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On the stability of some periodic orbits of a new type for twist maps

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Abstract

We study a two-parameter family of twist maps defined on the torus. This family essentially determines the dynamics near saddle-centre loops of four-dimensional real analytic Hamiltonian systems. A saddle-centre loop is an orbit homoclinic to a saddle-centre equilibrium (related to pairs of pure real, $\pm\nu$, and pure imaginary, $\pm\omega\mathrm{i}$, eigenvalues). We prove that given any period n we can find an open set of parameter values such that this family has an attracting n-periodic orbit of a special type. This has interesting consequences on the original Hamiltonian dynamics.

Mathematics Subject Classification: 37E40, 37E45, 37C25

1. Introduction

This paper is part of a series on the dynamics of a certain two-parameter family of twist maps (see [10–13, 1, 3, 4]). This family of maps, which we call saddle-centre loop maps, appears in the study of four-dimensional real analytic Hamiltonian systems in the following way. Suppose the system has an equilibrium of saddle-centre type (namely, it has pure real, $\pm\nu \neq 0$, and pure imaginary, $\pm\omega\mathrm{i} \neq 0$, eigenvalues) and additionally there exists an orbit Γ homoclinic to this equilibrium. The union of Γ and the equilibrium is called a saddle-centre loop. Under a mild hypothesis on the topology of the energy level set of the saddle-centre loop (see [11]), it is possible to define a Poincaré map to a transverse section to the saddle-centre loop. This construction, due to Lerman [17] and Mielke *et al* [18], is analogous to the more familiar

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one of a Poincaré map to a periodic orbit. The leading order term of the Poincaré map to the saddle-centre loop, that we call saddle-centre loop map, restricted to the energy level set of the saddle-centre loop is given by the following area preserving map:

$$F(z_1, z_2) = \begin{pmatrix} \alpha & 0 \\ 0 & \dfrac{1}{\alpha} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

where $z \in \mathbb{R}^2$ has sufficiently small norm, $\theta \overset{\mathrm{def}}{=} -\gamma \log(\|z\|^2/2)$, $\gamma \overset{\mathrm{def}}{=} \omega/\nu > 0$ and $\alpha \geqslant 1$ is a parameter obtained from the flow of the system linearized at the orbit Γ (it is analogous to a Floquet coefficient of a periodic orbit). For examples of computations of α, see [13, 5]. The origin $z = (0, 0)$ represents the intersection of Γ with the transverse section and it is, by definition, a fixed point of the saddle-centre loop map.

In [12, 4] it has been proved that many properties of the saddle-centre loop map extend to the real Poincaré map and thus to the Hamiltonian flow. For instance, in [12] it is shown that for certain parameter values (α, γ) the stability (instability) of the origin under iterations of the saddle-centre loop map implies a sort of stability (instability) of the saddle-centre loop under the action of the Hamiltonian flow. In [4] it is shown that if the origin is unstable under iterations of the saddle-centre loop map (which necessarily happens if $\gamma(\alpha - \alpha^{-1}) > 1$, see [12]), then the set of orbits that escape from it may contain some special points that in some sense 'attract' most of the escaping orbits. The reason for using the strange word 'attract' in this conservative setting will become clearer below. In order to motivate and explain the problem studied in this paper, we need first to present a result from [4].

The logarithmic singularity of the saddle-centre loop map can be removed with the following choice of polar coordinates that blow up the origin:

$$z_1 = \sqrt{2}\mathrm{e}^{I/2\gamma} \cos(I - \phi),$$
$$z_2 = \sqrt{2}\mathrm{e}^{I/2\gamma} \sin(I - \phi).$$

In these coordinates the saddle-centre loop map is written as

$$F : \begin{cases} \phi' = \mu(\phi) + I', \\ I' = \gamma \log J(\phi) + I, \end{cases} \tag{1}$$

where

$$J(\phi) = \alpha^2 \cos^2(\phi) + \alpha^{-2} \sin^2(\phi),$$
$$\mu(\phi) = \arctan\left(\frac{\tan(\phi)}{\alpha^2}\right), \qquad \text{with } \mu(0) = 0.$$

Using these coordinates we can easily identify the punctured plane with a cylinder $(\phi, I) \in S^1 \times \mathbb{R}$ where $S^1 = \mathbb{R}/\pi\mathbb{Z}$, or $\phi = \phi \bmod \pi$. We denote as $\hat{F}$ the map induced by F (see (1)) on this cylinder. The area preserving property of the saddle-centre loop map implies that $\hat{F}$ preserves the following measure on the cylinder:

$$\mu(A) = \int_A \mathrm{e}^{I/\gamma}\, \mathrm{d}\phi \wedge \mathrm{d}I. \tag{2}$$

Map $\hat{F}$ is invariant under vertical translations by π, namely $\hat{F}(\phi, I + \pi) = \hat{F}(\phi, I) + \pi$. Thus, $\hat{F}$ induces a map on the 2-torus, that we denote as $\bar{F}$, just by taking $I = I \bmod \pi$. Notice that the measure μ (2) is not invariant under I-translations by π which implies that it does not define a measure on the quotient 2-torus $\phi \bmod \pi$, $I \bmod \pi$. So, the original measure preserving property of $\hat{F}$ is lost in the quotient construction of $\bar{F}$. In this context a natural side question, that is not further addressed in this paper, is: does $\bar{F}$ preserve any measure on the 2-torus which is absolutely continuous with respect to the Lebesgue measure? For the rather special case of $\alpha = 1$, the answer is trivially yes, the invariant measure is simply $\mathrm{d}\phi \wedge \mathrm{d}I$. If map $\hat{F}$ has a homotopically nontrivial invariant curve, then the answer is again yes. In this case an annulus bounded by this invariant curve and its translation by π define a fundamental domain for the 2-torus which is invariant under $\hat{F}$. The measure (2) restricted to this fundamental domain is invariant under $\bar{F}$. If $\hat{F}$ does not admit a rotational invariant curve, then it is proved below that for certain values of (α, γ), map $\bar{F}$ has attractive periodic points which implies the nonexistence of an invariant measure absolutely continuous with respect to the Lebesgue measure. For other values of (α, γ), the existence of such an invariant measure is an open, and probably hard, question, although numerical studies indicate that for every pair (α, γ) such that $\bar{F}$ does not have rotational invariant circles there are 'attractors', which may be periodic sinks, Hénon-like attractors (see [3]), or something else.

Throughout this paper we assume the hypothesis that $\hat{F}$ does not have rotational invariant circles. In this case $\bar{F}$ can simultaneously have certain properties of a conservative and a dissipative system. For instance, all periodic points of $\bar{F}$ that are also periodic points of $\hat{F}$ have a conservative character, i.e. $\bar{F}$ can have elliptic points surrounded by invariant curves and also hyperbolic saddles with eigenvalues λ, λ^{-1}. Periodic points of $\bar{F}$ which represent vertical I-translations for $\hat{F}$ have necessarily a dissipative or expansive character. In [1–3], the reader can find examples of more complicated invariant sets of $\bar{F}$ with dissipative dynamics like, for instance, Hénon attractors. The consequences of the existence of attractors for $\bar{F}$ on the dynamics of the original Hamiltonian flow is not completely understood, yet. In [4] we prove several results in this direction. The main difficulty we find is that we still do not have a good description for the dynamics of $\hat{F}$ or $\bar{F}$ for most values of the parameters (α, γ). In order to better explain this point, let us get back to the analogy between the dynamics near a saddle-centre loop and the dynamics near a periodic orbit. For the saddle-centre loop the role played by the linear part of the Poincaré map to a periodic orbit is replaced by $\hat{F}$. The dynamics of a linear map is very simple for any value of its eigenvalues while the dynamics of $\hat{F}$, and its dependence on (α, γ), is very complicated. For instance, when the dynamics of the linear map is unstable, the set of orbits that escape from the periodic orbit has a very simple description given by the unstable manifold theorem. In the analogous case where the origin of the saddle-centre loop map is unstable, the unstable set of the saddle-centre loop is very complicated and still not fully understood. An interesting partial result proved in [4] says that if the origin of the saddle-centre loop map is unstable and $\bar{F}$ has a periodic topological sink, then there is a set of points of positive measure that escapes under the action of the original Hamiltonian flow from any sufficiently small neighbourhood of the saddle-centre loop following (or clustering around) a finite number of escaping orbits that correspond to the periodic sink of $\bar{F}$. In [4] we present some very particular pairs of values of (α, γ) where we can find explicitly attractive fixed points of $\bar{F}$. Our goal in this paper is to show that $\bar{F}$ has attractive periodic orbits of all periods and all 'vertical rotation numbers' (see definition below) for open sets of values of (α, γ). This is an important step in the understanding of the local dynamics near a saddle-centre loop.

This paper is organized as follows. In the next section we present the statements of our main results. In section 3 and in appendix A we present the proofs of our results.

2. Main results

First we present some definitions. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$, defined by equation (1) with $(\phi, I) \in \mathbb{R}^2$, be a lift of $\hat{F}$ to the plane. We recall that $\hat{F} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ and $\bar{F} : \mathrm{T}^2 \to \mathrm{T}^2$ are the quotient maps induced by F, where $S^1 \equiv \mathbb{R}/(\pi\mathbb{Z})$ and $\mathrm{T}^2 \equiv \mathbb{R}^2/(\pi\mathbb{Z})^2$:

$$\bar{F} : \begin{cases} \phi' = \mu(\phi) + I' \bmod \pi \\ I' = \gamma \log J(\phi) + I \bmod \pi \end{cases} \quad \text{and} \quad \hat{F} : \begin{cases} \phi' = \mu(\phi) + I' \bmod \pi, \\ I' = \gamma \log J(\phi) + I. \end{cases}$$

It is clear that $\hat{F}$ is homotopic to the identity on the cylinder and $\bar{F}$ is homotopic to the following linear map, denoted as LM, on the torus:

$$LM(\phi, I) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ I \end{pmatrix} (\bmod \pi)^2. \tag{3}$$

Let $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ and $p : \mathbb{R}^2 \to T^2$ be the associated covering maps ($\pi(\phi, I) = (\phi \bmod \pi, I)$ and $p(\phi, I) = (\phi \bmod \pi, I \bmod \pi)$). We define $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ as the standard projections, respectively, in the ϕ and I directions. We also use p_1 and p_2 for the standard projections on the cylinder.

We say that a point $x \in \mathrm{T}^2$ is n-periodic for $\bar{F}$ with vertical rotation number $\rho_{\mathrm{V}} = m/n$ if

$$\bar{F}^n(x) = x \qquad \text{and} \qquad \frac{p_2 \circ F^n(\tilde{x}) - p_2(\tilde{x})}{n\pi} = \frac{m}{n}, \qquad \text{for any } \tilde{x} \in p^{-1}(x).$$

A simple fact about these points is the following proposition.

Proposition 1. *Let x be a periodic point for $\bar{F}$ with nonzero vertical rotation number. If $\rho_{\mathrm{V}} > 0$ then the product of the eigenvalues associated with x is <1 (the periodic point is dissipative), and if $\rho_{\mathrm{V}} < 0$ then this product is >1 (the periodic point is expansive).*

Proof. To prove this we just have to observe that

$$\begin{aligned} \det[D\bar{F}^n(x)] &= \prod_{i=0}^{n-1} \det[D\bar{F}(\bar{F}^i(x))] \\ &= \det[D\bar{F}(\bar{F}^{n-1}(x))] \det[D\bar{F}(\bar{F}^{n-2}(x))] \cdots \det[D\bar{F}(x)] \\ &= \frac{1}{J(\phi_{n-1})} \cdots \frac{1}{J(\phi)} = \frac{1}{J(\phi_{n-1}) \cdots J(\phi)} = \frac{1}{\mathrm{e}^{m\pi/\gamma}}, \end{aligned} \tag{4}$$

where $x = (\phi, I)$ and $\bar{F}^i(x) = (\phi_i, I_i)$. This happens because as x is n-periodic with $\rho_{\mathrm{V}} = m/n$, we have

$$\begin{aligned} m\pi + I &= \gamma \log J(\phi_{n-1}) + I_{n-1} = \sum_{i=0}^{n-1} \gamma \log J(\phi_i) + I \\ &= \gamma \log[J(\phi_{n-1}) \cdots J(\phi)] + I \Rightarrow J(\phi_{n-1}) \cdots J(\phi) = \mathrm{e}^{m\pi/\gamma}. \end{aligned}$$ □

So if $m > (<)\ 0$, then x is dissipative (expansive). As we are looking for sinks, we can restrict ourselves to points with $\rho_{\mathrm{V}} > 0$ (the case of sources is analogous).

Before we start the discussion about the stability of periodic orbits of $\bar{F}$ with $\rho_{\mathrm{V}} > 0$, it is necessary to present two results proved in [2] that show that these orbits exist. We recall that $\hat{F}$ has a rotational invariant curve (RIC) if there is a homotopically nontrivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $\hat{F}(\gamma) = \gamma$.

Theorem 1. *Given $k \in \mathbb{N}^*$, $\exists N > 0$, such that $\bar{F}$ has at least 2-periodic orbits with $\rho_{\mathrm{V}} = k/N$, if and only if $\hat{F}$ does not have RICs.*

Theorem 2. *If $\bar{F}$ has a periodic orbit with $\rho_{\mathrm{V}} = k/N > 0$, then for every rational k'/N' such that $0 < k'/N' < k/N$, $\bar{F}$ has at least two periodic orbits with vertical rotation number $\rho'_{\mathrm{V}} = k'/N'$.*

As we mentioned in the introduction, $\hat{F}$ does not have RICs if, for instance, $\gamma(\alpha - \alpha^{-1}) > 1$ [12]. Moreover, a simple computation shows the following proposition.

Proposition 2. *For any given positive integer θ, $\bar{F}$ has a hyperbolic fixed point with $\rho_{\mathrm{V}} = \theta$ and another one with $\rho_{\mathrm{V}} = -\theta$, provided $\alpha > \mathrm{e}^{\theta\pi/\gamma}$.*

So as a corollary of theorem 2 and this proposition, we get that for any given rational $m/n > 0$, $\bar{F}$ has n-periodic orbits with $\rho_{\mathrm{V}} = m/n$ for all sufficiently large α, depending also on γ.

Now we come to the main results proved in this paper. The following theorem is crucial to show that $\bar{F}$ has topological sinks.

Theorem 3. *Given $k \in \mathbb{Z}^*$ and $n \in \mathbb{N}^*$, the n-periodic points of $\bar{F}$ with vertical rotation number $\rho_{\mathrm{V}} = k/n$ are finite.*

As a simple corollary of the above theorem, we get the following one.

Corollary 1. *For all $n \in \mathbb{N}^*$, $\#\{x \in \mathrm{T}^2 : \bar{F}^n(x) = x$ and $\rho_{\mathrm{V}}(x) \neq 0\} < \infty$.*

Proof. We just need to show that for a fixed $n > 0$, there exists $K > 0$ such that no point in the torus can have $|\rho_{\mathrm{V}}| > K/n$. In this case, $\#\{x \in \mathrm{T}^2 : \bar{F}^n(x) = x$ and $\rho_{\mathrm{V}}(x) \neq 0\} \leqslant \sum_{\substack{k=-K \\ (k\neq 0)}}^{K} \#A_n^k < \infty$, where A_n^k is the set of n-periodic points of $\bar{F}$ with $\rho_{\mathrm{V}} = k/n$. The existence of such K is a consequence of the inequality $|I' - I| \leqslant 2\gamma \log\alpha$ that we obtain from equation (1). $\square$

Theorem 3 implies that the n-periodic points of $\bar{F}$ that have $\rho_{\mathrm{V}} \neq 0$ are isolated. So, we can use a simple topological index argument to prove the following lemma.

Lemma 1. *The index of any periodic point of $\bar{F}$ with nonzero vertical rotation number can only assume the following values: $-1, 0, 1$. If the periodic point has real negative eigenvalues, then the index is 1. Moreover, if the periodic point has positive vertical rotation number and real positive eigenvalues, then it is a topological sink if and only if it has index 1.*

Using this lemma we prove the following theorem.

Theorem 4. *Given a rational $m/n > 0$, for all $\alpha > \exp[(\lfloor m/n \rfloor + 1)\pi/\gamma]$, $\bar{F}$ has at least two n-periodic orbits with $\rho_{\mathrm{V}} = m/n$ such that their topological indices are -1 and 1.*

Finally, from a bifurcation argument, we get the main result of this paper.

Theorem 5. *For any given $\gamma > 0$ and $m/n > 0$, there exists an interval $I_s = I_{\mathrm{s}}(m/n, \gamma) \subset \mathbb{R}$ such that for $\alpha \in I_s$, $\bar{F}$ has an n-periodic point with $\rho_{\mathrm{V}} = m/n$, which is a topological sink.*

So we have periodic attractors for $\bar{F}$ with all rational vertical rotation numbers. We just have to choose the appropriate parameters.

The proof of theorem 3 is based on the analyticity of $\hat{F}$ and on the invariance of measure μ. The proof of theorem 4 is by contradiction. We suppose that all n-periodic points for $\bar{F}$ with $\rho_{\mathrm{V}} = m/n$ have index zero. Using that there is a finite number of these points, we construct a map C^k close to $\bar{F}$ ($k \geqslant 1$) without n-periodic points with $\rho_{\mathrm{V}} = m/n$. This leads to a contradiction. Theorem 5 is a consequence of a simple bifurcation argument.

3. Proofs

3.1. Proof of theorem 3

What we have to prove is that

$$\#\{x \in \mathrm{T}^2 : F^n(\tilde{x}) = \tilde{x} + (s,k)\pi, \text{ for some } s \in \mathbb{Z}, \text{ where } \tilde{x} \in p^{-1}(x)\} < \infty.$$

First, let

$$A_n^k = \{\tilde{x} \in [0,\pi]^2 : F^n(\tilde{x}) = \tilde{x} + (s,k)\pi, \text{ for some } s \in \mathbb{Z}\}.$$

Using that the square $[0,\pi]^2 \subset \mathbb{R}^2$ is a fundamental domain of the torus, $\mathrm{T}^2 \equiv \mathbb{R}^2/(\pi\mathbb{Z})^2$, we get that to prove the theorem it is enough to show that $\#A_n^k < \infty$.

Let $N \in \mathbb{N}^*$ be such that

$$N > \sup_{\tilde{x}\in[0,\pi]^2} \frac{|p_1 \circ F^n(\tilde{x})|}{\pi} + 1. \tag{5}$$

Defining $A_{ns}^k = \{\tilde{x} \in [0,\pi]^2 : F^n(\tilde{x}) = \tilde{x} + (s,k)\pi\}$, we obtain that

$$A_n^k = \bigcup_{s=-N}^{N} A_{ns}^k, \qquad \text{because from (5), if } |s| > N \text{ then } A_{ns}^k = \emptyset.$$

So, it is enough to show that $\#A_{ns}^k$ is finite for every $|s| \leqslant N$. For each $s \in \{-N,\ldots,-1,0,1,\ldots,N\}$ let us define the following function:

$$f(\phi, I) = F^n(\phi, I) - (s,k)\pi.$$

The map $f : \mathbb{R}^2 \to \mathbb{R}^2$ induces an analytic diffeomorphism $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ that is homotopic to the identity on the cylinder.

Let

$$P_0(\hat{f}) = \{x \in S^1 \times \mathbb{R} : f(\tilde{x}) = \tilde{x}, \text{ where } \pi(\tilde{x}) = x\}.$$

If $x \in P_0(\hat{f})$ then $\pi^{-1}(x) \cap [0,\pi]^2 \in A_{ns}^k$ (of course $\pi^{-1}(x) \cap [0,\pi]^2$ is a single point or the empty set). And if $\tilde{x} \in A_{ns}^k$, then $\pi(\tilde{x}) \in P_0(\hat{f})$. So we have the following inequality:

$$\#P_0(\hat{f}) \geqslant \#A_{ns}^k. \tag{6}$$

Analysis of $P_0(\hat{f})$. $f(\tilde{x}) = \tilde{x} \Leftrightarrow F^n(\tilde{x}) = \tilde{x} + (s,k)\pi$. As F is a twist map and $\partial_I(p_1 \circ F) = 1$ (see (1)), it is clear that $\exists\, M > 0$ such that if $\tilde{x} \in \mathbb{R} \times [M, \infty[$, then $p_1 \circ F^n(\tilde{x}) > p_1(\tilde{x}) + s\pi$ and if $\tilde{x} \in \mathbb{R}\times\,]-\infty, -M]$ then $p_1 \circ F^n(\tilde{x}) < p_1(\tilde{x}) + s\pi$. So $P_0(\hat{f}) \subset S^1 \times [-M, M]$ $\Rightarrow P_0(\hat{f})$ is a compact set (it is closed by definition). In the following we will define a new function $\hat{G} : S^1 \times \mathbb{R} \to \mathbb{R}$ such that $\hat{G}^{-1}(0) = P_0(\hat{f})$.

First let $G : \mathbb{R}^2 \to \mathbb{R}$ be the following function:

$$G(\phi, I) = (p_1 \circ f(\phi, I) - \phi)^2 + (p_2 \circ f(\phi, I) - I)^2. \tag{7}$$

Using $G(\phi + \pi, I) = G(\phi, I)$, we can define a function on the cylinder $\hat{G} : S^1 \times \mathbb{R} \to \mathbb{R}$, such that $\hat{G} \circ \pi = G$. Notice that

$$\hat{G}^{-1}(0) = \pi[G^{-1}(0)] = P_0(\hat{f}) \subset S^1 \times [-M, M].$$

So, from (6) we obtain the following inequality:

$$\#\hat{G}^{-1}(0) \geqslant \#A_{ns}^k. \tag{8}$$

Now we have the following remarkable lemma where the analyticity of G is crucial. This lemma is a simple consequence of some general well-known results in algebraic geometry and singularity theory (see, for instance, [19], lemmas 3.1 and 3.3).

Lemma 2. *Let G be a real-valued nonidentically null analytic function defined on a real analytic connected two-dimensional manifold with no boundary such that $G^{-1}(0)$ is compact. Then $G^{-1}(0)$ is the union of a finite number (maybe zero) of isolated points and a one-cycle (maybe empty). Moreover, this one-cycle is the union of a finite number of closed curves and each closed curve is made of finitely many analytic arcs.*

In figure 1 we show some possibilities for the set $G^{-1}(0)$.

Applying this lemma to $\hat{G}^{-1}(0)$ we get that if $\#\hat{G}^{-1}(0) = \infty$, then as $\hat{G}^{-1}(0)$ is compact, it must contain a closed curve that we denote by η. The definition of $\hat{G}$ implies that η satisfies

$$\hat{F}^n(\eta) = \eta + (0, k)\pi. \tag{9}$$

Now there are two possibilities for η.

(a) η is homotopically nontrivial. Then, let η^- be the connected component of η^c (the complement of η) in the cylinder that is below η. As the measure μ (see (2)) is invariant under $\hat{F}$, using (9) we get

$$\mu(\eta^-) = \mu(\hat{F}^n(\eta^-)) = e^{k\pi/\gamma}\mu(\eta^-),$$

which is impossible because $k \neq 0$.

(b) η is homotopically trivial. In this case, let η^- be the connected component of η^c in the cylinder that is bounded (the set inside η). Again we have

$$\mu(\eta^-) = \mu(\hat{F}^n(\eta^-)) = e^{k\pi/\gamma}\mu(\eta^-),$$

which is also impossible because $k \neq 0$. (Here the density function of μ plays a fundamental role.)

So, we have the following implications:

$$\#\hat{G}^{-1}(0) < \infty \Rightarrow \#P_0(\hat{f}) < \infty \Rightarrow \#A^k_{ns} < \infty \Rightarrow \#A^k_n < \infty, \tag{10}$$

and the proof is complete. □

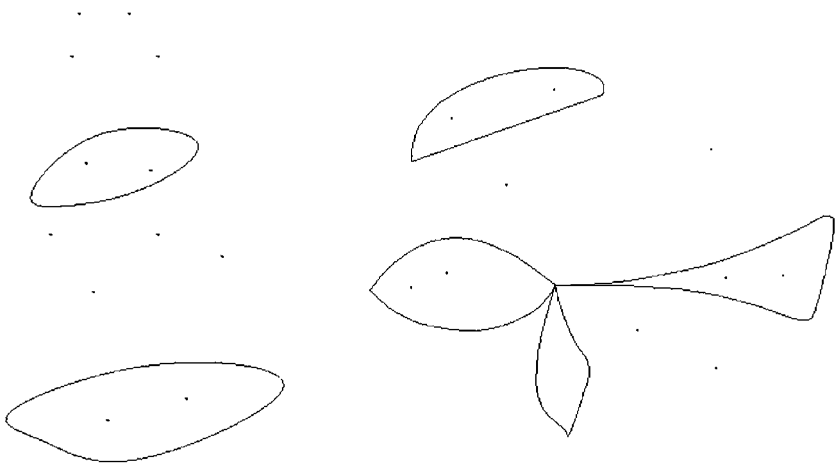

Figure 1. Diagram showing possibilities for the set $G^{-1}(0)$.

3.2. Proof of lemma 1

Given an n-periodic point x for $\bar{F}$ such that $\rho_V(x) \neq 0$, proposition 1 says that $0 < \det(D\bar{F}^n(x)) \neq 1$ and if $\rho_V > 0$ then $\det(D\bar{F}^n(x)) < 1$. If the periodic point is additionally supposed to be hyperbolic, then the lemma follows from an analysis of the linearized map (see [14], section 8.4). So, in the following we assume that this is not the case, namely its eigenvalues are either $(1, \lambda)$, with $\lambda > 0$, $\lambda \neq 1$, or $(-1, \lambda)$, with $\lambda < 0$, $\lambda \neq -1$. In any case the periodic point has a one-dimensional centre manifold and either a one-dimensional unstable manifold (if $|\lambda| > 1$) or a one-dimensional stable manifold (if $|\lambda| < 1$). For simplicity of writing we suppose that $0 < \lambda < 1$. The proofs for the other cases are similar. We can do a linear change of variables such that the centre manifold W^c is tangent to the horizontal axis (it is given by the graph of a C^k-function $x_1 \to h_c(x_1) = x_2$ where k can be chosen arbitrarily large) and the stable manifold W^s is tangent to the vertical axis (it is given by the graph of a C^∞-function $x_2 \to h_s(x_2) = x_1$). Now doing the C^k-change of variables,

$$y_1 = x_1 - h_s(x_2),$$
$$y_2 = x_2 - h_c(x_1),$$

we obtain that in a small neighbourhood of the origin (which after the coordinate changes coincides with the fixed point x of $\bar{F}^n$), the map $\bar{F}^n$ is written as

$$(y_1, y_2) \longrightarrow (y_1[1 + \cdots], y_2[\lambda + \cdots]). \tag{11}$$

In these coordinates W^c coincides with the horizontal axis and W^s with the vertical axis. The crucial step in the proof of the lemma is to use that x is an isolated fixed point of $\bar{F}^n$ (see theorem 3) to show that in a sufficiently small neighbourhood of the origin, the dynamics of $\bar{F}^n$ (see (11)) restricted to W^c satisfies one of the following conditions:

(a) $y_1 < 0 \Rightarrow y_1' \stackrel{\text{def}}{=} \bar{F}^n(y_1, 0) < y_1$ and $y_1 > 0 \Rightarrow y_1' > y_1$,
(b) $y_1 < 0 \Rightarrow y_1' > y_1$ and $y_1 > 0 \Rightarrow y_1' > y_1$,
(c) $y_1 < 0 \Rightarrow y_1' < y_1$ and $y_1 > 0 \Rightarrow y_1' < y_1$,
(d) $y_1 < 0 \Rightarrow y_1' > y_1$ and $y_1 > 0 \Rightarrow y_1' < y_1$.

Then a simple computation (using that $\bar{F}^n(y) - y$, with $\|y\|$ sufficiently small, is horizontal if and only if $y_2 = 0$ and is vertical if $y_1 = 0$) shows that the index of the periodic point in case (a) is -1, in cases (b) and (c) is zero, and in case (d) is 1 (see also [7], proposition 4.1, p 127 for a similar result in a more general context). If $\lambda > 1$ we arrive at essentially the same result, the index can be either -1, 0 or 1, and if $\lambda < 0$ the same analysis shows that the index is always 1. Finally the statement about the asymptotic stability of the periodic point if $\rho_V > 0$ and its eigenvalues are real and positive, which implies $0 < \lambda < 1$, is a consequence of a theorem (see, for instance, [6], section 2.8, theorem 8), saying that in this case the periodic point is a sink if and only if it is a sink for the dynamics restricted to the centre manifold. Notice that this happens only in case (d). This result is also a consequence of theorems on normal hyperbolic manifolds contained, for instance, in [20, 8].

3.3. Proof of theorem 4

From theorem 3 and the choice of α, there is a finite number of fixed points for $\bar{F}^n$ such that $\rho_V = m/n$. Let $\{P_1, P_2, \ldots, P_K\} = \{x \in \mathrm{Fix}(\bar{F}^n) : \rho_V(x) = m/n\}$. Suppose that all these points have topological index zero, namely $\mathrm{ind}(P_i, \bar{F}^n) = 0$, for $i = 1, 2, \ldots, K$. Then they all fall in cases (b) or (c) of the proof of lemma 1. Moreover, each of these points has a one-dimensional C^k centre manifold, where $k \geqslant 1$ can be chosen arbitrarily large. The dynamics of $\bar{F}^n$ on the centre manifold must be as illustrated in figure 2 (a rigorous justification of this

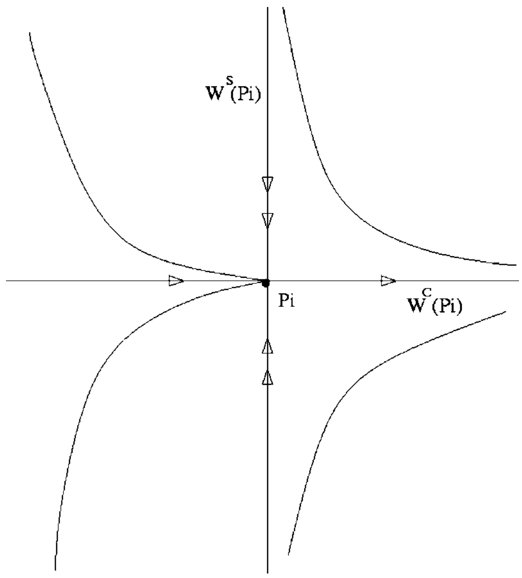

Figure 2. Diagram showing the dynamics of $\bar{F}^n$ in a neighbourhood of a point P_i.

figure, which is not necessary in the following, can be done using the results in [20, 8]). Now, let F be the lift of $\bar{F}$, given by (1) and let x_1 be the set of preimages of P_1 by the covering map $p : \mathbb{R}^2 \to T^2$. Let $x_2 = F(x_1), \ldots, x_n = F(x_{n-1})$ be the iterates of x_1 by F and let $W^{\mathrm{c}}_{j+1} = F^j(W^{\mathrm{c}}_1)$, $j = 0, \ldots, n-1$ be the images of W^{c}_1 by F, where W^{c}_1 is the preimage by p of a centre manifold of $\bar{F}^n$ at P_1. Let V_1 be the preimage by p of a small neighbourhood of P_1 and let $V_2 = F(V_1), \ldots, V_n = F(V_{n-1})$. The diameter of the connected components of V_1 can be chosen sufficiently small such that we can find C^k-coordinate systems on each connected component of $V_1, \ldots, V_n$ such that $F : V_j \to V_{j+1}$ is written as (11) with $\lambda < 1$. In these coordinates $W^{\mathrm{c}}_1, \ldots, W^{\mathrm{c}}_n$ locally coincide with the y_1-axis. Now, let $\psi : \mathbb{R} \to \mathbb{R}$ be an even C^k-function with compact support such that: $\psi'(s) \leqslant 0$ for $s \geqslant 0$, $\psi(s) = 1$ for $|s| \leqslant 1$ and $\psi(s) = 0$ for $|s| \geqslant 2$. Let $H_{1\delta\epsilon} : \mathbb{R}^2 \to \mathbb{R}^2$ be the map that is the identity outside V_1 and that in each connected component of V_1 is given by

$$(y_1, y_2) \longrightarrow \left(y_1 + \epsilon\psi\left(\frac{y_1}{\delta}\right)\psi\left(\frac{y_2}{\delta}\right), y_2\right),$$

where $\delta > 0$ and $\epsilon > 0$ are sufficiently small. Notice that $H_{1\delta\epsilon} - Id$, where Id is the identity map, is doubly π-periodic in the coordinates (ϕ, I). Notice that $H_{1\delta\epsilon}$ differs from Id only inside a square of size 4δ centred at $(y_1, y_2) = (0, 0)$ and the C^k norm of $H_{1\delta\epsilon} - Id$ tends to zero as $\epsilon \to 0$. This construction can be repeated for all P_i, $i = 1, \ldots, K$. Now, we define the composition map $F_{\mathrm{per}} = H_{1\delta\epsilon} \circ \cdots \circ H_{K\delta\epsilon} \circ F$. For $\delta > 0$ small, F_{per} differs from F only in small neighbourhoods of $p^{-1}(P_i)$, $i = 1, \ldots K$, and the C^k norm of $F_{\mathrm{per}} - F$ tends to zero as $\epsilon \to 0$. This implies that for any given $\bar{\epsilon} > 0$ we can choose $\delta > 0$ and $\epsilon > 0$ sufficiently small to construct a diffeomorphism $F_{\mathrm{per}} : \mathbb{R}^2 \to \mathbb{R}^2$ that is $(\bar{\epsilon} - C^k)$-close to F, such that $\bar{F}^n_{\mathrm{per}}$ does not have any fixed point with $\rho_{\mathrm{V}} = m/n$ (the definition of $H_{i\delta\epsilon}$ 'destroys' the fixed point P_i), and such that the following properties of F are preserved:

(a) $F_{\mathrm{per}}(\phi + i\pi, I + j\pi) = F_{\mathrm{per}}(\phi, I) + ([i + j]\pi, j\pi)$, so, in the same way as F, F_{per} induces maps on the cylinder $\hat{F}_{\mathrm{per}}$ and on the torus $\bar{F}_{\mathrm{per}}$ such that $\hat{F}_{\mathrm{per}}$ is $(\bar{\epsilon} - C^k)$-close to $\hat{F}$ and $\bar{F}_{\mathrm{per}}$ is $(\bar{\epsilon} - C^k)$-close to $\bar{F}$;

(b) $\bar{F}_{\text{per}}$ is homotopic to LM (see (3));

(c) $\partial_I(p_1 \circ F_{\text{per}}) > \frac{1}{2}$ (recall that $\partial_I(p_1 \circ F) = 1$);

(d) $\bar{F}_{\text{per}}$ has two hyperbolic fixed points $\tilde{Q}_1$ and $\tilde{Q}_2$ such that $\rho_{\text{V}}(\tilde{Q}_1) = \lfloor m/n \rfloor + 1$ and $\rho_{\text{V}}(\tilde{Q}_2) = -\lfloor m/n \rfloor - 1$ (recall that F has a similar property due to proposition 2 and the assumption $\alpha > \exp[(\lfloor m/n \rfloor + 1)\pi/\gamma]$.

We remark that this perturbation argument used to destroy the zero index fixed points could be done without the use of centre manifold theory, for instance, as suggested to us by the referee. We just kept our proof because a centre manifold argument was already necessary in the proof of lemma 1 (see the statement on topological sinks). Now we have the following claim.

Claim 1. *It is impossible for a map F_{per} to exist satisfying the above properties such that $\bar{F}_{\text{per}}$ does not have n-periodic points with $\rho_{\text{V}} = m/n$.*

The proof of this claim will be given in appendix A. It is a small modification in the argument used to prove theorem 2 in [2].

So this contradicts the hypothesis that $\text{ind}(P_i, \bar{F}^n) = 0$, for $i = 1, 2, \ldots, K$, which implies that at least one fixed point P_i of $\bar{F}^n$ must have either index 1 or -1. Finally, as a consequence of the Lifschitz fixed-point formula (see, e.g., [9]) and the fact that $\bar{F}$ is homotopic to LM (3), we obtain that

$$\sum_{x \in [\text{Fix}(\bar{F}^n):\rho_{\text{V}}(x)=m/n]} \text{ind}(x, \bar{F}^n) = 0.$$

So, from lemma 1, if there is one point with index -1, we must have one with index 1 and vice versa. □

3.4. Proof of theorem 5

Given $\gamma > 0$ and $m/n > 0$, we know from theorem 4 that the set $\{\alpha \in \mathbb{R} : \bar{F}$ has at least one n-periodic orbit with $\rho_{\text{V}} = m/n\}$ is closed and contains the interval $[\exp[(\lfloor m/n \rfloor + 1)\pi/\gamma], +\infty[$. So there is a function $\alpha_s = \alpha_s(m/n, \gamma) > 1$ such that for $\alpha > \alpha_s$, $\bar{F}$ has at least one n-periodic orbit with $\rho_{\text{V}} = m/n$ and index equal to 1 and for $\alpha = \alpha_s$ all n-periodic orbits with $\rho_{\text{V}} = m/n$ that exist must have zero index. Now we discuss the following claim.

Claim 2. *When $\alpha = \alpha_s$ there is a point $P \in \{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = m/n\}$ such that the eigenvalues of $D\bar{F}^n(P)$ are $\{1, \mathrm{e}^{-\pi m/\gamma}\}$ and given a sufficiently small disk $D \subset \mathrm{T}^2$ centred at P, there exists a number $\delta > 0$, such that for all $\alpha \in (\alpha_s, \alpha_s + \delta)$ we have*

$$\partial D \cap \left\{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = \frac{m}{n}\right\} = \emptyset, \tag{12}$$

and for all $\alpha_s < \alpha \underset{\circ}{<} \alpha_s + \delta$, there is at least one n-periodic point for $\bar{F}$ with $\rho_{\text{V}} = m/n$ and index equal to 1 in $\mathring{D}$ (the interior of D).

To prove this claim, first let $\{P_1, P_2, \ldots, P_K\} = \{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = m/n$ and $\alpha = \alpha_s\}$. Now suppose that for any arbitrarily small disk D_i centred at P_i and $\forall\delta > 0$, there is an $\alpha \in (\alpha_s, \alpha_s + \delta)$, such that $\partial D_i \cap \{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = m/n\} \neq \emptyset$. Taking the limit $\delta \to 0$, we get that for $\alpha = \alpha_s$, $\partial D_i \cap \{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = m/n\} \neq \emptyset$, which is a contradiction because the points P_i are isolated. So there is a constant $r > 0$ and a number $\delta = \delta(r) > 0$ such that if $\text{diam}(D_i) < r$ and $\alpha \in (\alpha_s, \alpha_s + \delta)$ we have $\partial D_i \cap \{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = m/n\} = \emptyset$ and further

$$\bigcup_{i=1}^{K} \mathring{D}_i \supset \left\{x \in \text{Fix}(\bar{F}^n) : \rho_{\text{V}}(x) = \frac{m}{n}\right\}.$$

Now we choose some $i_0 \in \{1, \ldots, K\}$ such that for $\alpha \in (\alpha_s, \alpha_s + \delta)$ there is at least one point with index 1 inside D_{i_0}, that we will just call D.

The choice of α_s and expression (12) imply that

$$\sum_{x \in D \cap [\mathrm{Fix}(\bar{F}^n), \rho_V = m/n]} \mathrm{ind}(x, \bar{F}^n) = 0, \qquad \text{for any } \alpha \in (\alpha_s, \alpha_s + \delta).$$

So, for $\alpha_s < \alpha < \alpha_s + \delta$, there is also at least one n-periodic point with $\rho_V = m/n$ and index equals -1 in $\overset{\circ}{D}$. Moreover, as the disk D can be chosen arbitrarily small, the eigenvalues of $D\bar{F}^n$ calculated at any point of $\overset{\circ}{D}$ must be both positive, which implies that the n-periodic point with $\rho_V = m/n$ and index equal 1 that is inside D must be a topological sink (see lemma 1). □

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We thank D C Panazzolo for his comments and help concerning lemma 2.

Appendix A

Appendix A.1. Proof of claim 1

Here we prove claim 1 of theorem 4. The theorem we prove is valid in a more general context, so we have to start with some standard definitions and background theory.

(a) Let (ϕ, I) denote the coordinates for the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where ϕ is defined modulo 1. Let $(\tilde{\phi}, \tilde{I})$ denote the coordinates for the universal cover of the cylinder, $\mathbb{R}^2$. For all maps $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, we define

$$(\phi', I') = \hat{f}(\phi, I) \qquad \text{and} \qquad (\tilde{\phi}', \tilde{I}') = f(\tilde{\phi}, \tilde{I}),$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $\hat{f}$.

(b) $D_r^1(\mathbb{R}^2) = \{f : \mathbb{R}^2 \to \mathbb{R}^2 / f$ is a C^1-diffeomorphism of the plane, $\tilde{I}'(\tilde{\phi}, \tilde{I}) \overset{\tilde{I} \to \pm\infty}{\to} \pm\infty$, $\partial_{\tilde{I}} \tilde{\phi}' > 0$ (twist to the right), $\tilde{\phi}'(\tilde{\phi}, \tilde{I}) \overset{\tilde{I} \to \pm\infty}{\to} \pm\infty$ and f is a lift of a C^1-diffeomorphism $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\}$.

(c) $\mathrm{Dif}_r^1(S^1 \times \mathbb{R}) = \{\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} / \hat{f}$ is induced by an element of $D_r^1(\mathbb{R}^2)\}$.

Now we recall some topological results for twist maps due to Le Calvez (for proofs, see [15, 16]). Let $\hat{f} \in \mathrm{Dif}_r^1(S^1 \times \mathbb{R})$ and $f \in D_r^1(\mathbb{R}^2)$ be its lifting. For every pair (p, q), $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we define the following sets:

$$\tilde{K}(p, q) = \{(\tilde{\phi}, \tilde{I}) \in \mathbb{R}^2 : p_1 \circ f^q(\tilde{\phi}, \tilde{I}) = \tilde{\phi} + p\} \qquad \text{and} \qquad K(p, q) = \pi \circ \tilde{K}(p, q). \tag{13}$$

Then we have the following lemma.

Lemma 3. *For every $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(p, q) \supset C(p, q)$, a connected compact set that separates the cylinder.*

For every $q \geqslant 1$ and $\bar{\phi} \in \mathbb{R}$ let

$$\mu_q(t) = f^q(\bar{\phi}, t), \qquad \text{for } t \in \mathbb{R}. \tag{14}$$

We say that the first encounter between μ_q and the vertical line through some $\phi_0 \in \mathbb{R}$ is for

$$t_F \in \mathbb{R} \text{ such that: } t_F = \min\{t \in \mathbb{R} : p_1 \circ \mu_q(t) = \phi_0\},$$

and the last encounter is defined in the same way:

$$t_L \in \mathbb{R} \text{ such that: } t_L = \max\{t \in \mathbb{R} : p_1 \circ \mu_q(t) = \phi_0\}.$$

Of course we have $t_F \leqslant t_L$.

Lemma 4. *For all $\phi_0, \bar{\phi} \in \mathbb{R}$, let $\mu_q(t) = f^q(\bar{\phi}, t)$, as in (14). So we have the following inequalities: $p_2 \circ \mu_q(t_L) \leqslant p_2 \circ \mu_q(\bar{t}) \leqslant p_2 \circ \mu_q(t_F)$, for all $\bar{t} \in \mathbb{R}$ such that $p_1 \circ \mu_q(\bar{t}) = \phi_0$.*

For all $s \in \mathbb{Z}$ and $N \in \mathbb{N}^*$, we can define the following functions on S^1:

$$\mu^-(\phi) = \min\{p_2(Q) : Q \in K(s, N) \text{ and } p_1(Q) = \phi\},$$
$$\mu^+(\phi) = \max\{p_2(Q) : Q \in K(s, N) \text{ and } p_1(Q) = \phi\},$$

and we can define similar functions for $\hat{f}^N(K(s, N))$:

$$\nu^-(\phi) = \min\{p_2(Q) : Q \in \hat{f}^N \circ K(s, N) \text{ and } p_1(Q) = \phi\},$$
$$\nu^+(\phi) = \max\{p_2(Q) : Q \in \hat{f}^N \circ K(s, N) \text{ and } p_1(Q) = \phi\}.$$

Lemma 5. *Defining* $\mathrm{Graph}\{\mu^\pm\} = \{(\phi, \mu^\pm(\phi)) : \phi \in S^1\}$ *we have*

$$\mathrm{Graph}\{\mu^-\} \cup \mathrm{Graph}\{\mu^+\} \subset C(s, N),$$

so for all $\phi \in S^1$, we have $(\phi, \mu^\pm(\phi)) \in C(s, N)$.

We have the following simple corollary to lemma 4.

Corollary 2. *$\hat{f}^N(\phi, \mu^-(\phi)) = (\phi, \nu^+(\phi))$ and $\hat{f}^N(\phi, \mu^+(\phi)) = (\phi, \nu^-(\phi))$.*

Now we define a large class of maps, already studied in [2], such that our result will apply. Let $TQ \subset D_r^1(\mathbb{R}^2)$ be such that for all $T \in TQ$ we have

$$T : \begin{cases} \tilde{\phi}' = T_\phi(\tilde{\phi}, \tilde{I}), \\ \tilde{I}' = T_I(\tilde{\phi}, \tilde{I}), \end{cases} \qquad \text{with } \partial_{\tilde{I}}\tilde{\phi}' = \partial_{\tilde{I}} T_\phi(\tilde{\phi}, \tilde{I}) > 0 \text{ and}$$

$$\begin{aligned} T_I(\tilde{\phi}+1, \tilde{I}) &= T_I(\tilde{\phi}, \tilde{I}), \\ T_I(\tilde{\phi}, \tilde{I}+1) &= T_I(\tilde{\phi}, \tilde{I}) + 1, \\ T_\phi(\tilde{\phi}+1, \tilde{I}) &= T_\phi(\tilde{\phi}, \tilde{I}) + 1, \\ T_\phi(\tilde{\phi}, \tilde{I}+1) &= T_\phi(\tilde{\phi}, \tilde{I}) + 1. \end{aligned} \tag{15}$$

Every $T \in TQ$ induces a map $\hat{T} \in \mathrm{Dif}_r^1(S^1 \times \mathbb{R})$ homotopic to the identity and a map $\bar{T} : \mathrm{T}^2 \to \mathrm{T}^2$ homotopic to LM (see (3)). It is clear that both F and F_{per} belong to TQ (see proof of theorem 4).

Now we are ready to prove the following theorem.

Theorem 6. *Given $T \in TQ$ and a rational $k/N > 0$, if $\bar{T}$ has fixed points Q_1 and Q_2 with vertical rotation numbers $\rho_{\mathrm{V}}(Q_1) = \lfloor k/N \rfloor + 1$ and $\rho_{\mathrm{V}}(Q_2) = -\lfloor k/N \rfloor - 1$, then $\bar{T}$ has at least one N-periodic orbit with $\rho_{\mathrm{V}} = k/N$.*

Remark. As $F_{\mathrm{per}} \in TQ$ and $\bar{F}_{\mathrm{per}}$ has two hyperbolic fixed points $\tilde{Q}_1$ and $\tilde{Q}_2$ such that $\rho_{\mathrm{V}}(\tilde{Q}_1) = \lfloor m/n \rfloor + 1$ and $\rho_{\mathrm{V}}(\tilde{Q}_2) = -\lfloor m/n \rfloor - 1$, theorem 6 implies that $\bar{F}_{\mathrm{per}}$ has at least one n-periodic orbit with $\rho_{\mathrm{V}} = m/n$, which contradicts the choice of $\bar{F}_{\mathrm{per}}$ and proves theorem 4.

Proof. If we do not have an N-periodic orbit with $\rho_V = k/N$, then there are two possibilities.

(a) For any fixed $s \in \mathbb{Z}$, we have the following: $p_2 \circ \hat{T}^N(Q) - p_2(Q) < k$, for any $Q \in C(s, N)$. This implies that $\nu^+(\phi) - \mu^-(\phi) < k$, for all $\phi \in S^1$. So there is a homotopically nontrivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ that separates $\hat{T}^N(C(s, N))$ from $C(s, N) + (0, k)$. If we call γ^- the connected component of γ^c which is below γ we get $\hat{T}^{iN}(\gamma^-) - (0, ik) \subset \gamma^-$. But this contradicts the fact that $\rho_V(Q_1) > k/N$.

(b) The second possibility is the following: for any fixed $s \in \mathbb{Z}$, $p_2 \circ \hat{T}^N(Q) - p_2(Q) > k$, for all $Q \in C(s, N)$. As above we get a contradiction, now using Q_2.

So, for all $s \in \mathbb{Z}$ there must be points $R_1, R_2 \in C(s, N)$ such that

$$p_2 \circ \hat{T}^N(R_1) - p_2(R_1) > k,$$
$$p_2 \circ \hat{T}^N(R_2) - p_2(R_2) < k.$$

Now the result follows from the fact that $C(s, N)$ is connected. □

References

[1] Addas Zanata S 2000 On the dynamics of twist maps of the torus *Doctoral Thesis* IMEUSP (in Portuguese)

[2] Addas Zanata S 2001 On the existence of a new type of periodic and quasi-periodic orbits for twist maps of the torus *Preprint*

[3] Addas Zanata S 2001 Elliptic islands, Hénon attractors and infinitely many sinks for a dissipative family of twist maps of the torus

[4] Addas Zanata S and Grotta Ragazzo C 2001 Conservative dynamics: unstable sets for saddle-center loops *Preprint*

[5] Addas Zanata S and Grotta Ragazzo C 2001 A critical number in scattering and escaping problems in classical mechanics *Phys. Rev.* E **64** 046216-1-11

[6] Carr J 1981 *Applications of Centre Manifold Theory* (New York: Springer)

[7] Chow S, Mallet Paret J and Yorke J 1981 *A Periodic Orbit Index which is a Bifurcation Invariant* (*Springer Lecture Notes Math.* **1007**) pp 109–131

[8] Fenichel N 1974 Asymptotic stability with rate conditions *Ind. Univ. Math. J.* **23** 1109–37

[9] Franks J 1982 *Homology and Dynamical Systems (CBMS Regional Conference Series in Mathematics vol 49)* (Providence, RI: American Mathematical Society)

[10] Grotta Ragazzo C 1994 Nonintegrability of some Hamiltonian systems, scattering and analytic continuation *Commun. Math. Phys.* **166** 255–77

[11] Grotta Ragazzo C 1997 Irregular dynamics and homoclinic orbits to Hamiltonian saddle-centers *Commun. Pure Appl. Math.* **L** 105–47

[12] Grotta Ragazzo C 1997 On the stability of double homoclinic loops *Commun. Math. Phys.* **184** 251–72

[13] Grotta Ragazzo C 1997 Stability of homoclinic orbits and diffusion in phase space *Phys. Lett.* A **230** 183–9

[14] Katok A and Hasselblatt B 1995 *Introduction to Modern Theory of Dynamical Systems* (Cambridge: Cambridge University Press)

[15] Le Calvez P 1986 Existence d'orbites quasi-périodiques dans les attracteurs de Birkhoff *Commun. Math. Phys.* **106** 383–94

[16] Le Calvez P 1991 Propriétés dynamiques des difféomorphismes de l'anneau et du tore *Astérisque* **204** 1–131

[17] Lerman L M 1991 Hamiltonian systems with loops of a separatrix of a saddle-center *Sel. Math. Sov.* **10** 297–306

[18] Mielke A, Holmes P and O'Reilly O 1992 Cascades of homoclinic orbits to, and chaos near, a Hamiltonian saddle center *J. Dyn. Diff. Eqns.* **4** 95–126

[19] Milnor J 1968 *Singular Points of Complex Hypersurfaces* (*Annals of Mathematics Studies* no 61) (Princeton, NJ: Princeton University Press)

[20] Pugh C and Shub M 1970 Linearizarion of normally hyperbolic diffeomorphisms and flows *Inv. Math.* **10** 187–98