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Conservative dynamics: unstable sets for saddle-center loops

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Abstract

We consider two-degree-of-freedom Hamiltonian systems with a saddle-center loop, namely an orbit homoclinic to a saddle-center equilibrium (related to pairs of pure real, $\pm v$, and pure imaginary, $\pm \omega i$, eigenvalues). We study the topology of the sets of orbits that have the saddle-center loop as their α and ω limit set. A saddle-center loop, as a periodic orbit, is a closed loop in phase space and the above sets are analogous to the unstable and stable manifolds, respectively, of a hyperbolic periodic orbit.

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1. Introduction

This paper concerns a problem quite similar to the one of describing the dynamics near an unstable hyperbolic periodic orbit of a Hamiltonian system with two degrees of freedom. In order to explain the problem let us consider as a model the

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two-degree-of-freedom Hamiltonian system given by

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 - v^2 q_1^2 + \omega^2 q_2^2 + b q_1^2 q_2^2 + \frac{q_1^4 + q_2^4}{2}\right). \tag{1}$$

The origin of the system $(q_1, q_2, p_1, p_2) = (0, 0, 0, 0)$ is a saddle-center equilibrium (namely, it is associated to pure real, $\pm v \neq 0$, and pure imaginary, $\pm \omega i \neq 0$, eigenvalues). This system has an orbit Γ homoclinic to the origin contained in the plane $\{q_1, p_1, q_2 = p_2 = 0\}$ that satisfies $q_1 > 0$ (there is another one that satisfies $q_1 < 0$, see Fig. 1). The union of Γ and the origin is called saddle-center loop and is denoted as $\bar{\Gamma}$. The homoclinic orbit Γ is approximated by a family of periodic orbits Γ_E, one for each energy E, as $E \to 0$. Each Γ_E has energy $E < 0$ and is also contained in the plane $\{q_1, p_1, q_2 = p_2 = 0\}$ (see Fig. 1). To each periodic orbit Γ_E we can define a transversal section and an associated Poincaré map. Conservation of energy implies that the study of the dynamics near Γ_E restricted to the energy level E can be reduced to the study of the dynamics of an area preserving, usually, twist map from a two-dimensional disc to itself. The periodic orbit Γ_E is represented by a trivial fixed point of this map. The iso-energetic stability of the periodic orbit Γ_E under the flow is equivalent to the stability of this fixed point under iterations of the map. The example above shows that it is quite natural to try to define a Poincaré map to the saddle-center loop $\bar{\Gamma}$ as it is done for the periodic orbits Γ_E that accumulates on it. Indeed, this can be done. The construction of a Poincaré map to a saddle-center loop has ideas that go back to Conley [11,12], Churchill et al. [9,10] (see also earlier work by the same authors), Llibre et al. [25], and it was first done by Lerman [24], and subsequently, but independently, by Mielke et al. [26] (generalizations of this idea to higher dimensions are presented in [22,23]). Several results on the dynamics of a Poincaré map to a saddle-center loop can be found in the references above and below. As for the periodic orbits Γ_E, a certain type of stability of $\bar{\Gamma}$ is related to the

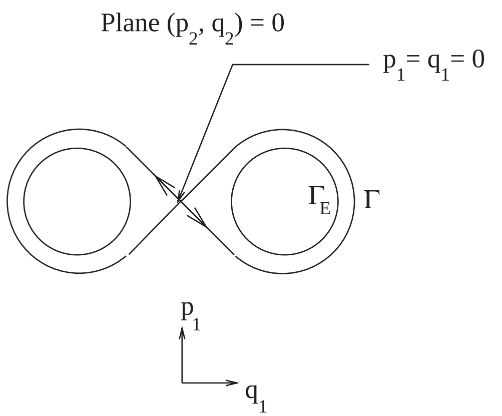

Fig. 1. Diagram showing a pair of orbits of system (1) homoclinic to the equilibrium at the origin. These orbits are contained in the invariant plane $\{q_1, p_1, q_2 = p_2 = 0\}$. Γ_E is a periodic orbit with $E < 0$.

stability of a trivial fixed point of the Poincaré map to $\bar{\Gamma}$ [16]. In the present paper we assume that this trivial fixed point is unstable. Our goal is to study the topology of the orbits of the Poincaré map to $\bar{\Gamma}$ that have the trivial fixed point as their α (or ω)-limit set. The answer to a similar question in the case of a hyperbolic periodic orbit is given by the unstable (or stable) manifold theorem.

The hypotheses assumed in this paper are the following. Let (M, Ω, H) be a real analytic Hamiltonian system where: M is a four-dimensional manifold, Ω is a symplectic form and H is a Hamiltonian function. We suppose that

(H1) (M, Ω, H) has an equilibrium point p of saddle-center type (the eigenvalues related to it are: $\pm v \neq 0$ and $\pm \omega i \neq 0$, $v > 0$, $\omega > 0$);
(H2) (M, Ω, H) has an orbit Γ homoclinic to p;

A third hypothesis requires some information on the energy level sets of H near p. The "Morse lemma" implies that in a neighborhood of the saddle-center critical point p there exists a coordinate system such that either H or $-H$ can be written as

$$H(p) - x_1^2 + x_2^2 + x_3^2 + x_4^2.$$

This implies that the intersection of the level set $H(x) = H(p)$ with a small ball B centered at p has two three-dimensional conical components, one with $x_1 > 0$ another with $x_1 < 0$, that intersect only at the critical point p. The third hypothesis is:

(H3) The intersection of Γ and a sufficiently small ball B is contained in only one of the above defined conical components.

This hypothesis ensures that any orbit with energy $H(p)$ sufficiently close to Γ remains close to Γ after a passage near the critical point p (it could be that after approaching p this orbit would follow the branch of the unstable manifold of p that did not belong to Γ). System (1), for instance, satisfies these three hypotheses. Under these hypotheses it is possible to define a transverse section to Γ and a Poincaré map to it. The restriction of this Poincaré map to the energy level $H(p)$, denoted by f, can be written in convenient coordinates as (see [15,24,26] or [16]):

$$f(x) = AR(-2\gamma \ln||x||)x + G(x) \overset{\text{def}}{=} F(x) + G(x), \tag{2}$$

where $x \in \mathbb{R}^2$ has sufficiently small norm; $\gamma = \omega/v$; $||G(x)|| < K_1||x||^2$, $||DG(x)|| < K_2||x||$, $K_1 > 0$ and $K_2 > 0$ are constants; and

$$R(\theta) \overset{\text{def}}{=} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad A \overset{\text{def}}{=} \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix},$$

with $\alpha \geqslant 1$. The origin, $x = (0, 0)$, denoted as $\underline{0}$, represents the intersection of Γ and the Poincaré section. The origin is by definition a fixed point of the Poincaré map f. Notice that F and f are continuous but not differentiable at the origin while G is differentiable and small near the origin.

The dominant part of the Poincaré map, given by F, is the composition of a twist map and a linear stretching map. The twist part, given by $R(-2\gamma \ln\|x\|)x$, comes from the passage of solutions near the saddle-center equilibrium. There, solutions rotate with frequency ω a number of times proportional to the time, $-(2/v)\ln\|y\|$, they spend near the equilibrium. The stretching part, given by A, is due to the travel of solutions near Γ.

Map F depends on two parameters $\gamma = \omega/v$ and α. Both numbers are symplectic invariants related to the saddle-center loop (for the invariant character of α see [5,14,21], and the discussion at the end of Section 2). The explicit computation of α for system (1) can be found in [17]. It was proved in [16] that if, for a given γ, the parameter α is sufficiently close to one then the origin is a stable fixed point of f. It was also proved in [16] that if $\gamma(\alpha - \alpha^{-1}) > 1$ then the origin is an unstable fixed point of f. In the rest of this paper we will suppose that: *the origin is an unstable fixed point of f*. As shown in [16] important properties of the dynamics of the flow near $\bar{\Gamma}$ can be obtained from the dynamics of f. Therefore, from now on we just consider the dynamics generated by f.

There are two main ideas behind our study of the dynamics of f. The first is that in order to prove a result for f it is enough to prove it for F and then use that these two maps are C^1 close near the origin. So, the crucial point in the study of the dynamics of f is to understand the dynamics of F. The second is that map F although highly nonlinear has a discrete dilation symmetry given by $F(e^{-k\pi/\gamma}x) = e^{-k\pi/\gamma}F(x)$, $k \in \mathbb{Z}$, that in some sense replaces the homogeneity property of linear maps.

Map F is an area preserving twist map on the plane with infinite twist at the origin. The dynamics of F is intimately related to the dynamics of a quotient map $\hat{F}$ defined on a two-dimensional torus in the following way. The mapping $\zeta : \mathbb{Z} \times \mathbb{R}^2\backslash\{0\} \to \mathbb{R}^2\backslash\{0\}$ given by $\zeta_k(x) = e^{k\pi/\gamma}x$ defines a properly discontinuous action of the group $\mathbb{Z}$ on the punctured plane $\mathbb{R}^2\backslash\{0\}$. So, the quotient of $\mathbb{R}^2\backslash\{0\}$ by the orbits of ζ is a two-dimensional manifold which is a two-torus denoted as T^2. Notice that each ring $e^{k\pi/\gamma} \leqslant \|x\| < e^{(k+1)\pi/\gamma}$, $k \in \mathbb{Z}$, is a fundamental region of ζ and the torus T^2 can be represented by one of these rings with its boundary points identified as $e^{k\pi/\gamma}x \sim e^{(k+1)\pi/\gamma}x$. We denote $\pi : \mathbb{R}^2\backslash\{0\} \to T^2$ the projection that associates to each point of $\mathbb{R}^2\backslash\{0\}$ its orbit under ζ. Notice that F restricted to the punctured plane commutes with $\zeta_k(\cdot)$, so it induces a diffeomorphism on T^2, $\hat{F} : T^2 \to T^2$, through the relation $\pi \circ F = \hat{F} \circ \pi$. Map $\hat{F}$ can be understood as a mapping from the fundamental region of ζ, $e^{-\pi/\gamma} \leqslant \|x\| < 1$, to itself. Since $\zeta_k(\cdot)$, $k \neq 0$, is not area preserving we conclude that $\hat{F}$ does not preserve the area form of $e^{-\pi/\gamma} \leqslant \|x\| < 1$. The dynamics of $\hat{F}$ is "dissipative".

One of our most interesting results for F, that implies a similar one for f, is the following. For some values of (γ, α) $\hat{F}$ has periodic attractors that correspond to orbits of F that escape exponentially fast from the origin. This implies that a set of points of positive measure escapes from a neighborhood of the origin under iterations of F following (or clustering around) a finite number of orbits that are given by the pre-image of the periodic attractors of $\hat{F}$ by π. In order to get a better understanding of this

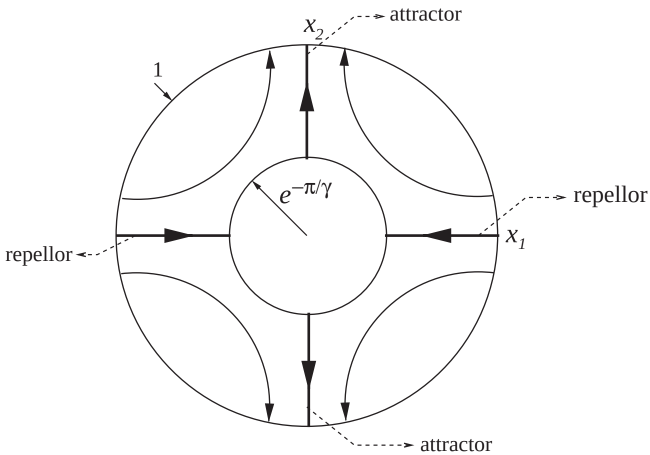

Fig. 2. Diagram showing iterates of the map induced by $(x_1, x_2) \to (2^{-1}x_1, 2x_2)$ on the two-torus T^2.

statement and to appreciate the novelty of $\hat{F}$ having discrete point set attractors let us use that any linear map commutes with ζ_k to analyze what happens to the map induced by a hyperbolic map $(x_1, x_2) \to (2^{-1}x_1, 2x_2)$ on the torus T^2 above. In this case the quotient map is dissipative and has four invariant curves represented by the intersection of the x_1 and x_2 axis with the fundamental region $e^{-\pi/\gamma} \leqslant ||x|| < 1$ (see Fig. 2). The two curves related to the x_1-axis are repellers and the two circles related to the x_2-axis are attractors. The dynamics in these invariant curves are topologically equivalent to rigid rotations. The quotient map does not have isolated point set attractors. Notice that iterates under the linear map of a small ball of initial conditions near the origin accumulate at the x_2-axis, which represents the unstable manifold of the fixed point. This is related to the fact that the x_2-axis attracts the iterates of the quotient map. This compression of a ball of initial conditions to a one dimensional structure represented by the unstable manifold of a saddle-point has important consequences in local and global dynamics (think, for instance, on the consequences of the so called "λ-lemma"). The zero dimensional structures that appear as attractors of $\hat{F}$ are in great contrast to this usual hyperbolic scenario. The global consequence on the dynamics of the original Hamiltonian system of these isolated point set attractors of $\hat{F}$ is an interesting problem for future studies.

This paper is organized as follows. In Section 2 we present the statements of our main results as well as proofs of some simple facts. In Section 3 we present the proofs of the theorems in Section 2.

2. Main results

Before introducing our main results concerning map f it is necessary to present some auxiliary results for F.

The logarithmic singularity of F can be removed with the following choice of polar coordinates that blow up the origin:

$$x_1 = e^{z/(2\gamma)}\cos(z-\phi), \quad x_2 = e^{z/(2\gamma)}\sin(z-\phi). \tag{3}$$

In these coordinates $x' = F(x)$, $x \neq \underline{0}$, writes as

$$\phi' = \mu(\phi) + z',$$

$$z' = z + \gamma \ln[J(\phi)], \tag{4}$$

where μ is a circle map given by

$$\mu(\phi) = \arctan\left(\frac{\tan(\phi)}{\alpha^2}\right), \quad \text{with } \mu(0) = 0$$

and

$$J(\phi) = \alpha^2\cos^2(\phi) + \alpha^{-2}\sin^2(\phi), \quad \text{with } \frac{d\mu}{d\phi}(\phi) = \frac{1}{J(\phi)}.$$

Using these coordinates we can easily identify the punctured plane with a cylinder $(\phi, z) \in S^1 \times \mathbb{R}$. We will denote by $\tilde{F}$ the map defined on the cylinder by Eq. (4). Analogously, we will denote by $\tilde{f}$ and $\tilde{\zeta}_k$ the maps obtained from f and ζ_k, respectively, through the change of coordinates (3). Maps $\tilde{f}$ and $\tilde{\zeta}_k$ are defined on a semi-infinite and an infinite cylinder, respectively. Map $\tilde{\zeta}_k$ represents a vertical translation on the cylinder and is given by $\tilde{\zeta}_k(\phi, z) = (\phi, z + 2\pi k)$. Notice that $\tilde{F}$ is a twist map that preserves the measure

$$\frac{e^{z/\gamma}}{2\gamma} d\phi \wedge dz \tag{5}$$

and that the expression of $\tilde{F}$ is quite similar to that of the well-known "standard map". In particular, both commute with $\tilde{\zeta}_k$ and induce twist maps on a two torus. The difference between them is that the standard map preserves the area form $d\phi \wedge dz$ that is also invariant with respect to $\tilde{\zeta}_k$ while the invariant measure (5) is not invariant under $\tilde{\zeta}_k$.

Remark. Before continuing it is convenient to emphasize some notation used in this paper: F is a map defined on the plane $\mathbb{R}^2$, $\hat{F}$ is a map defined on a two-torus T^2 (obtained from F through the quotient by orbits of ζ), and $\tilde{F}$ is defined on the cylinder $S^1 \times \mathbb{R}$ (obtained from F through the blow up coordinates (3)). Similarly, x, $\hat{x}$, and $\tilde{x}$ denote points on the plane, torus, and cylinder, respectively.

We say that F has a vertical periodic point $x \neq \underline{0}$ with vertical rotation number $\rho = m/n$, $n \in \mathbb{Z}_*$, $m \in \mathbb{Z}$, if $F^n(x) = e^{m\pi/\gamma}x$. The name vertical periodic point comes from the fact that $\hat{x} = \pi(x)$ is a periodic point of $\hat{F}$ and that the orbit of the corresponding point $\tilde{x}$ in the cylinder moves vertically as $z_n = z_0 + 2\pi m$, $\phi_n = \phi_0 \pmod{2\pi}$. In the physics literature, particularly in the context of the standard map, these vertical periodic orbits are called accelerator orbits [8, Section 5.5]. This notion of vertical rotation number naturally generalizes to an arbitrary orbit, not necessarily periodic, if we distinguish between the limits for forward and backward iterations of F. We say that an orbit of F, and also the corresponding orbits of $\hat{F}$ and $\tilde{F}$, has forward vertical rotation number $\rho_+ \in \mathbb{R}$ (or, equivalently, a point x of the orbit has forward vertical rotation number $\rho_+(x)$) if the following limit exists:

$$\rho_+ = \frac{\gamma}{\pi} \lim_{n\to\infty} \frac{\ln ||F^n(x)||}{n}. \tag{6}$$

Analogously, we say that an orbit of F has backward vertical rotation number $\rho_- \in \mathbb{R}$ if the following limit exists:

$$\rho_- = \frac{\gamma}{\pi} \lim_{n\to-\infty} \frac{\ln ||F^n(x)||}{n}. \tag{7}$$

If the orbit of x is such that $\rho_+(x) = \rho_-(x)$ then we just say it has vertical rotation number $\rho(x)$. For vertical periodic orbits it is always true that $\rho_+ = \rho_- = \rho$. For non-periodic orbits the limits ρ_+ and ρ_- can be different.

We say that $F(\tilde{F})$ has a rotational invariant curve if it has a homotopically non-trivial invariant simple closed curve in the punctured plane (in the cylinder). It is clear that if F has a rotational invariant curve then all orbits of F have null vertical rotation numbers. The next theorem, which is a consequence of a more general result proved in [2], shows that this is the only situation where all orbits of F have trivial vertical rotation numbers.

Theorem 1. *Suppose that F does not have any rotational invariant circle. Then there exists $\bar{\rho} > 0$ such that for each $\rho \in [-\bar{\rho}, \bar{\rho}]$ there exists an orbit of F with vertical rotation number ρ. If $\rho \in]-\bar{\rho}, \bar{\rho}[$ is rational then there are at least two vertical periodic orbits of F with vertical rotational number ρ.*

Remark. When ρ is irrational the above theorem implies the existence of a compact, $\hat{F}$-invariant set on the 2-torus such that the forward and backward vertical rotation numbers are equal and assume the same value at all its points.

Before we state our results for f we need to introduce some more definitions. Map f is defined in a small neighborhood U of the origin $\underline{0}$ of the plane. We define the unstable (stable) set W_f^{u} (W_f^{s}) of the fixed point $\underline{0}$ of f as the set of points $x \in U$ such that $f^n(x) \to \underline{0}$ as $n \to -\infty$ ($n \to \infty$). We define $W_f^{\mathrm{u}\rho}$ ($W_f^{\mathrm{s}\rho}$), $\rho \geqslant 0$ ($\rho \leqslant 0$), as the set of points $x \in U$ that have backward (forward) vertical rotation number equal to ρ (we dropped the subscript $\pm$ of $\rho_\pm$ since ρ_- is always associated to W_f^{u} and ρ_+ is always associated to W_f^{s}), namely

$$\rho = \frac{\gamma}{\pi} \lim_{n\to-\infty} \frac{\ln||f^n(x)||}{n} \quad \left(\rho = \frac{\gamma}{\pi} \lim_{n\to\infty} \frac{\ln||f^n(x)||}{n}\right).$$

For a given $\tau \geqslant 0$ and for each $C \geqslant 1$ we define the sets $W_f^{\mathrm{u}\rho\tau C}$ ($W_f^{\mathrm{s}\rho\tau C}$) as the subset of points x in $W_f^{\mathrm{u}\rho}$ ($W_f^{\mathrm{s}\rho}$), such that for all $n<0$ ($n>0$) the following inequality holds:

$$\frac{1}{C} e^{n(\rho+\tau)\pi/\gamma}||x|| \leqslant ||f^n(x)|| \leqslant C e^{n(\rho-\tau)\pi/\gamma}||x||,$$

$$\left(\frac{1}{C} e^{n(\rho-\tau)\pi/\gamma}||x|| \leqslant ||f^n(x)|| \leqslant C e^{n(\rho+\tau)\pi/\gamma}||x||\right). \tag{8}$$

The same definitions hold for F and are denoted by W_F^{u}, W_F^{s}, etc.

The apparent artificial definition of the sets $W_f^{\mathrm{u}\rho\tau C}$ and $W_F^{\mathrm{u}\rho\tau C}$ deserves some comments. The idea in the proof of the theorems below is that near $\underline{0}$ maps F and f are close. So their unstable and stable sets may have the same properties. A first problem with orbits of points $x \in W_F^{\mathrm{s}\rho}$ is that although their iterates go exponentially fast to zero as $n \to \infty$ they can have values of $||F^n(x)||$ very large for some values of n. The bounds in Eq. (8) are enough to overcome this sort of difficulty. A second problem is that the intersection of $W_F^{\mathrm{u}\rho}$ (or $W_f^{\mathrm{u}\rho}$) with any closed bounded ball centered at the origin may not be compact. The issue related to this lack of compactness also appears when we try to prove that the whole unstable manifold of a hyperbolic fixed point, considered as a submanifold of the manifold where the dynamics is defined, is homeomorphic to the unstable space of its corresponding linear part, considered as a subspace of the tangent space at the fixed point. It is always true that a compact part of the unstable space is homeomorphic to a compact part of the unstable manifold. But, in general, it is false that the unstable space is homeomorphic to the unstable manifold (due to the way the latter is generally folded). Notice that $W_F^{\mathrm{u}\rho\tau C}$ is compact for all τ and C, and for any fixed $\tau > 0$

$$W_F^{\mathrm{u}\rho} = \bigcup_{C \geqslant 1} W_F^{\mathrm{u}\rho\tau C}.$$

So, our definition of $W_F^{\mathrm{u}\rho\tau C}$ is a way to select compact parts of $W_F^{\mathrm{u}\rho}$. The same properties hold for $W_f^{\mathrm{u}\rho\tau C}$, $W_F^{\mathrm{s}\rho\tau C}$, etc.

For our next theorems it is important to know that the sets $W_F^{\mathrm{u}\rho\tau C}$ and $W_F^{\mathrm{s}\rho\tau C}$ are nonempty for several values of ρ, τ and C. Notice that if ρ is rational then any vertical periodic orbit of F with vertical rotation number $\rho > 0$ (the case with $\rho < 0$ is similar), or any orbit asymptotic to it, is contained in $W_F^{\mathrm{u}\rho 0 C}$, for some $C \geqslant 1$. Since, $W_F^{\mathrm{u}\rho\tau C} \subset W_F^{\mathrm{u}\rho\tau' C}$ if $\tau \leqslant \tau'$ we conclude that any periodic orbit with $\rho > 0$ is contained in $W_F^{\mathrm{u}\rho\tau C}$ for all $\tau \geqslant 0$. An analogous result for invariant sets with irrational vertical rotation number is given by the following lemma, proved in Section 3.1.

Lemma 1. *Given an irrational number $\rho \in]-\bar{\rho}, \bar{\rho}[$, where $\bar{\rho}$ is defined in Theorem* 1, *there is a compact $\hat{F}$-invariant set with vertical rotation number ρ such that all* *its points belong to either* $W_F^{s\rho 0C}$, *if* $\rho < 0$, *or* $W_F^{u\rho 0C}$, *if* $\rho > 0$, *for some* $C > 0$ *sufficiently large*.

Theorem 1, Lemma 1, and the remarks above this lemma, have the following corollary.

Corollary 1. *Suppose that F does not have any rotational invariant circle. Then there exists* $\bar{\rho} > 0$ *as in Theorem* 1 *such that for every* $\rho \in]-\bar{\rho}, 0[$ $(\rho \in]0, \bar{\rho}[)$ *there exists a C depending on* ρ *such that* $W_F^{s\rho 0C}$ $(W_F^{u\rho 0C})$ *is non-empty.*

We say that $\tilde{F}$ has a hyperbolic orbit $\tilde{x}_n$, $n \in \mathbb{Z}$, if the tangent space at $\tilde{x}_n$ has a splitting $E_n^- \oplus E_n^+$ such that $D\tilde{F}(\tilde{x}_n)E_n^- = E_{n+1}^-$, $D\tilde{F}(x_n)E_n^+ = E_{n+1}^+$, and there exist $\lambda_- < 1$ and $\lambda_+ > 1$ such that $||D\tilde{F}(\tilde{x}_n)|_{E_n^-}|| \leqslant \lambda_-$ and $||D\tilde{F}(\tilde{x}_n)|_{E_n^+}^{-1}|| \leqslant \lambda_+^{-1}$. The same definition of hyperbolic orbit holds for F. Using the periodicity of $\tilde{F}$ with respect to z, to obtain uniform estimates, we get from standard results in hyperbolic dynamics (the "Hadamard–Perron" theorem, see [20, Lemma 6.2.7 and Theorem 6.28]) that there are local smooth manifolds W_n^+ (W_n^-), $n \in \mathbb{Z}$, containing $\tilde{x}_n$ with the same dimension as E_n^+ (E_n^-) such that $\tilde{F}(W_n^+) = W_{n+1}^+$ $(\tilde{F}(W_n^-) = W_{n+1}^-)$ and if $w \in W_n^+$ $(w \in W_n^-)$ then $||\tilde{F}^{-1}(w) - \tilde{x}_{n-1}|| < \bar{\lambda}_+^{-1}||w - \tilde{x}_n||$ $(||\tilde{F}(w) - \tilde{x}_{n+1}|| < \bar{\lambda}_-||w - \tilde{x}_n||)$ where $1 < \bar{\lambda}_+ \leqslant \lambda_+$ $(\lambda_- \leqslant \bar{\lambda}_- < 1)$. In order to prove the existence of hyperbolic orbits for f it is convenient to extend it to the whole plane. In the next section, we show that f indeed, has such an extension f_ε from a neighborhood $V_\varepsilon \subset U$ of $\underline{0}$ such that f_ε is ε-C^1-close to F (obviously $\mathrm{diam}(V_\varepsilon) \to 0$ as $\varepsilon \to 0$). We denote by $\tilde{f}_\varepsilon$ the corresponding ε-C^1-close extension of $\tilde{f}$ to the infinite cylinder. This extension and a standard result on persistence of hyperbolic sets (see [20, Proposition 6.2.21]) imply the following theorem.

Theorem 2. *Suppose that* $\tilde{F}$ *has a hyperbolic orbit that corresponds to an orbit of F in* $W_F^{u\rho}$ $(W_F^{s\rho})$. *Then* $\tilde{f}$ *has also a hyperbolic orbit that corresponds to an orbit of f in* $W_f^{u\rho}$ $(W_f^{s\rho})$. *Moreover, compact parts of the invariant manifolds associated to the hyperbolic orbits of* $\tilde{F}$ *and* $\tilde{f}$ *get arbitrarily* C^1*-close when restricted to half cylinders* $\{(\phi, z) : z < Z\}$ *as the constant Z tends to minus infinity.*

Remark. To say that two orbits x_n, w_n, $n > 0$, of f and F, respectively, have their corresponding orbits on the cylinder $\tilde{x}_n$, $\tilde{w}_n$, $n > 0$, ε-close, namely $||\tilde{x}_n - \tilde{w}_n|| < \varepsilon$, $n > 0$, means that x_n and w_n satisfy $||x_n - w_n|| < \varepsilon \underline{C}||x_n||$, where $\underline{C} > 0$ does not depend on n and ε. This is a consequence of the change of coordinates formulas (3). So, for a constant $\varepsilon > 0$ the Euclidean distance between x_n and w_n will decrease if $||x_n|| \to 0$.

Theorem 2 concerns the relation between single orbits of F and f in $W_F^{u\rho}$ $(W_F^{s\rho})$ and $W_f^{u\rho}$ $(W_f^{s\rho})$ and their respective invariant manifolds. The next two theorems are about the relation of the sets $W_F^{u\rho}$ $(W_F^{s\rho})$ and $W_f^{u\rho}$ $(W_f^{s\rho})$ as a whole. Unfortunately, these theorems contain a restriction on the vertical rotation number that is not natural. However, this restriction makes the proofs quite simple and it seems difficult to prove Theorem 3 only under the natural condition $|\rho|>0$. Notice that the geometry of the sets $W_F^{\mathrm{u}\rho}$ and $W_F^{\mathrm{s}\rho}$ is not known and it is probably very complex.

Theorem 3. *Suppose that $\rho\in\mathbb{R}$ is such that*

$$0<|\rho|-\frac{\gamma}{2\pi}\ln[\alpha(1+2\gamma)]\stackrel{\mathrm{def}}{=}2\bar{\tau} \tag{9}$$

and $W_F^{\mathrm{u}\rho}$ ($W_F^{\mathrm{s}\rho}$) is nonempty. For each $\tau\in[0,\bar{\tau}]$ let I_τ denotes the set of values of C such that $W_F^{\mathrm{u}\rho\tau C}$ ($W_F^{\mathrm{s}\rho\tau C}$) is a nonempty topological subspace of $\mathbb{R}^2$.

Then for each $\tau\in[0,\bar{\tau}]$ and $C\in I_\tau$ there exists a closed ball B, centered at the origin of the plane, with positive radius depending on C, and a map $h: W_F^{\mathrm{u}\rho\tau C}\cap B\to W_f^{\mathrm{u}\rho}$ ($h: W_F^{\mathrm{s}\rho\tau C}|\cap B\to W_f^{\mathrm{s}\rho}$), with the following properties:

(i) *h is a homeomorphism over its image;*
(ii) *h is close to the identity near the origin, namely there exists $K_3>0$ such that $||h(x)-x||<K_3||x||^2$;*
(iii) *h conjugates f and F on $W_F^{\mathrm{u}\rho\tau C}\cap B$ ($W_F^{\mathrm{s}\rho\tau C}\cap B$), namely*

$$h\circ F|_{W_F^{\mathrm{u}\rho\tau C}\cap B}=f\circ h \quad (h\circ F|_{W_F^{\mathrm{s}\rho\tau C}\cap B}=f\circ h),$$

where $F|_{W_F^{\mathrm{u}\rho\tau C}\cap B}$ denotes the restriction of F to $W_F^{\mathrm{u}\rho\tau C}\cap B$;
(iv) *h and h', corresponding to two distinct pair of values of (τ,C), coincide on the intersection of their domains.*

Theorem 3 implies that $W_f^{\mathrm{u}\rho}$ is non-empty if $W_F^{\mathrm{u}\rho}$ is non-empty. The next theorem shows that every point in $W_f^{\mathrm{u}\rho}$ corresponds to some point in $W_F^{\mathrm{u}\rho}$. In order to give a more precise statement let us define a subset $W_F^{\mathrm{u}\rho\infty}$ ($W_F^{\mathrm{s}\rho\infty}$) of $W_F^{\mathrm{u}\rho}$ ($W_F^{\mathrm{s}\rho}$) in the following way. Any given point $x\in W_F^{\mathrm{u}\rho}$ ($x\in W_F^{\mathrm{s}\rho}$) belongs to some $W_F^{\mathrm{u}\rho\tau C}$ ($x\in W_F^{\mathrm{s}\rho\tau C}$) for some $\tau\in]0,\bar{\tau}]$ and $C\in I_\tau$. For a given x, let us fix such a pair τ, C (the following definition does not depend on the choice of τ and C due to property (iv) in Theorem 3). Under the hypotheses of Theorem 3 let B and $h_{\tau C}$ be the ball and the map given in this theorem for these values of τ and C. The point x above belongs to $W_F^{\mathrm{u}\rho\infty}$ ($W_F^{\mathrm{s}\rho\infty}$) if there exists an integer $n\geqslant 0$ ($n\leqslant 0$) such that $F^{-n}(x)\in B$ and $f^j\circ h_{\tau C}\circ F^{-n}(x)\in U$ for all $j=1,2,\dots,n$ ($j=-1,-2,\dots,n$). For this point x we associate the point

$$h(x)=f^n\circ h_{\tau C}\circ F^{-n}(x) \tag{10}$$

which belongs to $W_f^{\mathrm{u}\rho}$ ($W_f^{\mathrm{s}\rho}$). The set $W_F^{\mathrm{u}\rho\infty}$ ($W_F^{\mathrm{s}\rho\infty}$) is given by the totality of points in $x\in W_F^{\mathrm{u}\rho}$ ($x\in W_F^{\mathrm{s}\rho}$) that have the above property. Notice that $W_F^{\mathrm{u}\rho\tau C}\cap B\subset W_F^{\mathrm{u}\rho\infty}$ ($W_F^{\mathrm{s}\rho\tau C}\cap B\subset W_F^{\mathrm{s}\rho\infty}$) for all $\tau\in[0,\bar{\tau}]$ and $C\in I_\tau$. The function $h: W_F^{\mathrm{u}\rho\infty}\to W_f^{\mathrm{u}\rho}$

($h: W_F^{s\rho\infty} \to W_f^{s\rho}$) given by (10) is clearly injective. The next theorem shows that it is also surjective.

Theorem 4. *Suppose that the hypotheses of Theorem* 3 *are verified. Then the function* $h: W_F^{u\rho\infty} \to W_f^{u\rho}$ ($h: W_F^{s\rho\infty} \to W_f^{s\rho}$) *given by* (10) *satisfies*:

(i) *h is a bijection*;
(ii) *h restricted to* $W_F^{u\rho\tau C} \cap B$ ($W_F^{s\rho\tau C} \cap B$) *is the map defined in Theorem* 3 *on the same set*.

The sets $W_F^{u\rho\tau C}$ and $W_F^{s\rho\tau C}$ for various values of ρ, τ, and C, can have a very complex topology. In [1] it is proved the existence of homoclinic bifurcations for the invariant manifolds of vertical periodic orbits of $\hat{F}$. This implies that for rational vertical rotation numbers ρ the sets $W_F^{u\rho}$ and $W_F^{s\rho}$ can have complicated hyperbolic invariant sets, Hénon attractors, etc. Numerical investigations show that not only the topology of the sets W_F^{u} and W_F^{s} is complicated but also their dependence on variations of (γ, α). We are far from a good understanding of the topology of and the dynamics in W_F^{u} and W_F^{s}. Nevertheless, Theorem 1 shows that when F has no invariant curve (what is true, for instance, if $\gamma(\alpha - \alpha^{-1}) > 1$ [16]) then $W_F^{u\rho}$ and $W_F^{s\rho}$ are not empty for intervals of values of ρ. Then Corollary 1 ensures that for all values of ρ in this interval and for $\tau = 0$ there are values of C where $W_F^{u\rho 0C}$ and $W_F^{s\rho 0C}$ are not empty. In order to apply Theorems 2 and/or 3 it is still necessary to check the hyperbolicity hypothesis and/or condition (9), respectively. This can be done explicitly for several vertical periodic points of $\hat{F}$, as for example, for the following pair. The stability properties of these points, which is important in our next theorems, are also discussed. The fact that the action of F on the plane preserves area and the action of $x \to e^{\pi/\gamma}x$ expands the area form by a factor $e^{2\pi/\gamma}$ imply that the eigenvalues λ_1 and λ_2 of $D\hat{F}^n(\hat{x})$ at an n-periodic point $\hat{x}$ of $\hat{F}$ with $\rho = \pm m/n$ satisfy $\lambda_1\lambda_2 = e^{\mp 2\pi m/\gamma}$. So a periodic point with $\rho > 0$ can never be a source and one with $\rho < 0$ can never be a sink. Explicit hyperbolic periodic sinks and saddle points of low period can be found easily. For instance, imposing $z_1 = z_0 + 2\pi$ and $\phi_1 = \phi_0 + 2\pi$ in equation (4) we find two equations for (ϕ_0, z_0):

$$\ln J(\phi_0) = \ln[\alpha^2 \cos^2(\phi_0) + \alpha^{-2} \sin^2(\phi_0)] = \frac{2\pi}{\gamma}, \quad z_0 = \phi_0 - \mu(\phi_0). \tag{11}$$

For $\alpha = e^{\pi/\gamma}$ these equations have two solutions $(\phi_0, z_0) = (0, 0)$ and $(\phi_0, z_0) = (\pi, 0)$. Both periodic points have an eigenvalue equal to one and vertical rotation number $\rho = 1$. Notice that for these orbits and for $\gamma < 1.9$ inequality (9) is verified, so Theorem 3 can be applied (indeed in this case Theorem 3 can be applied to all sets given by Corollary 1 that have rotation number satisfying inequality (9)). If γ is kept constant and α is increased then these periodic points unfold saddle-node bifurcations. It is easy to check that the periodic point that appears from this bifurcation with ϕ-coordinate slightly larger than zero is a hyperbolic sink and the one with ϕ-coordinate slightly smaller than zero is a hyperbolic saddle. Depending on the values of the parameters α and γ map $\hat{F}$ may have several hyperbolic sinks with positive vertical rotation number. In [1,6] the reader finds more information about the existence of periodic and non-periodic attractors for $\hat{F}$. In fact, in [6], we prove that for every positive rational number $\frac{p}{q}$, there is an open set in the parameter space (γ, α) such that $\hat{F}$ has a topological sink with vertical rotation number $\frac{p}{q}$.

In [4], we study how the extreme points of the vertical rotation interval behave as a function of the twist mapping. As in this paper we are considering a particular family of mappings, the supremum of the vertical rotation interval is a function of (γ, α):

$$\rho_V^{\max}(\gamma, \alpha) = \sup\left[\lim_{n\to\infty} \frac{p_2 \circ \tilde{F}^n(x) - p_2(x)}{\pi.n}\right],$$

where the supremum is taken over all $x \in \mathbb{R}^2$ such that the above limit exists. It can be proved that this function is continuous (see [3]) and it is easy to see that for a fixed $\gamma > 0$ $(\alpha > 1)$, $\lim \rho_V^{\max}(\gamma, \alpha) = \infty$, as $\alpha \to \infty$ $(\gamma \to \infty)$. Suppose now that for some (γ_0, α_0) $\rho_V^{\max}$ is not locally constant. Then the continuity of $\rho_V^{\max}$ and a method developed in [6] imply that by an arbitrarily small perturbation in the parameters we can create topological sinks for $\hat{F}$. So as an attempt to prove the density of topological attractors in the parameter space, in the future we will try to understand what happens when $\rho_V^{\max}$ is locally constant.

In particular, what we said above implies the following. From results due to Birkhoff we know that the subset of the parameter space (γ, α) for which $\hat{F}$ has at least one rotational invariant curve is closed. This can also be proved by noticing that, from Theorem 1, this set is equal to $[\rho_V^{\max}]^{-1}(0)$, which is obviously closed. In this way, given any point (γ^*, α^*) in the boundary of $[\rho_V^{\max}]^{-1}(0)$ we can create topological sinks for $\hat{F}$ by an arbitrarily small perturbation in the parameters. So when $\hat{F}$ has at least one rotational invariant curve, there are 2 possibilities:

(1) By any sufficiently small perturbation in the parameters, $\hat{F}$ still has at least one rotational invariant curve.
(2) By arbitrarily small perturbations in the parameters we can create topological sinks for $\hat{F}$ with positive vertical rotation number (the origin looses stability under iterations of F).

Now we turn to our main theorems. Before stating them we need to define a function on points of W_F^{u} to distinguish among them those that come from attractors of $\hat{F}$. Let us denote by $m(V)$ the Lebesgue measure of a set $V \subset \mathbb{R}^2$ and by B_k an open ball of radius $e^{-\pi k/\gamma}$, $k \in \mathbb{Z}$, centered at the origin of $\mathbb{R}^2$. We define the "density" $\sigma_F(x)$ of a point $x \in W_F^{\mathrm{u}}$ as

$$\sigma_F(x) = \inf_V \liminf_{k\to\infty} \frac{m(B_k \cap [\bigcup_{n=0}^{\infty} F^{-n}(V)])}{m(B_k)}, \tag{12}$$

where the infimum is taken over all open neighborhoods V of x. Notice that given a small neighborhood V of x, $\sigma_F(x) \in [0,1]$ is an estimate of the density of points y in a small ball B_k that have some positive iterate $F^n(y)$ passing near x, namely $F^n(y) \cap V \neq \emptyset$, for some $n \geqslant 0$. The same definition holds for $\sigma_f(x)$ with the obvious modification that all sets B_k, V, must be contained in U, the domain of definition of f. It is easy to check that $\sigma_F(x) = \sigma_F(F(x))$ so σ_F is the same for all points in the same orbit. This property also holds for f. We remark that our definition of σ_F resembles the definition of lower derivatives for set functions (see [28], Chapter 8, [29]). The definition of σ can be extended to any area preserving map with an unstable fixed point. The particular family of balls B_k chosen above can also be changed or generalized considerably without changing the value of the density σ. In order to get some familiarity with our definition of σ let us compute a similar quantity $\sigma_L(x)$ for a linear hyperbolic map $L:(x_1,x_2) \to (2x_1, 2^{-1}x_2)$ and $x = (1,0)$. We change the family of balls above by a family of squares B_k of side $1/k$ centered at the origin. Let V_b be a rectangle centered at $(1,0)$ of width $b>0$ small and height equal to 1. Then, for k large, $m(B_k \cap L^{-n}(V_b)) = 0$ if $n<j$, where $j>0$ is the smallest integer such that $2^{-j} \leqslant [k(2-b)]^{-1}$, and

$$\frac{m(B_k \cap \bigcup_{n=j}^{\infty} L^{-n}(V_b))}{m(B_k)} \leqslant \frac{1}{m(B_k)} \frac{b}{k} \sum_{n=j}^{\infty} 2^{-n} = kb22^{-j} < \frac{2b}{2-b}.$$

Taking the limit as $k \to \infty$ and then $b \to 0$ we get $\sigma_L(\hat{x}) = 0$. From the same type of argument we can get that any point in the local unstable manifold of a hyperbolic point of a two-dimensional diffeomorphism has density equals to zero with respect to any family of squares. This is a consequence of the geometrical fact presented in the introduction (see Fig. 2) that the map induced by a linear hyperbolic map on the torus T^2 has one-dimensional attractors and no isolated attracting points.

In the next section we will prove the following two theorems. They show that the unstable set of a saddle-center loop may have a finite number of orbits such that a relatively large set of solutions initially close to the saddle-center loop escape from it through trajectories that follow these orbits.

Theorem 5. *Let $x \in W_F^{\mathrm{u}}$ and $\hat{x} = \pi(x)$ be a topological periodic sink of $\hat{F}$. Then $\sigma_F(x)$ is strictly positive. Moreover if S is the set of points in the annulus $\mathscr{A} = \{x : e^{-\pi/\gamma} \leqslant \|x\| < 1\}$ that correspond to points in the basin of attraction of $\hat{x}$ with respect to iterates of $\hat{F}$ then*

$$\sigma_F(x) = \frac{m(S)}{m(\mathscr{A})}.$$

Theorem 6. *Let $\hat{x}$ be a hyperbolic periodic sink of $\hat{F}$. Then there exists $x \in W_F^{\mathrm{u}}$, with $\hat{x} = \pi(x)$ and $\|x\|$ sufficiently small, and $y \in W_f^{\mathrm{u}}$ corresponding to x according to Theorem 2 such that $\sigma_f(y) \geqslant \sigma_F(x)$. In case $\hat{x}$ is only a topological periodic sink,* *there exists* $y \in W_f^{u\rho}$ *such that* $\sigma_f(y) > 0$, *where* ρ *is the vertical rotation number associated to* $\hat{x}$.

In order to illustrate the consequences of Theorems 5 and 6 in the dynamics of F (and also of f), we show in Fig. 3 three sets of iterates of map F. The three sets were generated by the same initial conditions, namely a small ball of points centered at the origin, have the same number of iterates, and are presented on the same scale. The value $\gamma = 2$ was kept constant in all cases and the values of parameter α were 2, 2.3, and 3.2, in Figs. 3(a), (b), and (c), respectively. Mapping $\hat{F}$ has no (at least expressive) sinks in cases 3(a) and 3(c) and has a low period sink in case 3(b). The escaping points in Fig. 3(b) are much less spread than in the other two figures, since according to Theorem 5 the iterates accumulate on the pre-image of the sink by map π.

We finish this section with some remarks on the way to compute the function $\sigma_f(x)$ in a coordinate system that is not the special one used in this paper. Let $u : \mathbb{R}^2 \to \mathbb{R}^2$ be a twice continuously differentiable change of coordinates such that $x = u(z) = Pz + \mathcal{O}(||z||^2)$. In the new coordinates it is easy to check, using the same ideas as in Section 3.2, that both F and f can be written as

$$z' = P^{-1}APP^{-1}R(-2\gamma \ln||Pz||)Pz + \mathcal{O}(||z||^2).$$

Let us define

$$B = P^{-1}AP, \quad \Lambda(\theta) = P^{-1}R(\theta)P, \quad M = P^{\dagger}P, \tag{13}$$

where $P^{\dagger}$ denotes the transpose of matrix P. So, in any coordinate system f can be written as

$$f(z) = B\Lambda(\Theta)z + \mathcal{O}(||z||^2), \quad \Theta(z) = -2\gamma \ln ||z||_M, \tag{14}$$

where

(a) M is a symmetric positive matrix such that $(\cdot, M\cdot)$ defines an inner product on $\mathbb{R}^2$ and $||z||_M^2 = (z,z)_M = (z, Mz)$;
(b) $\Lambda(\Theta)$ is an isometry of $\{\mathbb{R}^2, (\cdot, M\cdot)\}$ (the matrix identity $\Lambda(\Theta)^{\dagger}M\Lambda(\theta) = M$ holds);
(c) B is a symmetric positive transformation of $\{\mathbb{R}^2, (\cdot, M\cdot)\}$ (the matrix identity $B^{\dagger}M = MB$ holds).

Notice that the metric $(\cdot, M\cdot)$ and the transformations $\Lambda(\theta)$ and B are uniquely determined by f. This is a consequence of the invariance of the logarithmic singularity of f under differentiable change of variables and the uniqueness property of the polar decomposition of non-singular transformations of a vector space with inner product. As we said the dominant part F of f is characterized by two numbers γ and α. In any coordinate system the number γ is related to the dilation invariance of

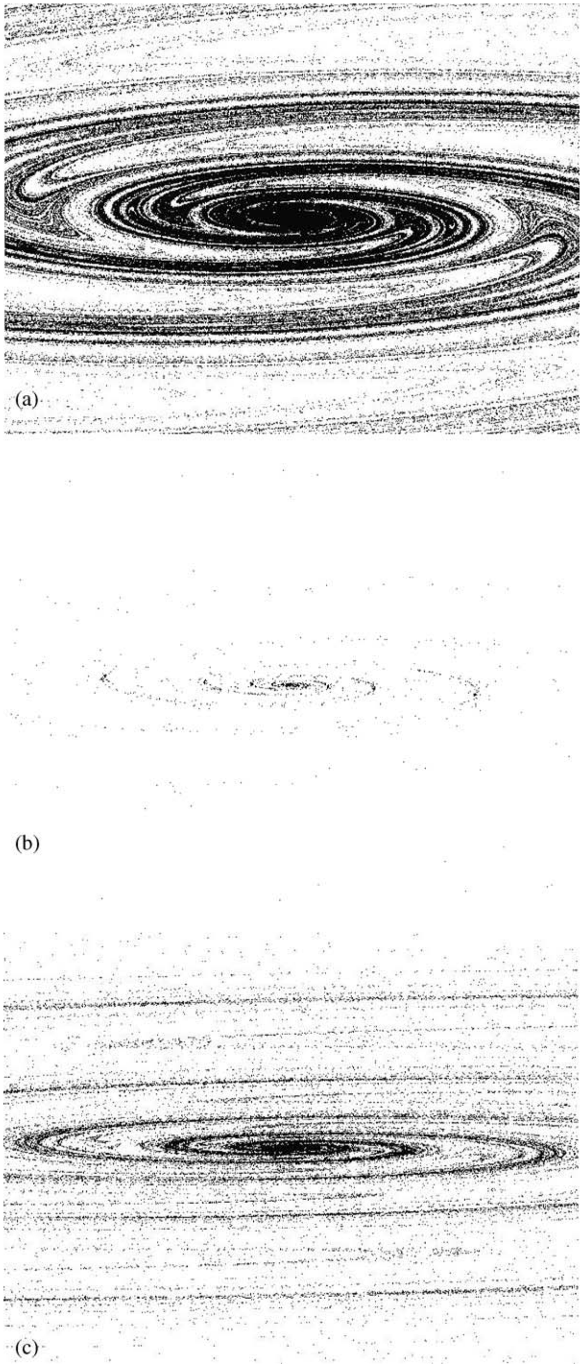

Fig. 3. Three sets of iterates of map F. For all three sets: the value of γ is 2, the initial conditons are in a small ball centered at the origin, the number of iterates is the same, and the figure scale is the same. The parameter α in (a), (b), and (c), is respectively: 2 ("no sink"), 2.3 ("low period sink"), and 3.2 ("no sink").

the dominant part of f. A coordinate independent geometric interpretation of α is given by the following argument. Let us consider the ellipsis $||z||_M = 1$ and its image under the dominant part $\underline{F}$ of f, $||\underline{F}^{-1}(z)|| = 1$. Let Q_1 be the area inside $||z||_M = 1$, Q_2 be the area inside $||\underline{F}^{-1}(z)|| = 1$ but outside $||z||_M = 1$, and $\Delta = Q_1/Q_2$ be the ratio between them. We claim that $\alpha \geqslant 1$ is the number given by

$$\frac{\alpha + \alpha^{-1}}{2} = \frac{\mathrm{Tr}\, B}{2} = \frac{1}{\cos(\Delta\pi/2)}. \tag{15}$$

To prove this we first show by explicit computation that (15) is true when M is the identity. Then we use that Δ is invariant under any linear isomorphism P and $\mathrm{Tr} A = \mathrm{Tr}\, PAP^{-1} = \mathrm{Tr}\, B$.

Finally, in order to compute σ_f in any coordinate system z it is necessary first to find the matrix M and then replace the definition of the family of balls B_k appearing in Eq. (12) by $B_k = \{z : ||z||_M \leqslant e^{-\pi k/\gamma}\}$. If we start from the original Hamiltonian flow then matrix M is determined by the quadratic part of the Hamiltonian function at the saddle-center equilibrium (see for instance [5]).

3. Proofs of the main results

3.1. Proof of Lemma 1

At first, let us choose rational numbers ρ_1, ρ_2, such that $-\bar{\rho} < \rho_1 < \rho < \rho_2 < \bar{\rho}$. From Theorem 1 there are periodic orbits Q_1 and Q_2 such that their vertical rotation numbers are, respectively, ρ_1 and ρ_2. Let $Q = Q_1 \cup Q_2$. Now we blow-up each $x \in Q$ to a circle S_x. Let T^2_Q be the compact manifold (with boundary) thereby obtained; T^2_Q is the compactification of $T^2 \backslash Q$, where S_x is a boundary component where x was deleted. Now we extend $\hat{F} : T^2 \backslash Q \to T^2 \backslash Q$ to $\hat{F}_Q : T^2_Q \to T^2_Q$ by defining $\hat{F}_Q : S_x \to S_x$ via the derivative; we just have to think of S_x as the unit circle in $T_x T^2$ and define

$$\hat{F}_Q(v) = \frac{D\hat{F}_x(v)}{||D\hat{F}_x(v)||},$$

for $v \in S_x$. $\hat{F}_Q$ is continuous on T^2_Q because $\hat{F}$ is C^1 on T^2 (this construction is due to Bowen, see [7]). Now we have the following

Theorem 7. *The map $\hat{F}_Q : T^2_Q \to T^2_Q$ is isotopic to a pseudo-Anosov homeomorphism $\hat{G}_Q : T^2_Q \to T^2_Q$.*

The above theorem is proved using Nielsen–Thurston theory of classification of homeomorphisms of surfaces, see [2] for a proof and [13,19] for more information on the theory. As $\hat{G}_Q : T^2_Q \to T^2_Q$ is pseudo-Anosov, it admits a Markov partition, see for example, [13,30]. Now let $\mathbb{R}^2_Q$ be a cover of T^2_Q which simply is $\mathbb{R}^2$ with an infinite family of holes deleted and let $G_Q : \mathbb{R}^2_Q \to \mathbb{R}^2_Q$ be a lift of $\hat{G}_Q$. It can be proved, see Proposition 1.1 of [19], that $]\rho_1, \rho_2[$ is contained in the vertical rotation set of $\hat{G}_Q$. Also, using the fact that $\hat{G}_Q$ admits a Markov partition, one can prove that (see [13,30]) $\exists C_1 > 0$ and a point $x_1 \in \mathbb{R}^2_Q$ such that ($p_2 : R^2_Q \to \mathbb{R}$ is the projection on the vertical direction):

$$||p_2 \circ G^n_Q(x_1) - p_2(x_1) - n\rho|| \leqslant C_1, \quad \forall n > 0.$$

Now by a result due to Handel on shadowing of pseudo-Anosov homeomorphisms (see [18]), there exists a point $x_2 \in \mathbb{R}^2_Q$ and a constant $C_2 > 0$, such that

$$||p_2 \circ G^n_Q(x_1) - p_2 \circ F^n_Q(x_2)|| \leqslant C_2, \quad \forall n > 0.$$

So we get that $||p_2 \circ F^n_Q(x_2) - p_2(x_2) - n\rho|| \leqslant C'$, for a sufficiently large $C' > 0$ and all $n > 0$. Now, if we define $\hat{x}_2 \in T^2$ as the projection of x_2, we get that the ω-limit set of the orbit of $\hat{x}_2$ by $\hat{F}$ satisfies the lemma hypothesis with $C = 2C'$.

3.2. *Some propositions concerning Theorem 2*

As we have already mentioned, Theorem 2 is a consequence of standard results in hyperbolic dynamics and C^1 bounds on the difference between F and f (that will be given in Proposition 2). So, in the following we study the C^1 closeness of F and f. Let $B_\psi \subset \mathbb{R}^2$ be a ball of radius ψ centered at the origin. From [16] we have that f can be written as the composition of two maps $f = g \circ \ell$, where $g : B_\psi \to \mathbb{R}^2$ is an analytic mapping for some $\psi > 0$ and $\ell : B_\psi \to B_\psi$. Map ℓ is the "local map" associated to the passage of trajectories of the flow near the saddle-center singularity and g is the "global map" associated to the trajectories of the flow near the homoclinic orbit Γ. Map ℓ is given by (see [16] Eqs. (6)–(8)):

$$x' = \ell(x) = R(\theta(x))x, \tag{16}$$

where

$$\theta(x) = -[\partial_I v(I)] \ln \frac{|v(I)|}{\delta^2}, \quad I = \frac{|x|^2}{2},$$

$$v(I) = \gamma I + \bar{v}(I) > 0 \quad \text{for } I \in (0, \psi],$$

$\bar{v}(I) = \mathcal{O}(I^2)$ is an analytic function from $(-\psi, \psi)$ to $\mathbb{R}$, and $\delta > 0$ is some constant. The expression for θ can be written in a more convenient form as

$$\theta(x) = -2\gamma \ln||x|| + c_1 + I[a(I)\ln(I) + b(I)],$$

where a and b are real analytic functions in $(-\psi,\psi)$, and c_1 is some constant. Therefore ℓ can be written as

$$\ell(x) = R(-2\gamma \ln||x|| + c_1)x + \eta(x),$$

where

$$\eta(x) = R(-2\gamma \ln||x|| + c_1)[R\{I[a(I)\ln(I) + b(I)]\} - \text{Identity}]x.$$

Notice that $\eta(x) = \mathcal{O}(||x||^2)$, η is continuously differentiable in a neighborhood of $(0,0)$ and $||D\eta(x)|| < c_2||x||$, for $||x||$ small. Notice also that the derivative of the function $x \to R(-2\gamma \ln||x||)x$ is

$$R(-2\gamma \ln||x||)(\text{Identity} + 2\gamma\Lambda(x)),$$

where

$$\Lambda(x) = \frac{1}{||x||^2}\begin{pmatrix} x_1x_2 & x_2^2 \\ -x_1^2 & -x_1x_2 \end{pmatrix}. \tag{17}$$

Matrix $\Lambda(x)$ has norm 1 for any $x \neq \underline{0}$ since $\Lambda(x)\Lambda(x)^\dagger$ is a matrix with determinant zero and trace one. This implies that the norm of derivative (17) is less than or equal to $1 + 2\gamma$ for $x \neq \underline{0}$. So, although the derivative of ℓ is not defined at $\underline{0}$ we have that it is bounded in a neighborhood of the origin, where the following inequality holds:

$$||D\ell(x)|| < 1 + 2\gamma + c_2||x||, \tag{18}$$

for some $c_2 > 0$. Now, let us write $g(x) = AR(c_3)x + \bar{g}(x)$ where $AR(c_3)$ is the polar decomposition of $Dg(\underline{0})$ and $\bar{g}(x) = \mathcal{O}(||x||^2)$. Then $f = g \circ \ell$ can be written as

$$f(x) = AR(-2\gamma \ln||x|| + c_1 + c_3)x + AR(c_2)\eta(x) + \bar{g}(\ell(x)).$$

Rescaling x as $x \to x\exp[(c_1 + c_3)/(\gamma 2)]$ we can eliminate the constant term in the argument of the rotation matrix of the dominant term of f. So, in the following we will just omit this constant. Using that $||D\ell||$ is bounded near zero and $||D\bar{g}(x)|| = \mathcal{O}(||x||)$ we conclude that there exists a constant $c_4 > 0$ such that $||D\bar{g} \circ \ell(x)|| < c_4||x||$. Therefore we can write

$$f(x) = F(x) + G(x), \quad \text{with } F(x) = AR(-2\gamma \ln||x||)x, \tag{19}$$

where

$$G(x) = Dg(\underline{0})\eta(x) + \bar{g}(\ell(x)) \tag{20}$$

is continuously differentiable in a neighborhood of the origin and satisfies

$$||G(x)|| < K_1||x||^2, \quad ||DG(x)|| < K_2||x||, \tag{21}$$

for some positive constants $K_1 > 0$ and $K_2 > 0$. Now we have the following proposition concerning the extension of f to the whole plane.

Proposition 1. *Given an $\varepsilon > 0$ there exists $\delta > 0$ and a homeomorphism $f_\varepsilon : \mathbb{R}^2 \to \mathbb{R}^2$*

$$f_\varepsilon(x) = F(x) + G_\varepsilon(x)$$

with the following properties:

(i) *f_ε is C^∞ except at the origin and it is analytic outside the sets $x = \underline{0}$, $\delta \leqslant ||x|| \leqslant 2\delta$*;
(ii) *f_ε coincides with f inside a ball of radius δ centered at the origin*;
(iii) *f_ε coincides with F outside a ball of radius 2δ centered at the origin*;
(iv) *f_ε is $\varepsilon - C^1$-close to F on the plane, namely* $\sup\{||G_\varepsilon(x)|| + ||DG_\varepsilon(x)|| : x \in \mathbb{R}^2\} < \varepsilon$ *and, moreover, for $||x|| < \delta$, G_ε satisfies inequalities* (21).

Proof. Given $\delta > 0$ let $\xi : R \to \mathbb{R}$ be a C^∞ function such that $\xi(s) = 1$ if $|s| \leqslant \delta$, $\xi(s) = 0$ if $|s| \geqslant 2\delta$, and the derivative, ξ', of ξ satisfies $|\xi'(s)| < \tilde{c}_1/\delta$ for some $\tilde{c}_1 > 0$. Then define $f_\varepsilon(x) = F(x) + G_\varepsilon(x)$ where $G_\varepsilon(x) = \xi(||x||)G(x) = (0,0)$ if $\xi(||x||) = 0$. The proposition follows from inequalities (21) that imply

$$||\xi(x)G(x)|| < K_1 4\delta^2,$$

$$||(D\xi(||x||))G(x) + \xi(||x||)DG(x)|| < \frac{\tilde{c}_1}{\delta} K_1 4\delta^2 + K_2 2\delta. \qquad \square \tag{22}$$

Let us denote the extension f_ε written in coordinates (3) by $\tilde{f}_\varepsilon$. Writing the change of coordinates (3) in a concise way as $x = H(\tilde{x})$, with $\tilde{x} = (\phi, z)$, we define $\tilde{G}_\varepsilon = H^{-1} \circ (F \circ H + G_\varepsilon \circ H) - \tilde{F}$, where $\tilde{F} = H^{-1} \circ F \circ H$. Proposition 1 can be written in these new coordinates as the following.

Proposition 2. *Given an $\varepsilon > 0$ there exists $\delta > 0$ and a homeomorphism*

$$\tilde{f}_\varepsilon(\tilde{x}) = \tilde{F}(\tilde{x}) + \tilde{G}_\varepsilon(\tilde{x}) \tag{23}$$

with the following properties:

(i) *$\tilde{f}_\varepsilon$ is C^∞ in the whole cylinder and real analytic outside the annulus $2\gamma \ln \delta \leqslant z \leqslant 2\gamma \ln(2\delta)$*,
(ii) *$\tilde{f}_\varepsilon$ coincides with $\tilde{f}$ for $z \leqslant 2\gamma \ln \delta$*,
(iii) *$\tilde{f}_\varepsilon$ coincides with $\tilde{F}$ for $z \geqslant 2\gamma \ln(2\delta)$*,
(iv) *$\tilde{f}_\varepsilon$ is $\varepsilon - C^1$-close to $\tilde{F}$ on the cylinder with $\varepsilon = 2(c_7 + c_8)\delta$, namely* $\sup\{||\tilde{G}_\varepsilon(\tilde{x})|| + ||D\tilde{G}_\varepsilon(\tilde{x})|| : \tilde{x} \in S^1 \times \mathbb{R}\} < \varepsilon$ *and, moreover, $||\tilde{G}_\varepsilon(\tilde{x})|| \leqslant c_7 e^{z/(2\gamma)}$ and $||\tilde{D}G_\varepsilon(\tilde{x})|| \leqslant c_8 e^{z/(2\gamma)}$ where c_7 and c_8 are positive numbers that depend on γ, α,* and δ.

Proof. This proposition is a consequence of Proposition 1, the change of variables formula (3), and a simple but long computation which will be omitted. $\square$

3.3. Proof of Theorem 3

Let f_ε be the extension of f given by Proposition 1. The idea in the proof of Theorem 3 is to show that for a given but fixed $\varepsilon > 0$, independent of τ and C, the sequence of functions $h_{\varepsilon n} = f_\varepsilon^n \circ F^{-n} : W_F^{\mathrm{u}\rho\tau C} \to \mathbb{R}^2$ converges as $n \to \infty$ to a function h_ε that has properties (i)–(iv) of Theorem 3 after replacing f by f_ε. Then, using that $f(x)$ and $f_\varepsilon(x)$ coincide for $||x|| < \delta$ and that h_ε is close to the identity near the origin we conclude that in a sufficiently small ball B centered at the origin the function h appearing in the theorem is given by the restriction of h_ε to B. The diameter of this ball may depend on C and may vanish as $C \to \infty$. We remark that the proof in the case where $W_F^{\mathrm{u}\rho\tau C}$ is replaced by $W_F^{\mathrm{s}\rho\tau C}$ is essentially the same. It is enough to change the definition $h_{\varepsilon n} = f_\varepsilon^n \circ F^{-n}$ by $h_{\varepsilon n} = f_\varepsilon^{-n} \circ F^n$ and follow the same steps. So in the following we just consider the case of $W_F^{\mathrm{u}\rho\tau C}$ and prove the theorem for the extension f_ε of f.

Proposition 3. *Function F is Lipschitz with Lipschitz constant $k_1 \leqslant \alpha(1 + 2\gamma)$. Moreover, for any given $\varepsilon_1 > 0$ there exists $\varepsilon > 0$ such that f_ε is Lipschitz with Lipschitz constant $k_2 \leqslant \alpha(1 + 2\gamma) + \varepsilon_1$.*

Proof. For F we have

$$\begin{aligned} ||F(x) - F(y)|| &= ||AR(-2\gamma \ln||x||)x - AR(-2\gamma \ln||y||)y|| \\ &\leqslant ||A|| \; ||\mathrm{R}(-2\gamma \ln||x||)x - R(-2\gamma \ln||y||)y||. \end{aligned}$$

Using that $||A|| = \alpha$ and that $x \to R(-2\gamma \ln||x||)x$ is continuous and has a derivative with norm less than $1 + 2\gamma$ if $||x|| \neq 0$ (see Eq. (17) and the paragraph below) we conclude that F is Lipschitz and has Lipschitz constant less than $\alpha(1 + 2\gamma)$. The estimate on the Lipschitz constant of f_ε is a consequence of Proposition 1 item (iv). $\square$

The next lemma contains the main point of the proof.

Lemma 2. *Suppose that ρ, α, and γ satisfy inequality* (9) *of Theorem* 3 and $\bar{\tau}$ *is defined as in* (9). *Then there exists $\varepsilon > 0$, depending on ρ, α, and γ, such that for any given $r > 0$, $\tau \in [0, \bar{\tau}]$, and $C \in I_\tau$, the sequence $h_{\varepsilon n}(x) = f_\varepsilon^n \circ F^{-n}(x)$ converges to $h_\varepsilon(x)$ uniformly with respect to x, for $x \in W_F^{\mathrm{u}\rho\tau C}$ and $||x|| \leqslant r$. Moreover, the limit function $h_\varepsilon : W_F^{\mathrm{u}\rho\tau C} \to \mathbb{R}^2$ is continuous and satisfies $f_\varepsilon \circ h_\varepsilon = h_\varepsilon \circ F$.*

Remarks. The topology in $W_F^{\mathrm{u}\rho\tau C}$ is that induced by the topology of $\mathbb{R}^2$.

Since in the rest of this section we will only work with the extension f_ε of f we will omit the index ε both in f_ε and h_ε.

Proof. The following argument was taken from [27, Section 3]. In order to prove the lemma it is enough to show that h_n, $n \in \mathbb{N}$, is a Cauchy sequence, namely given an

$\varepsilon_3>0$ there exists $N>0$ such that $||h_{n+j}(x)-h_n(x)||<\varepsilon_3$ if $n\geqslant N$, $j\geqslant 0$, for all $x\in W_F^{\mathrm{u}\rho\tau C}$ and $||x||\leqslant r$. Using that the Lipschitz constant of f is k_2 we get

$$\begin{aligned}||h_{n+j}(x)-h_n(x)|| &= ||f^n\circ f^j\circ F^{-j-n}(x)-f^n\circ F^{-n}(x)||\\ &\leqslant k_2^n||f^j\circ F^{-j-n}(x)-F^{-n}(x)|| = k_2^n||f^j(y)-F^j(y)||,\end{aligned}$$

where $y=F^{-j-n}(x)$. Now, denoting $f\circ f$ as ff, etc., we get

$$\begin{aligned}||f^j-F^j|| &= ||f^{j-1}f-f^{j-1}F+f^{j-2}fF-f^{j-2}FF\ldots+fF^{n-1}-FF^{n-1}||\\ &\leqslant ||f^{j-1}f-f^{j-1}F||+||f^{j-2}fF-f^{j-2}FF||\ldots+||fF^{j-1}-FF^{j-1}||\\ &\leqslant k_2^{j-1}||G||+k_2^{j-2}||GF||+\cdots+k_2^0||GF^{j-1}|| = \sum_{l=1}^{j} k_2^{j-l}||GF^{l-1}||.\end{aligned}\tag{24}$$

From these inequalities, Proposition 1, $||G_\varepsilon(x)||<K_1||x||^2$ (inequality (21)), and the definition of $W_F^{\mathrm{u}\rho\tau C}$ (inequality (8)) we get

$$\begin{aligned}||h_{n+j}(x)-h_n(x)|| &\leqslant k_2^n||f^j(y)-F^j(y)||\leqslant k_2^n\sum_{l=1}^{j}k_2^{j-l}||GF^{l-1}(y)||\\ &\leqslant K_1k_2^{n+j}\sum_{l=1}^{j}k_2^{-l}||F^{l-1}(y)||^2 = K_1k_2^{n+j}\sum_{l=1}^{j}k_2^{-l}||F^{-j-n+l-1}(x)||^2\\ &\leqslant C^2||x||^2K_1k_2^{n+j}e^{-2(j+n+1)(\rho-\tau)\pi/\gamma}\sum_{l=1}^{j}k_2^{-l}e^{2l(\rho-\tau)\pi/\gamma}\\ &= C^2||x||^2K_1e^{-2(\rho-\tau)\pi/\gamma}e^{-\beta(n+j)}\sum_{l=1}^{j}e^{\beta l}\\ &= \frac{C^2||x||^2K_1e^{-2(\rho-\tau)\pi/\gamma}}{1-e^{-\beta}}e^{-\beta n}(1-e^{-\beta j}),\end{aligned}\tag{25}$$

where

$$\beta \stackrel{\text{def}}{=} \frac{2(\rho-\tau)\pi}{\gamma}-\ln k_2.$$

Inequality (9), the definition of $\bar\tau$ in (9), the inequality $0\leqslant\tau\leqslant\bar\tau$, and Proposition 3 imply that we can choose an $\varepsilon>0$ sufficiently small, not depending on r, τ, and C, such that $\beta>0$. So, we consider f with this choice of ε. Then it is clear from the inequality above that given any $r>0$ and any $\varepsilon_3>0$ we can find an integer N sufficiently large such that $||h_{n+j}(x)-h_n(x)||<\varepsilon_3$ for all $||x||\leqslant r$, $n>N$, and $j>0$. The statements in the lemma about the continuity of h and the property $f\circ h=h\circ F$

follow from the uniform convergence of the limit $h_n \to h$ with respect to x and the continuity of f and F. $\square$

Lemma 3. *Let h be the function defined in Lemma 2. Then there exists $K_3 > 0$ such that $h(x) = x + R(x)$ where $||R(x)|| < K_3 ||x||^2$ for $x \in W_F^{\mathrm{u}\rho\tau C}$.*

Proof. Let $R_n(x) = h_n(x) - x$ for $x \in W_F^{\mathrm{u}\rho\tau C}$. Notice that $R_n \to R$ as $n \to \infty$. Moreover, if $y = F^{-n}(x)$ then

$$||R_n(x)|| = ||f^n \circ F^{-n}(x) - x|| = ||f^n(y) - F^n(y)|| \leqslant \sum_{l=1}^{n} k_2^{n-l} ||GF^{l-1}(y)||,$$

where we used inequality (24). Thus the same reasoning as used in (25) leads us to

$$\begin{aligned} ||R_n(x)|| &\leqslant \sum_{l=1}^{n} k_2^{n-l} ||GF^{l-1-n}(x)|| \leqslant K_1 k_2^n \sum_{l=1}^{n} k_2^{-l} ||F^{l-1-n}(x)||^2 \\ &\leqslant C^2 K_1 k_2^n e^{-2(n+1)(\rho-\tau)\pi/\gamma} \sum_{l=1}^{n} k_2^{-l} e^{2l(\rho-\tau)\pi/\gamma} ||x||^2 \\ &= \frac{C^2 K_1 e^{-2(\rho-\tau)\pi/\gamma}}{1 - e^{-\beta}} (1 - e^{-\beta n}) ||x||^2. \end{aligned}$$

Using that $\beta > 0$ as in Lemma 2 and taking the limit as $n \to \infty$ in this last inequality we get $||R(x)|| \leqslant [(C^2 K_1 e^{-2\rho\pi/\gamma})/(1 - e^{-\beta})] ||x||^2 \overset{\mathrm{def}}{=} K_3 ||x||^2$. $\square$

Lemma 4. *Let $h : W_F^{\mathrm{u}\rho\tau C} \to \mathbb{R}^2$ be the function defined in lemma 2 and let Σ be the set of image points of h. Then there exists a continuous function $h^{-1} : \Sigma \to W_F^{\mathrm{u}\rho\tau C}$ which is the inverse of h (again the topology in Σ is that induced by the topology of $\mathbb{R}^2$).*

Proof. Let $\psi_n = F^n \circ f^{-n}$ be a sequence of functions defined over Σ and $r > 0$ be any given number. We claim that the limit $||\psi_n \circ h(x) - x|| \to 0$ as $n \to \infty$ converges uniformly with respect to $x \in W_F^{\mathrm{u}\rho\tau C}$ for $||x|| \leqslant r$. Therefore $\psi_n \to h^{-1}$ as $n \to \infty$ and h^{-1} is continuous.

Lemma 2 implies

$$||\psi_n \circ h(x) - x|| = ||F^n \circ f^{-n} \circ h(x) - x|| = ||F^n \circ h \circ F^{-n}(x) - x||.$$

Defining $y = F^{-n}(x)$, using that the Lipschitz constant of F is k_1 (Proposition 3), using Lemma 3, and the definition of $W_F^{\mathrm{u}\rho\tau C}$ we get

$$\begin{aligned} ||\psi_n \circ h(x) - x|| &= ||F^n \circ h(y) - F^n(y)|| \leqslant k_1^n ||h(y) - y|| \leqslant K_3 k_1^n ||y||^2 \\ &= K_3 k_1^n ||F^{-n}(x)||^2 \leqslant K_3 C^2 ||x||^2 k_1^n e^{-2n(\rho-\tau)\pi/\gamma} \leqslant K_3 C^2 ||x||^2 e^{-n\beta}, \end{aligned}$$

where in the last inequality we used that $k_1 \leqslant k_2$. Since $\beta > 0$ we prove the claim and also the lemma. $\square$

We still have to show that the set Σ is contained in $W_f^{\mathrm{u}\rho}$. This is essentially a consequence of Lemmas 2 and 3 that imply

$$\begin{aligned}\frac{\gamma}{\pi}\lim_{n\to-\infty}\frac{\ln\|f^n\circ h(x)\|}{n} &= \frac{\gamma}{\pi}\lim_{n\to-\infty}\frac{\ln\|h\circ F^n(x)\|}{n}\\ &= \frac{\gamma}{\pi}\lim_{n\to-\infty}\frac{\ln\|F^n(x)+R[F^n(x)]\|}{n} = \frac{\gamma}{\pi}\lim_{n\to-\infty}\frac{\ln\|F^n(x)\|}{n}\\ &\quad + \frac{\gamma}{\pi}\lim_{n\to-\infty}\frac{1}{n}\ln\left\{\frac{\|F^n(x)+R[F^n(x)]\|}{\|F^n(x)\|}\right\} = \rho.\end{aligned}$$

Finally, property (iv) is an immediate consequence of our definition of $h(x) = \lim_{n\to\infty} f^n\circ F^{-n}(x)$.

3.4. *Proof of Theorem 4*

Property (ii) stated in Theorem 4 and the injectivity of mapping h, given in Eq. (10), are immediate consequences of the definition of h. So, in order to prove the theorem it is enough to show that h is surjective. Again we restrict attention to the case where h is defined on $W_F^{\mathrm{u}\rho\infty}$ the other one (h defined on $W_F^{\mathrm{s}\rho\infty}$) is analogous. Here f stands for the mapping defined on U and not for its extension as in the previous section.

Let $\psi_n = F^n\circ f^{-n}$ be a sequence of functions defined on $W_f^{\mathrm{u}\rho}$, for $n = 0, 1, \ldots$. Given $y \in W_f^{\mathrm{u}\rho}$ and $\tau \in]0, \bar{\tau}]$, where $\bar{\tau}$ is defined in Eq. (9), there exists a value of C such that $f^n(y)$ satisfies inequality (8). Then an argument similar to the one used to prove Lemma 2 implies that the limit $\psi(y) = \lim_{n\to\infty}\psi_n(y) = x \in \mathbb{R}^2$ exists and $F\circ\psi = \psi\circ f$.

Lemma 5. *$\psi(y) \in W_F^{\mathrm{u}\rho}$ for every $y \in W_f^{\mathrm{u}\rho}$.*

Proof. Let $R_n(y_j) = \psi_n(y_j) - y_j$, where $y_j = f^j(y_0)$, $j \leqslant 0$, and $y_0 = y$ is any given point in $W_f^{\mathrm{u}\rho}$. Let us choose a $\tau \in]0, \bar{\tau}]$ such that $\tau < \rho/3$, and let C be a sufficiently large number such that $f^l(y_0)$, $l \leqslant 0$, satisfies inequality (8) for these values of τ and C. We claim that $\|R_n(y_j)\| \leqslant K' e^{2j(\rho-\tau)\pi/\gamma}$, for any $j \leqslant 0$, $n \geqslant 0$, where K' is a number that does not depend on j and n. Indeed, as in the proof of Lemmas 2 and 3 we have

$$\begin{aligned}\|R_n(y_j)\| &= \|F^n\circ f^{-n}(y_j) - y_j\| = \|F^n(y_{j-n}) - f^n(y_{j-n})\|\\ &\leqslant \sum_{l=1}^{n} k_1^{n-l}\|Gf^{l-1}(y_{j-n})\| \leqslant K_1 k_1^n \sum_{l=1}^{n} k_1^{-l}\|f^{l-1-n+j}(y_0)\|^2\end{aligned}$$

$$\begin{aligned}
&\leqslant C^2 K_1 k_1^n e^{-2(n-j+1)(\rho-\tau)\pi/\gamma} \sum_{l=1}^{n} k_1^{-l} e^{2l(\rho-\tau)\pi/\gamma} ||y_0||^2 \\
&= \frac{C^2 K_1 e^{-2(\rho-\tau)\pi/\gamma}}{1-e^{-\beta}} (1-e^{-\beta n}) ||y_0||^2 e^{2j(\rho-\tau)\pi/\gamma} \\
&\leqslant \frac{C^2 K_1 e^{-2(\rho-\tau)\pi/\gamma}}{1-e^{-\beta}} ||y_0||^2 e^{2j(\rho-\tau)\pi/\gamma} = K' e^{2j(\rho-\tau)\pi/\gamma},
\end{aligned}$$

where we used the same reasoning as is in inequality (24), that $k_1 < k_2$ (Proposition 3), and the definition of $\beta > 0$ as in Lemma 2. Then, taking the limit as $n \to \infty$ we get $R(y_j) = \psi(y_j) - y_j$, with $||R(y_j)|| \leqslant K' e^{2j(\rho-\tau)\pi/\gamma}$, where K' does not depend on j. Now, the relation $F^j \circ \psi(y_0) = \psi \circ f^j(y_0)$ implies

$$\begin{aligned}
\frac{\gamma}{\pi} \lim_{j \to -\infty} \frac{\ln ||F^j \circ \psi(y_0)||}{j} &= \frac{\gamma}{\pi} \lim_{j \to -\infty} \frac{\ln ||\psi \circ f^j(y_0)||}{j} = \frac{\gamma}{\pi} \lim_{j \to -\infty} \frac{\ln ||y_j + R(y_j)||}{j} \\
&= \rho + \frac{\gamma}{\pi} \lim_{j \to -\infty} \frac{1}{j} \ln \left\{ \frac{||y_j + R(y_j)||}{||y_j||} \right\}.
\end{aligned}$$

In order to finish the proof we have to show that this last limit is zero. This is a consequence of $||R(y_j)|| \leqslant K' e^{2j(\rho-\tau)\pi/\gamma}$, inequality (8), and our choice of $\tau < \rho/3$, which imply

$$0 \leqslant \frac{||R(y_j)||}{||y_j||} \leqslant \frac{K' e^{2j(\rho-\tau)\pi/\gamma}}{C^{-1} e^{j(\rho+\tau)\pi/\gamma} ||y_0||} = \frac{K' e^{j(\rho-3\tau)\pi/\gamma}}{||y_0||} \to 0 \quad \text{as } j \to -\infty. \qquad \square$$

Now, $\psi(y) \in W_F^{\mathrm{u}\rho}$ implies that $\psi(y) \in W_F^{\mathrm{u}\rho\tau C}$ for some $\tau \in [0, \bar{\tau}]$, $C \in I_\tau$. Then, for a sufficiently large n, $F^{-n}(\psi(y)) \in B$, where B is the ball given in Theorem 3 for this values of τ and C. Moreover, by Theorem 3 and Lemma 4, inside B the inverse of h is given by ψ. Therefore, for $0 \leqslant j \leqslant n$

$$f^j \circ h \circ F^{-n} \circ \psi(y) = f^j \circ h \circ \psi \circ f^{-n}(y) = f^{j-n}(y) \in W_f^{\mathrm{u}\rho}. \tag{26}$$

This implies that $\psi(y) \in W_F^{\mathrm{u}\rho\infty}$. Finally, the definition of $h : W_F^{\mathrm{u}\rho\infty} \to W_f^{\mathrm{u}\rho}$ and the reasoning in Eq. (26) with $j = n$ imply that $y = h \circ \psi(y)$. So, h is surjective and its inverse is given by ψ.

3.5. *Proof of Theorem 5*

In order to simplify the notation we will suppose that $\hat{x}$ is a fixed point of $\hat{F}$ with vertical rotation number equal to one. The proof in the case where $\hat{x}$ is an n-periodic orbit of $\hat{F}$ is similar after replacing F by F^n, $\hat{F}$ by $\hat{F}^n$, etc.

For a given open neighborhood V of $x \in W_F^{\mathrm{u}}$ and $k \in \mathbb{N}$ we define

$$\sigma_{FVk}(x) = \frac{m(B_k \cap [\bigcup_{n=0}^{\infty} F^{-n}(V)])}{m(B_k)}. \tag{27}$$

If $V \subset V'$ then $\sigma_{FVk}(x) \leqslant \sigma_{FV'k}(x)$ for all $k \geqslant 0$. So, in order to prove Theorem 5 it is enough to show that given any neighborhood V' of x we can find another one $V \subset V'$ such that $\liminf_{k\to\infty} \sigma_{FVk}(x) = \sigma_F(x)$. This set V is constructed in the following way. The fact that $\hat{x} = \pi(x)$ is a sink and $\zeta_{-1} \circ F(x) = x$ imply that we can find an arbitrarily small neighborhood V of x such that $F(\bar{V}) \subset \zeta_1(V)$. Let A_k and S_k be the following sets:

$$A_k = B_k \cap \left[\bigcup_{n=0}^{n=\infty} F^{-n}(V) \right] = \{ y \in B_k : F^n(y) \in V, \text{ for some } n \geqslant 0 \}$$

and

$$S_k = B_k \cap \left[\bigcup_{n\in\mathbb{Z}} \zeta_n(S) \right] = B_k \cap (\pi^{-1}\hat{S}),$$

where $\hat{S}$ is the basin of attraction of $\hat{x}$. Notice that

$$\frac{m(S_k)}{m(B_k)} = \frac{m(S)}{m(\mathscr{A})}, \quad \text{and} \quad A_k \subset S_k$$

which imply

$$\frac{m(A_k)}{m(B_k)} \leqslant \frac{m(S)}{m(\mathscr{A})}$$

and $\liminf_{k\to\infty} \sigma_{FVk}(x) \leqslant \sigma_F(x)$. Now, we will prove the opposite inequality. Given any $\varepsilon > 0$ there exists a compact subset S_ε of S such that $m(S_\varepsilon) > m(S) - \varepsilon$. Let

$$S_{\varepsilon k} = B_k \cap \left[\bigcup_{n\in\mathbb{Z}} \zeta_n(S_\varepsilon) \right] = B_k \cap (\pi^{-1}\hat{S}_\varepsilon).$$

We claim that $S_{\varepsilon k} \subset A_k$ if k is sufficiently large. Indeed, if $y \in S$ then there exists $n_y > 0$ and $k_y \in \mathbb{Z}$ such that $F^{n_y}(y) \in \zeta_{k_y}(V)$. This implies that for $k > k_y$

$$F^{n_y+k-k_y}(\zeta_{-k}(y)) \in V \Rightarrow \zeta_{-k}(y) \in A_k.$$

Due to the compacity of S_ε the number $K(\varepsilon) = \sup_{y\in S_\varepsilon} k_y$ is finite. Therefore $\zeta_{-k}(S_\varepsilon) \subset A_k$ for $k > K_\varepsilon$ which implies the claim. So, given $\varepsilon > 0$ there exists $K = K(\varepsilon)$

such that for all $k \geqslant K$

$$\frac{m(S)-\varepsilon}{m(\mathscr{A})} \leqslant \frac{m(S_\varepsilon)}{m(\mathscr{A})} = \frac{m(S_{\varepsilon k})}{m(B_k)} \leqslant \frac{m(A_k)}{m(B_k)}.$$

This implies $\liminf_{k\to\infty} \sigma_{FVk}(x) \geqslant \sigma_F(x)$ which proves the theorem.

3.6. *Proof of Theorem 6*

First we consider the hyperbolic case:

As in the proof of Theorem 5 we suppose that $\hat{x}$ is a fixed point of $\hat{F}$ with vertical rotation number equal to one.

Let $\hat{V}$ be a small neighborhood of $\hat{x}$ such that $\hat{F}(\hat{V}) \subset \hat{V}$. Let $\{x_k, k \in \mathbb{Z}\} = \pi^{-1}(\hat{x})$ and $\{V_k, k \in \mathbb{Z}\} = \pi^{-1}(\hat{V})$ be such that $x_k \in V_k$. Theorem 2 and Proposition 2 imply that we can choose $\|x_0\|$ sufficiently small such that to each x_k, $k \leqslant 0$, there corresponds a y_k according to Theorem 2 such that $y_k \in V_k$ and $f(\zeta_{-1}\bar{V}_k) \subset V_k$. This last inclusion and Theorem 2 imply that $f^j(\zeta_{-j}V_0) = f^j(V_{-j}) \stackrel{\text{def}}{=} V_0^j$, $j \geqslant 0$, is a sequence of nested neighborhoods of y_0 such that $\operatorname{diam}(V_0^j) \to 0$ as $j \to \infty$. Let $\sigma_{fV_0^jk}(y_0)$ be defined as in (27) after replacing F by f. We will show that

$$\lim_{j\to\infty} \liminf_{k\to\infty} \sigma_{fV_0^jk}(y_0) \geqslant m(S)/m(\mathscr{A}) = \sigma_F(y_0). \tag{28}$$

This inequality and the fact $\sigma_{fV'k}(y_0) \geqslant \sigma_{fVk}(y_0)$ if $V' \supset V$ imply the theorem. So in the following we prove inequality (28).

Given $\varepsilon > 0$ let S_ε and $S_{\varepsilon k}$ be the sets defined in the proof of Theorem 5. Then there exist positive integers $l(\varepsilon) > 0$, $i(\varepsilon) > 0$, such that

$$F^l(S_{\varepsilon k}) \subset \left(\bigcup_{n \in \mathbb{Z}} V_n \right) \cap B_{k+i}.$$

This, the compacity of $S_{\varepsilon k}$, and Proposition 2 imply that there exists an integer $\bar{k}(\varepsilon)$ such that

$$f^l(S_{\varepsilon k}) \subset \left(\bigcup_{n \leqslant 0} V_n \right) \cap B_{k-i},$$

for all $k > \bar{k}$. Using that $f^n(V_{-n}) \subset V_0^j$, for $n \geqslant j$, we get that

$$S_{\varepsilon k} \subset \left[\bigcup_{n \geqslant 0} f^{-n}(V_0^j) \right] \cap B_k$$

if $k > \max\{\bar{k}, i+j\} \overset{\text{def}}{=} K$. Therefore, for any given $\varepsilon > 0$ and $j > 0$ there exists $K(\varepsilon, j)$ such that

$$\sigma_{fV_0^jk}(y_0) = \frac{m(B_k \cap [\bigcup_{n=0}^{\infty} F^{-n}(V_0^j)])}{m(B_k)} \geqslant \frac{m(S_{\varepsilon k})}{m(B_k)} = \frac{m(S_\varepsilon)}{m(\mathscr{A})} \geqslant \frac{m(S) - \varepsilon}{m(\mathscr{A})}$$

for all $k > K$. This implies that $\lim_{j\to\infty} \liminf_{k\to\infty} \sigma_{fV_0^jk}(y_0) \geqslant (m(S)/m(\mathscr{A})) - \varepsilon$ for all $\varepsilon > 0$. So inequality (28) is true.

To prove the assertion related to the existence of a topological sink for $\hat{F}$ we have to notice that:

There exists a small neighborhood $\hat{V}$ of $\hat{x}$ such that $\hat{F}(\hat{V}) \subset \hat{V}$. Again, as we did above, let $\{x_k, k \in \mathbb{Z}\} = \pi^{-1}(\hat{x})$ and $\{V_k, k \in \mathbb{Z}\} = \pi^{-1}(\hat{V})$ be such that $x_k \in V_k$. From the fact that $F(\zeta_{-1}\bar{V}_k) \subset V_k$ and $\|f(x) - F(x)\| < K_1\|x\|^2$ we get that if we choose $\|x_0\|$ sufficiently small, then $f(\zeta_{-1}\bar{V}_k) \subset V_k$, for all $k \leqslant 0$. The only thing that might be different from the hyperbolic case is that $V_0^j = f^j(\zeta_{-j}V_0)$, $j \geqslant 0$, is a sequence of nested neighborhoods such that it is not necessarily true that $\operatorname{diam}(V_0^j) \to 0$ as $j \to \infty$. So we have 2 possibilities:

(i) $\operatorname{diam}(V_0^j) \to 0$ as $j \to \infty$. Exactly as above we get that

$$\lim_{j\to\infty} \liminf_{k\to\infty} \sigma_{fV_0^jk}(y_0) \geqslant (m(S)/m(\mathscr{A})) > 0.$$

(ii) $\operatorname{diam}(V_0^j)$ does not converge to 0 as $j \to \infty$. In this case, as $\hat{x}$ is a topological sink for $\hat{F}$, we

Claim. *$\exists y \in V_0$ and an open neighborhood $y \in U_0 \subset V_0$, such that*:

(a) *$f(\zeta_{-1}\bar{U}_k) \subset U_k$, for all $k \leqslant 0$,*
(b) *$\operatorname{diam}(U_0^j) \to 0$ as $j \to \infty$, where $y \in U_0^j = f^j(\zeta_{-j}U_0)$.*

Clearly from what was done in the hyperbolic case and in the proof of Theorem 5, we get that $\sigma_f(y) > 0$.

The above claim is a consequence of the following results from [6]:

(1) If x is a q-periodic point for $\hat{F}$ with vertical rotation number $\frac{p}{q} > 0$, then $\det[D\hat{F}^q(x)] = e^{-p\pi/\gamma} < 1$.
(2) The topological index of a periodic point for $\hat{F}$ with non-null vertical rotation number can assume only the following values: $-1, 0, 1$.

So the theorem is proved.

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