Support of maximizing measures for typical $\mathcal{C}^0$ dynamics on compact manifolds

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Abstract

Given a compact manifold X, a continuous function $g : X \to \mathbb{R}$, and a map $T : X \to X$, we study properties of the T-invariant Borel probability measures that maximize the integral of g.

We show that if X is a n-dimensional connected Riemaniann manifold, with $n \geq 2$, then the set of homeomorphisms for which there is a maximizing measure supported on a periodic orbit is meager.

We also show that, if X is the circle, then the "topological size" of the set of endomorphisms for which there are g maximizing measures with support on a periodic orbit depends on properties of the function g. In particular, if g is $\mathcal{C}^1$, it has interior points.

Key words: periodic orbits, ergodic optimization, maximizing measures

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Addas-Zanata and Tal are partially supported by CNPq, grants: 301485/03-8 and 304360/05-8 respectively

1 Introduction

The last few years have seen an increasing interest in the field of ergodic optimization, a discipline that resides in the intersection of dynamical systems, ergodic theory and optimization theory.

Given a compact metric space X and a continuous self-map $T : X \to X$, we can consider the space $\mathcal{M}_T$ of the T-invariant Borel probability measures on X. The Krylov-Bogolyubov theorem asserts that $\mathcal{M}_T$ is not an empty set.

For a given continuous function $g : X \to \mathbb{R}$, it is natural to consider the functional

$$\mu \to \int g d\mu, \tag{1}$$

which associates to every T–invariant measure μ, its g-average.

Ergodic optimization is concerned with, given T and g, finding the maximum points for the above functional. Since $\mathcal{M}_T$ is convex and compact in the weak * topology, there exists at least one $\nu \in \mathcal{M}_T$ (called a g-maximizing measure) such that

$$\int_X g \; d\nu = \sup_{\mu \in \mathcal{M}_T} \int_X g \; d\mu,$$

and since the ergodic measures are the extremal points of $\mathcal{M}_T$ and the functional (1) is linear in the space of Borel measures on X, we can always find an ergodic g-maximizing measure.

As stated in [5], " ... the fundamental question of ergodic optimization is: What do maximizing measures look like?" This is meant to be understood as what are the typical properties one can expect to find in a maximizing measure. Yet, there is an openness to the question, since typical will greatly depend on the context, and different contexts have been studied to date.

A point that has been commonly studied is the support of a maximizing measure. There are several papers dealing with when can we expect the support of a maximizing measure to be small, e.g. a periodic orbit. Most studies were done for a fixed function T with some special dynamical property, like hyperbolicity or expansiveness, and g was allowed to vary in a sufficiently regular class. Usually one finds that the set of g for which a maximizing measure has support in a periodic orbit is typical. See, for instance, [1], [2], [3], [8], and specially [5] and the references therein.

Another approach, in some sense complementary to the one described above, is to take g to be a fixed function, and allow the dynamics to vary in some space. This is the line we pursue in this work, and that was studied in [6] and [7].

In [6] the case of a homeomorphism of a compact n-dimensional connected Riemannian manifold X is considered.

To describe this result, first let a distance in the space of homeomorphisms of X, denoted by $Hom(X)$, be defined in the following natural way. Given $T, F \in Hom(X)$, we define

$$d_{Hom}(T, F) = \max_{x \in X} \; d(T(x), F(x)),$$

where d is the distance in X which comes from the Riemannian metric.

Note that d_{Hom} is topologically equivalent to

$$D_{Hom}(T,F) = \max\{\max_{x\in X} d(T(x),F(x)), \max_{x\in X} d(T^{-1}(x),F^{-1}(x))\},$$

a distance which makes $Hom(X)$ a complete metric space.

The main theorem proved in [6] is the following:

Theorem 1 *: For a fixed continuous function $g : X \to \mathbb{R}$, there exists a dense subset of $Hom(X)$ such that for all T in this subset, there is a g-maximizing T-invariant measure supported on a periodic orbit.*

The question we address here is a natural one: Is the dense subset described in the above theorem a residual set? The answer to the question is no. Before presenting our result, denote by $Hom_{Per(g)}(X) = \{T \in Hom(X)$: there exists a g-maximizing, T-invariant measure supported on a periodic orbit$\}$.

In section 2, we prove:

Theorem 2 *: Given a continuous locally non-constant function $g : X \to \mathbb{R}$, where the dimension of X, denoted n, is greater than or equal to 2, we have the following: The complement of $Hom_{Per(g)}(X)$ is a residual set.*

The hypothesis $n \geq 2$ is clearly fundamental, because the above result does not hold for homeomorphisms of the circle (there is a residual subset of homeomorphisms of the circle which have rational rotation number and so their invariant measures are all periodic).

We supposed that X is a compact connected Riemannian manifold because the proof of theorem 1 in [6] uses a property of X which is trivially true in the Riemannian manifold setting, but we note that, with minor changes, the proof of theorem 2 still stands if instead of assuming that X is a riemannian manifold, we just assume that X is a compact connected metric space, with the following properties:

- Given any finite set of points $\{x_1, x_2, ..., x_n\}$ and any two different points of this set, say x_i and x_j, there exists a simple continuous arc $\gamma \subset X$, whose endpoints are x_i and x_j, such that γ avoids all the other points of the set $\{x_1, x_2, ..., x_n\}$ and $diam(\gamma) = dist(x_i, x_j)$.
- Given any open connected set Ω of X and two points x, y in Ω, there exists a homeomorphism of X, denoted h, such that h is the identity outside Ω and $h(x) = y$.

The perturbations used in many parts of the proof of theorem 2 only make sense if the above properties holds.

In [7], a different case was examined, where $X = S^1$ and T was taken as an endomorphism of the circle, i.e., T was a continuous surjective self-map of

S^1 onto itself. If we denote by $E(S^1)$ the space of circle endomorphisms and consider the usual distance in $E(S^1)$,

$$d_{E(S^1)}(T,Q) = \max_{x\in S^1} \mid T(x) - Q(x) \mid,$$

then $E(S^1)$ is a complete metric space. The result proved in [7] is similar to theorem 1, namely:

Theorem 3 *: For a fixed continuous function $g : S^1 \to \mathbb{R}$, there exists a dense subset of $E(S^1)$ such that for all T in this subset, there is a g-maximizing T-invariant measure supported on a periodic orbit.*

We further study this context in section 3. Here a new phenomenon appears: unlike the case of a homeomorphism of a n-dimensional ($n \geq 2$) connected manifold, the topological "size" of the set of endomorphisms of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ possessing a g-maximizing measure supported in a periodic orbit depends on the function g chosen. In particular, we show that:

Theorem 4 *: Fix a continuous function $g : S^1 \to \mathbb{R}$. We have two possibilities:*

a) g is weakly monotone in an interval of S^1. Then the set of endomorphisms of S^1 such that a g-maximizing measure has support in a single fixed point has a nonempty interior in the space of degree-1 endomorphisms.

b) g has a dense subset of strict local maxima (and, so, of strict local minima). Then the set of endomorphisms possessing a g-maximizing periodic orbit is meager.

Clearly, if g is $\mathcal{C}^1$, then we are in possibility a) above.

The proof of this result is in section 3, the final section of this paper.

2 Proof of Theorem 2

From the theorem hypothesis, given a connected compact Riemannian manifold X and a continuous function $g : X \to \mathbb{R}$, let us consider the following sets:

$Hom_k(X) = \{T \in Hom(X)$: there exists a g-maximizing T-invariant measure supported on a periodic orbit of period k and all periodic orbits of period less then k are not maximizing$\}$. The set which we proved to be dense in [6] is $\underset{k\geq 1}{\cup} Hom_k(X)$.

Theorem 2 will follow from the following lemma:

Lemma 1 *: For every integer $k > 0$, $Hom_k(X)^c$ contains an open and dense subset of $Hom(X)$.*

Proof of the lemma:

Given $k > 0$, we can suppose that $Hom_k(X) \neq \emptyset$, otherwise the result is trivial. Let $T_0 \in Hom_k(X)$.

We claim that, arbitrarily $\mathcal{C}^0$ close to T_0, there is a homeomorphism T_1 such that:

- T_1 has a period-m point y, topologically non degenerate, with $m > k$.
- The g average of the y orbit is strictly larger then the g average of any period-k orbit, i.e., for any period-k point z,

$$\frac{1}{m}\sum_{i=0}^{m-1} g({T_1}^i(y)) > \frac{1}{k}\sum_{i=0}^{k-1} g({T_1}^i(z)).$$

The proof of the claim will be made below. In order to see that the lemma follows from this claim, first we note that, clearly, $T_1 \notin Hom_k(X)$. Also, note that the two properties stated above are valid in open sets of homeomorphisms, therefore T_1 must be an interior point of $Hom_k(X)^c$. But T_1 was found arbitrarily close to any T_0 in $Hom_k(X)$, so the interior points of $Hom_k(X)^c$ are dense and this proves the lemma.

To prove the claim, let $T \in Hom(X)$ be a $\mathcal{C}^0$-arbitrarily small perturbation of T_0, which has finitely many period-k orbits, and such that

$$\sup_{\mu\in\mathcal{M}_T} \int_X g\ d\mu \text{ is also arbitrarily close to } \sup_{\mu\in\mathcal{M}_{T_0}} \int_X g\ d\mu.$$

This can be done in such a way as to ensure that all g-maximizing measures for T are not supported on periodic orbits of period strictly smaller then k.

If $T \notin Hom_k(X)$, we have 2 possibilities:

i) at least one g-maximizing measure for T is supported on a periodic orbit with period larger then k. In this case, we perturb T near this orbit, so that it becomes topologically non-degenerate and still has average larger then the g-average on all its period-k orbits. If we denote this new homeomorphism by T_1, we get that every homeomorphism sufficiently close to T_1 will not belong to $Hom_k(X)$

ii) all g-maximizing measures for T are not periodic. We pick one ergodic g-maximizing measure for T (denoted μ) and consider a typical point x_μ and a finite piece of the positive orbit of x_μ, $\{x_\mu, T(x_\mu), ..., T^{n_\mu}(x_\mu)\}$, such that, for a given $\varepsilon > 0$,

$$dist(x_\mu, T^{n_\mu}(x_\mu)) < \varepsilon \text{ and } \left|\frac{1}{n_\mu}\sum_{i=0}^{n_\mu-1} g(T^i(x_\mu)) - \int_X g d\mu\right| < \varepsilon,$$

and such that $n_\mu > k$. After an application of the C^0-closing lemma, we produce a homeomorphism T_1, such that $D_{Hom}(T_1, T) < 2\varepsilon$, with a topologically non-degenerate periodic orbit $\{x_\mu, T(x_\mu), ..., T^{n_\mu-1}(x_\mu)\}$ with period n_μ, which has g-average $\frac{1}{n_\mu}\sum_{i=0}^{n_\mu-1} g(T^i(x_\mu))$. If ε is chosen sufficiently small, the g average of the orbit of x_μ is greater then the g-average on all periodic orbits for T_1 with period less than or equal to k. The construction ensures this holds for every homeomorphism sufficiently close to T_1.

In both cases above the claim is proved.

The remaining case is $T \in Hom_k(X)$. In this case, we present a construction which should be performed in disjoint neighborhoods of each of the (now finite) maximizing period-k orbits.

Let $Orb(x) = \{x, T(x), ..., T^{k-1}(x)\}$ be one period-k orbit ($T^k(x) = x$) such that

$$\frac{1}{k} \sum_{i=0}^{k-1} \delta_{T^i(x)}$$

is a maximizing measure for the integral of $g : X \to \mathbb{R}$. Let $\epsilon > 0$ be sufficiently small such that

- $B_\epsilon(T^i(x)) \cap B_\epsilon(T^j(x)) = \emptyset$ for all $0 \leq i, j \leq k-1,\ i \neq j$,
- $T^i(B_\epsilon(x)) \cap B_\epsilon(x) = \emptyset,$ for $i = 1, 2, ..., k-1$,
- The set
$$E = \left(\bigcup_{i=0}^{k-1} T^i(B_\epsilon(x)) \right) \cup \left(\bigcup_{i=0}^{k-1} B_\epsilon(T^i(x)) \right)$$
does not contain any periodic point of period less or equal to k, except the orbit of x.

Now let us look at the partial average

$$y \in \left\{ B_\epsilon(x) \cap T^{-k}(B_\epsilon(x)) \right\} \to \frac{1}{k} \sum_{i=0}^{k-1} g \circ T^i(y) \stackrel{def}{=} \phi(y).$$

Continuity implies that, there exists $0 < \delta < \epsilon$ such that $B_\delta(x) \subset B_\epsilon(x) \cap T^{-k}(B_\epsilon(x))$. Consider ϕ restricted to $B_\delta(x)$. As g is not locally constant, we have the following proposition:

Proposition 1 *: There exists a non-empty open set $V \subset B_\delta(x)$, $x \notin V$, such that $T^k(V) \cap V = \emptyset$ (in other words, T^k is fixed point free in V) and a mapping $\widetilde{T} \in Hom(X)$, $\widetilde{T} = T$ in V^c such that $\widetilde{\phi} : B_\delta(x) \to \mathbb{R}$ given by*

$$\widetilde{\phi}(y) = \frac{1}{k} \sum_{i=0}^{k-1} g \circ \widetilde{T}^i(y)$$

is not constant.

Proof:

If ϕ is not constant in $B_\delta(x)$, then $\widetilde{T} = T$ and the proposition is proved. So, let us suppose that ϕ is constant in $B_\delta(x)$. Pick some $x' \in B_\delta(x)$, $x' \neq x$, and let $V \subset B_\delta(x)$ be a small open ball centered at x', disjoint from x, such that $T^k(V) \cap V = \emptyset$. Let $x'' \in V$ be such that $g(x') \neq g(x'')$, which exists because g is not locally constant. Assume, without loss of generality, that $g(x'') > g(x')$. Finally, let $h : X \to X$ be a homeomorphism such that $h \equiv Id$ in $T(V)^C$, which satisfies $h(T(x'')) = T(x')$ and let $\widetilde{T} = h \circ T$. Clearly, $\widetilde{\phi}(x'') > \widetilde{\phi}(x) = \phi(x)$. ■

Let $\min \widetilde{\phi}\mid_{\overline{B_\delta(x)}} = \widetilde{\phi}(x_{\min}) < \max \widetilde{\phi}\mid_{\overline{B_\delta(x)}} = \widetilde{\phi}(x_{\max})$, for some $x_{\min}, x_{\max} \in \overline{B_\delta}(x)$. Let $x_1, x_2 \in B_\delta(x), x_1 \neq x_2$, be points sufficiently close to $x_{\max}$ so that

$$\begin{gathered}\widetilde{\phi}(x_i) > \widetilde{\phi}(x_{\min}) + 9/10(\widetilde{\phi}(x_{\max}) - \widetilde{\phi}(x_{\min})), i \in \{1,2\}, \\ x_1 \text{ and } x_2 \text{ are not in the same orbit and, } x \neq x_i \text{ for } i = 1,2 \\ x_{\min} \notin \{x_1, x_2, \widetilde{T}^k(x_1), \widetilde{T}^k(x_2)\}.\end{gathered} \tag{2}$$

Remember that, as $\widetilde{T}^k(V) \cap V = \emptyset$, $\widetilde{T}$ has only one period-k point in $B_\delta(x)$, x itself. Let $h' : X \to X$ be a homeomorphism such that $h' \equiv Id$ in $B_\epsilon(x)^C$, which satisfies $h'(\widetilde{T}^k(x_1)) = x_2$ and $h'(\widetilde{T}^k(x_2)) = x_1$ and let $Q = h' \circ \widetilde{T}$. To construct such a homeomorphism, let $\alpha_1, \alpha_2 \in B_\varepsilon(x)$ be disjoint simple arcs, such that:

- $endpoints(\alpha_1) = \{\widetilde{T}^k(x_1), x_2\}$ and $endpoints(\alpha_2) = \{\widetilde{T}^k(x_2), x_1\}$

The homeomorphism h' is supported in the union of disjoint open neighborhoods of these arcs (both contained in $B_\varepsilon(x)$). Condition (2) implies that h' can be chosen so that it is the identity in a neighborhood of $x_{\min}$.

Our construction made the orbit of x_1 period-$2k$ for Q (actually, this periodic orbit is topologically non-degenerate if h' is properly chosen) and, the integral of g with respect to the Dirac measure supported in the orbit of x_1 is greater then $\widetilde{\phi}(x_{\min}) + 9/10(\widetilde{\phi}(x_{\max}) - \widetilde{\phi}(x_{\min}))$. Now, by an arbitrarily small perturbation applied to Q, supported in $\bigcup_{i=0}^{k-1} T^i(B_\epsilon(x))$, we obtain a homeomorphism $\widetilde{Q}$ with finitely many (N) period-k orbits contained in $\bigcup_{i=0}^{k-1} T^i(B_\epsilon(x))$. We denote by $y_j, 1 \leq j \leq N$, the period-k points in $B_\epsilon(x)$. Moreover, $\widetilde{Q} \equiv Q$ in a neighborhood of the orbit of x_1 and $\widetilde{Q}^k(B_\delta(x)) \subset B_\epsilon(x)$. Now let $z \in B_\delta(x)$ be such that

$$\frac{1}{k}\sum_{i=0}^{k-1} g \circ \widetilde{Q}^i(z) < \widetilde{\phi}(x_{\min}) + \frac{1}{10}(\widetilde{\phi}(x_{\max}) - \widetilde{\phi}(x_{\min}))$$

and let $\theta > 0$ be such that

$$\begin{gathered}B_\theta(\widetilde{Q}^i(z)) \subset \widetilde{Q}^i(B_\delta(x)), \text{ for } i = 0,1,...,k-1; \\ \left(\bigcup_{i=0}^{2k-1} \{\widetilde{Q}^i(x_1)\}\right) \cap \left(\bigcup_{i=0}^{k-1} B_\theta(\widetilde{Q}^i(z))\right) = \emptyset; \\ \tfrac{1}{k}\sum_{i=0}^{k-1} g(z_i) < \widetilde{\phi}(x_{\min}) + \tfrac{1}{5}(\widetilde{\phi}(x_{\max}) - \widetilde{\phi}(x_{\min})), \text{ for any choice of} \\ z_i \in B_\theta(\widetilde{Q}^i(z)), \ i = 0,1,...,k-1.\end{gathered}$$

Finally, let $\sigma : X \to X$ be a homeomorphism, $\sigma \equiv Id$ in

$$\left(\bigcup_{i=0}^{k-1} T^i(B_\epsilon(x))\right)^C = \left(\bigcup_{i=0}^{k-1} Q^i(B_\epsilon(x))\right)^C = \left(\bigcup_{i=0}^{k-1} \widetilde{Q}^i(B_\epsilon(x))\right)^C,$$

such that:

1) $\sigma \equiv Identity$ in a neighborhood of the orbit of x_1;

2) For each $(i,j) \in \{0,1,...,k-1\} \times \{1,...,N\}$, $\sigma(\widetilde{Q}^i(y_j)) \in B_\theta(\widetilde{Q}^i(z))$;

If we denote $T_1 = \sigma \circ \widetilde{Q} \circ \sigma^{-1}$, we get that the period-k orbits for T_1 contained in $B_\epsilon(x)$ are $w_j = \sigma(y_j), 1 \leq j \leq N$. Since

$$T_1^i(w_j) = \sigma(\widetilde{Q}^i(y_j)) \in B_\theta(\widetilde{Q}^i(z)),$$

the g-average of these period-k points is smaller then the g-average of the period-$2k$ orbit of x_1, which can not be destroyed by small perturbations applied to T_1. Since T_1 and T differ only in $\bigcup_{i=0}^{k-1} T^i(B_\epsilon(x))$, which from the choice of $\epsilon > 0$, is a disjoint union, we get that

$$d_{Hom}(T, T_1) < \max_{0 \leq i \leq k-1} \; diam(T^i(B_\epsilon(x))) \overset{\epsilon \to 0}{\to} 0,$$

and this proves lemma 1. Remember that this construction must be performed in disjoint neighborhoods of each of the (finite) g -maximizing period-k orbits for T. ■

3 Proof of Theorem 4

First let us identify S^1 with the interval $[-\frac{1}{2}, \frac{1}{2})$.

Proof of case a): We will assume, without loss of generality, that g is non-decreasing in the interval $I = (-\delta, \delta)$ for some $0 < \delta < 1/2$. Let T be the endomorphism of the circle with degree one such that its lift $\tilde{T} : \mathbb{R} \to \mathbb{R}$ satisfies $\tilde{T}(x+1) = \tilde{T}(x) + 1$ and :

$$\tilde{T}(x) = \begin{cases} -\delta/2, \text{ if } -\frac{1}{2} \leq x \leq -\delta/4 \\ 2x, \text{ if } -\delta/4 < x < \frac{1-\delta/2}{2} \\ 1 - \delta/2, \qquad \text{otherwise} \end{cases}$$

(see figure 1).

It follows from continuity that, if Q is $\mathcal{C}^0$ sufficiently close to T, and if $\tilde{Q}$ is the lift of Q close to $\tilde{T}$, then the following must happen:

- There is a point x_0 in $I = (-\delta, \delta)$ such that $Q(x_0) = x_0$ and $\tilde{Q}(y) > y, \forall y \in (x_0, \delta)$.
- $Q([-3\delta/4, -\delta/4]) \subset (-3\delta/4, -\delta/4)$.
- For all y in $(x_0, 1 - 3\delta/4)$, we have $y < \tilde{Q}(y) < 1 - \delta/4$.

From these considerations, if y is a point in S^1, then there are, at most, a finite number of positive integers n such that $Q^n(y)$ does not belong to $(-3\delta/4, x_0]$. But this implies that the g Birkhoff average of any Q-orbit is smaller then or equal to $\operatorname*{Max}_{x \in (-3\delta/4, x_0)} \{g(x)\} = g(x_0)$, which is the g average for the atomic measure supported at the Q fixed point x_0, thus showing the case.

Proof of case b): This case is very similar to the arguments shown in the previous section.

Claim *: If T has a g maximizing measure supported in a periodic orbit of period k, then arbitrarily close to T there is a open set of endomorphisms B such that for every $Q \in B$, there is a Q periodic point y of period $2k$ satisfying

$$\frac{1}{2k}\sum_{i=0}^{2k-1} g \circ Q^i(y) > \max_{x \in S^1, Q^k(x)=x} \frac{1}{k}\sum_{i=0}^{k-1} g \circ Q^i(x).$$

As in the proof of theorem 2, this shows that the set of endomorphisms with a g maximizing measure supported on a periodic orbit is meager.

For simplicity, we will only show the previous claim for the case $k = 1$.

Given an endomorphism T with a g maximizing measure δ_{x_0}, let us choose $\overline{T}$ an endomorphism arbitrarily close to T such that $\overline{T}$ has finitely many fixed points, $\overline{T}'$ (= the differential of $\overline{T}$) exists at each of these points, and is not equal to 1.

The adaptations for $k > 1$ follow the same ideas below, but the perturbations should be performed in disjoint neighborhoods of each point in all the period-k orbits of $\overline{T}$, of which there are finitely many by the choice of $\overline{T}$.

Let $\delta^* > 0$ be such that for all x^* and x', fixed under $\overline{T}$, $\mid x^* - x' \mid > 2\delta^*$. Also, given $\varepsilon > 0$, let $0 < \delta < \min\{\delta^*/2, \varepsilon/2\}$ be such that for all $\bar{x}$ fixed under $\overline{T}$, if $I = (\bar{x} - \delta, \bar{x} + \delta)$, then $\max_{x,y \in I} \mid \overline{T}(x) - \overline{T}(y) \mid < min\{\delta^*/10, \varepsilon/2\}$.

The remaining arguments should be performed in δ-neighborhoods of all the $\overline{T}$-fixed points.

Since the set of strict local maxima and minima of g are dense, we can take a closed interval $J = [a, b] \subset I$ such that a and b are local maxima, and let x_1 be the minimum of g in J. We recall that, since we are dealing with case b), no local maximum of g can also be a local minimum, otherwise there would be an interval where g was constant. Therefore x_1 belongs to the interior of J. Let $K \subset (x_1, b)$ be another closed interval whose endpoints are again local maxima, and let x_5 be the minimum of g in K. Again x_5 belongs to the interior of K. Finally, let $L \subset (x_1, x_5) \cap K$ be another closed interval, again with maxima at the endpoints, and let x_3 be the minimum of g in L. Choose x_2 and x_4 points in L, local maxima, such that $x_2 < x_3 < x_4$ (see figure 2).

These choices yield

$$g(x_1) \leq g(x_5) \leq g(x_3) < min\{g(x_2), g(x_4)\}.$$

We have 2 possibilities, $\overline{T}'(\bar{x}) < 1$ or $\overline{T}'(\bar{x}) > 1$.

If $\overline{T}'(\bar{x}) < 1$ then there is an endomorphism Q_1 which differs from $\overline{T}$ only in the interval I, such that $\max_{x \in I} \mid \overline{T}(x) - Q_1(x) \mid < \varepsilon$ and such that the only fixed point for Q_1 in I is x_3. Furthermore $Q_1(x_2) = x_4, Q_1(x_4) = x_2$, and the periodic orbit of x_2 is attracting (see figure 3).

If $\overline{T}'(\bar{x}) > 1$, then there is an endomorphism Q_2 which differs from T only in the interval I, such that $\max_{x \in I} \mid \overline{T}(x) - Q_1(x) \mid < \varepsilon$ and such that the only fixed points for Q_2 in I are x_1, x_3 and x_5. Furthermore $Q_2(x_2) = x_4, Q_2(x_4) = x_2$, and the periodic orbit of x_2 is attracting (see figure 4).

After performing the above type of perturbations in δ-neighborhoods of all the $\overline{T}$-fixed points, we get an endomorphism Q^* such that for any Q contained in a sufficiently small $\mathcal{C}^0$ open neighborhood of Q^*, claim * holds.

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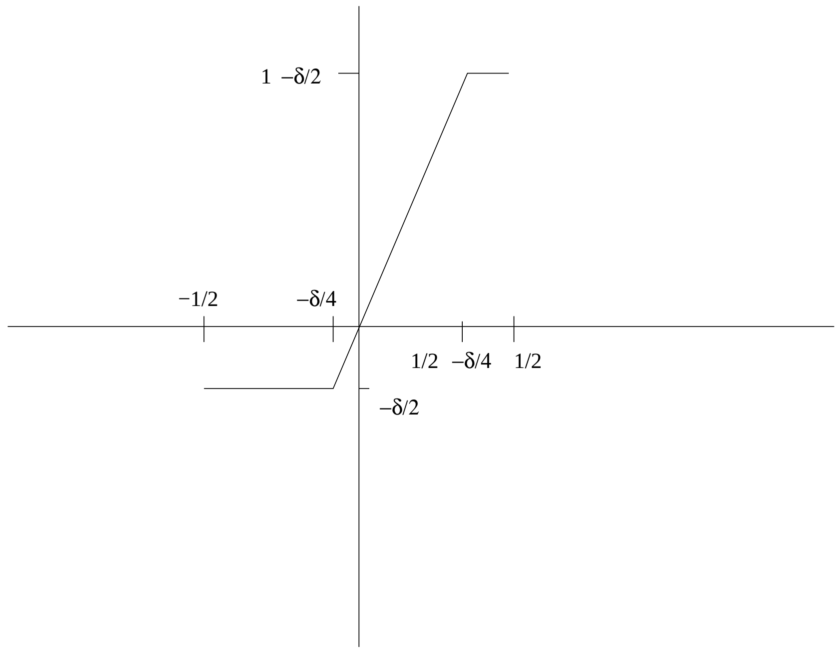
Figure captions.

Figure 1. Diagram showing the graph of T

Figure 2. Diagram showing the intervals I and J and points x_1, x_2, x_3, x_4, x_5

Figure 3. Diagram showing the graphs of T and Q_1

Figure 4. Diagram showing the graphs of T and Q_2


1 −δ/2
−1/2
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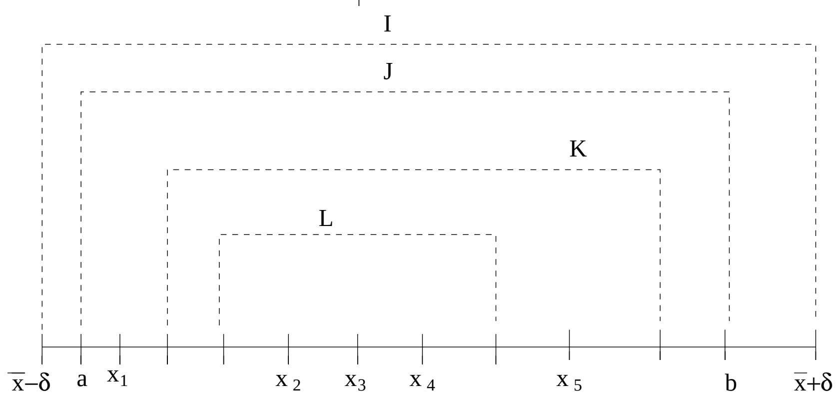
I
J
K
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x̄−δ
a
x1
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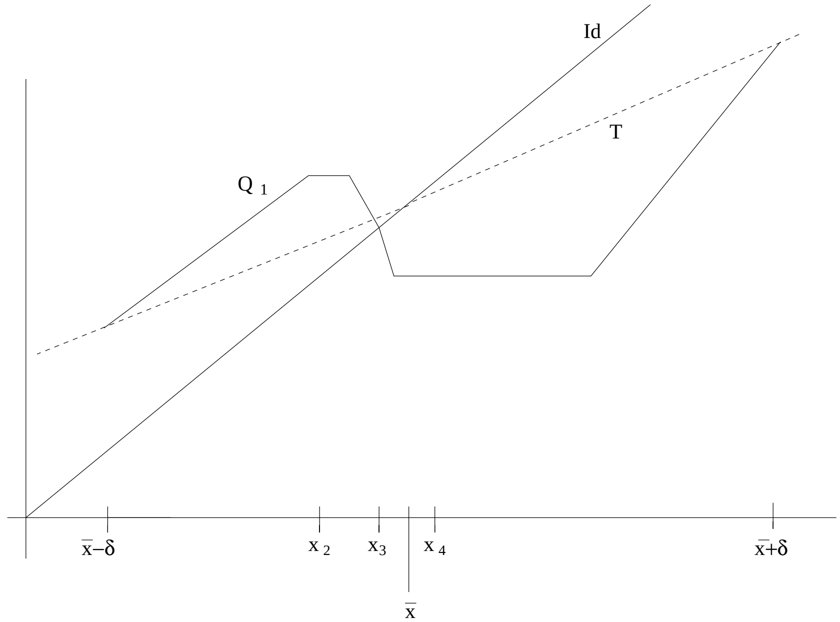
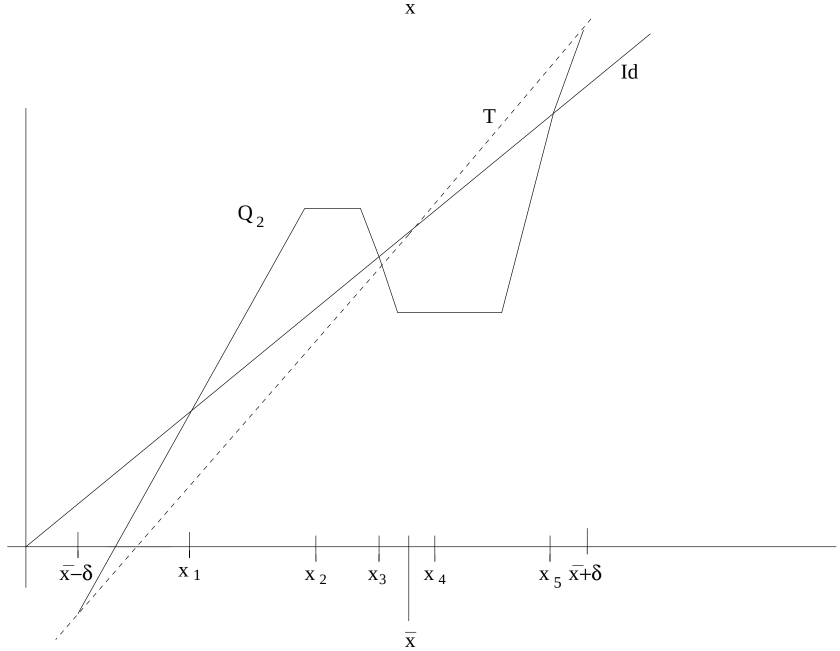
x5
b
x̄+δ


Id
T
Q 1
x̄−δ
x 2
x3
x 4
x̄+δ
x̄


Id
T
Q 2
x̄−δ
x 1
x 2
x3
x 4
x 5
x̄+δ
x̄