborhood of μ_{f_km} which also does not contain any other point of sequence (9). In the same way, let $\mu_{l\_km} \subset \overline{E}x'_{km,\alpha}$ be a simple arc whose endpoints are $\overline{x}_{km\_last}$ and $\overline{F}_\alpha(\overline{z}_{km})$, such that, apart from $\overline{x}_{km\_last}$, it does not contain any other point of sequence (9). Let $V_{l\_km} \subset \overline{E}x'_{km,\alpha}$ be an open neighborhood of μ_{l_km} which also does not contain any other point of sequence (9). As $\overline{E}n'_{km,\alpha}$ and $\overline{E}x'_{km,\alpha}$ are disjoint and free under $\overline{F}_\alpha$, the same is true for V_{f_km} and V_{l_km}. So, let us define the following homeomorphisms $\overline{h}_{V_{f\_km}} : \mathrm{T}^2 \to \mathrm{T}^2$ and $\overline{h}_{V_{l\_km}} : \mathrm{T}^2 \to \mathrm{T}^2$:

$$\begin{array}{c} \overline{h}_{V_{f\_km}} \mid_{(V_{f\_km})^c} = id \\ \overline{h}_{V_{f\_km}}(\overline{x}_{km\_first}) = \overline{z}_{km} \\ \text{and} \\ \overline{h}_{V_{l\_km}} \mid_{(V_{l\_km})^c} = id \\ \overline{h}_{V_{l\_km}}(\overline{F}_\alpha(\overline{z}_{km})) = \overline{x}_{km\_last} \end{array} \tag{10}$$

For $m = 1$, we perform an analogous construction. Let $\mu_{f\_k1} \subset \overline{E}n'_{k1,\alpha}$ be a simple arc whose endpoints are $\overline{r}_l$ and $\overline{z}_{k1}$, which, apart from $\overline{r}_l$, avoids all points of sequence (9). Let $V_{f\_k1} \subset \overline{E}n'_{k1,\alpha}$ be an open neighborhood of μ_{f_k1} which also does not contain any other point of sequence (9). In the same way, let $\mu_{l\_k1} \subset \overline{E}x'_{k1,\alpha}$ be a simple arc whose endpoints are $\overline{r}_f$ and $\overline{F}_\alpha(\overline{z}_{k1})$, which, apart from $\overline{r}_f$, avoids all points of sequence (9). Let $V_{l\_k1} \subset \overline{E}x'_{k1,\alpha}$ be an open neighborhood of μ_{l_k1} which also does not contain any other point of sequence (9). Then we define:

$$\begin{array}{c} \overline{h}_{V_{f\_k1}} \mid_{(V_{f\_k1})^c} = id \\ \overline{h}_{V_{f\_k1}}(\overline{r}_l) = \overline{z}_{k1} \\ \text{and} \\ \overline{h}_{V_{l\_k1}} \mid_{(V_{l\_k1})^c} = id \\ \overline{h}_{V_{l\_k1}}(\overline{F}_\alpha(\overline{z}_{k1})) = \overline{r}_f \end{array} \tag{11}$$

Denote by $\overline{h} : \mathrm{T}^2 \to \mathrm{T}^2$ the homeomorphism which agrees with all the $\overline{h}_{V_{f\_km}}$ and $\overline{h}_{V_{l\_km}}$, $1 \leq m \leq L$. From the choice of $\alpha_2 > 0$, we know that there is an isotopy supported in $\bigcup\limits_{k=1}^{N.q} \overline{B}_k$ between $\overline{F}_\alpha$ and a homeomorphism without fixed points of zero vertical rotation number. Let us denote the final mapping of this isotopy by $\overline{F}^*_\alpha$. By construction $\overline{F}^*_\alpha$ coincides with $\overline{F}_\alpha$ outside $\bigcup\limits_{k=1}^{N.q} \overline{B}_k$, so we get that the mapping $\overline{G} \overset{def}{=} \overline{h} \circ \overline{F}^*_\alpha$ has a periodic point with positive vertical rotation number, that comes from the orbit segment between $\overline{r}_f$ and $\overline{r}_f$, see Observation 1. The deformation $\overline{h}$ is supported in a disjoint union of free sets for $\overline{F}^*_\alpha$, so as $\overline{F}^*_\alpha$ has no fixed points of zero vertical rotation number, the same is true for $\overline{G}$.

As $\overline{G}$ coincides with $\overline{F}_\alpha$ outside $\bigcup\limits_{k=1}^{N.q} \overline{W}_k$, $\overline{G}$ also has periodic points with negative vertical rotation number. But this contradicts a generalization of some results of [12] to this context. To be more precise, it contradicts the next theorem due to H.E.Doeff (see [9], theorem 5.3).

Theorem 4 *: If $\overline{G} : \mathrm{T}^2 \to \mathrm{T}^2$ is a homeomorphism homotopic to $(\phi, I) \to (\phi+n.I \mod 1, I \mod 1)$, for a certain $n \in \mathbb{N}^*$ and there are points $\overline{x}_+, \overline{x}_- \in \mathrm{T}^2$ such that $\rho_V(\overline{x}_-) < 0 < \rho_V(\overline{x}_+)$, then $\overline{G}$ has fixed points of zero vertical rotation number.*

So it is not possible that $\rho^+_V(T_\alpha) \neq \frac{p}{q}$ for all $\alpha \neq 0$ and the proof is complete.□

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Figure captions.

Figure 1. Diagram showing the dynamics near a fixed point of zero index

Figure 2. Diagram showing the sets $\overline{W}_k \supset \overline{E}n_{k,0} \cup \overline{E}x_{k,0} \cup \overline{B}_k$

Figure 3. Diagram showing the sets $\overline{E}n'_{k,\alpha} \cup \overline{E}x'_{k,\alpha}$

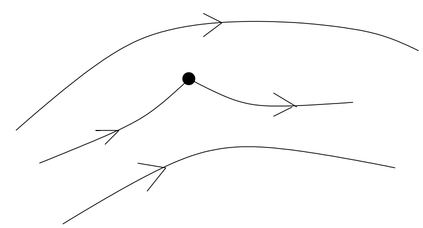

Figure 1

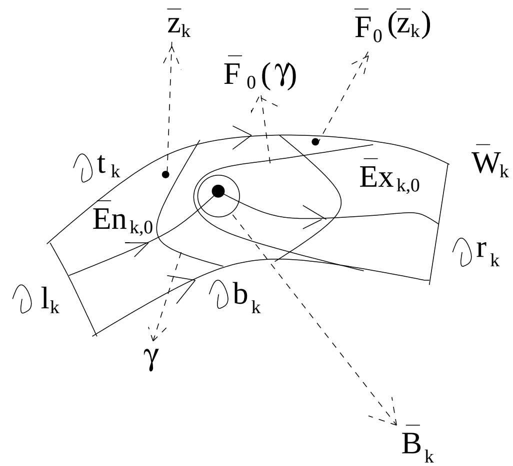
$\bar{z}_k$
$\bar{F}_0(\bar{z}_k)$
$\bar{F}_0(\gamma)$
t_k
$\bar{W}_k$
$\bar{E}x_{k,0}$
$\bar{E}n_{k,0}$
r_k
l_k
b_k
γ
$\bar{B}_k$

Figure 2

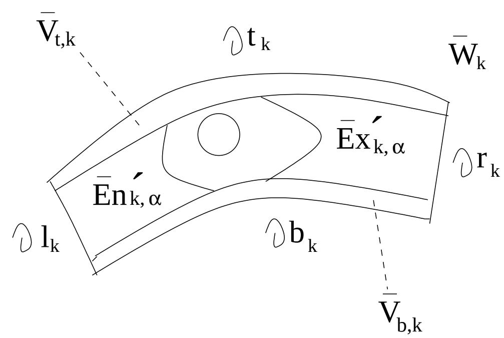

Figure 3