Some extensions of the Poincaré–Birkhoff theorem to the cylinder and a remark on mappings of the torus homotopic to Dehn twists

2005 Nonlinearity 18 2243

(http://iopscience.iop.org/0951-7715/18/5/018)

INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **18** (2005) 2243–2260 doi:10.1088/0951-7715/18/5/018

Some extensions of the Poincaré–Birkhoff theorem to the cylinder and a remark on mappings of the torus homotopic to Dehn twists

Salvador Addas-Zanata

Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, Cidade Universitária, 05508-090 São Paulo, SP, Brazil

E-mail: sazanata@ime.usp.br

Received 27 July 2004, in final form 15 June 2005
Published 8 July 2005
Online at stacks.iop.org/Non/18/2243

Recommended by C Liverani

Abstract

In this paper we prove some extensions of the Poincaré–Birkhoff theorem to $S^1 \times \mathbb{R}$. We consider mappings h of the cylinder which satisfy an 'infinity twist condition' and prove that under certain additional hypotheses, for every rational p/q, h has q-periodic orbits with rotation number p/q. And in many interesting cases these orbits have non-null topological indices so they appear at least in pairs. We also obtain, as a consequence of some of the above results, that in the area-preserving case, the subset of diffeomorphisms of the torus which have a periodic orbit is dense in the set of diffeomorphisms of the torus in any topology. Finally, we extend some theorems we already obtained for twist mappings to a more general setting.

Mathematics Subject Classification: 37C25, 37C50, 37E30, 37E45

1. Introduction and main results

The so called 'last geometric theorem of Poincaré' or Poincaré–Birkhoff theorem is one of the first results of its type in mathematics (see [5, 6, 29]). Roughly speaking, it says that an area-preserving, boundary component preserving homeomorphism $h : S^1 \times [0, 1] \to S^1 \times [0, 1]$ has at least two fixed points if the rotations in the boundaries have opposite signs. Since the first complete proofs of this result, given by Birkhoff in [5,6], many 'simpler proofs', generalizations and improvements have appeared. See for instance [4, 9, 11, 16, 17, 19, 22, 33], etc.

Our purpose here is to give some generalizations of the Poincaré–Birkhoff theorem to the cylinder and then apply these results to certain mappings on the torus.

Before stating the results we need some definitions.

Definitions.

- *Let* $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ *be a homeomorphism homotopic to the identity and let* $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ *be one of its lifts.*
- *Let* $p_1 : \mathbb{R}^2 \to \mathbb{R}$ *and* $p_2 : \mathbb{R}^2 \to \mathbb{R}$ *be the standard projections, respectively, in the* $\tilde{x}$ *and* $\tilde{y}$ *coordinates (*$p_1(\tilde{x}, \tilde{y}) = \tilde{x}$ *and* $p_2(\tilde{x}, \tilde{y}) = \tilde{y}$*). Projections of the cylinder are also denoted by* p_1 *and* p_2*.*
- *We say that* h *satisfies the 'infinity twist condition' (ITC) if for some lift* $\tilde{h}$ *the following holds:*

$$p_1 \circ \tilde{h}(\tilde{x}, \tilde{y}) \to \pm\infty, \qquad \text{as } \tilde{y} \to \pm\infty. \tag{1}$$

  Clearly, if (1) holds for some lift $\tilde{h}$*, then it holds for all lifts.*
- *Let* $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ *be the following covering mapping:*

$$\pi(\tilde{x}, \tilde{y}) = (\tilde{x} \bmod 1, \tilde{y}).$$

- *We say that* $(h, \tilde{h})$ *has a periodic orbit of rotation number* p/q*, for some rational number* p/q*, if there exists* $z \in S^1 \times \mathbb{R}$ *such that* $h^q(z) = z$ *and* $\tilde{h}^q(\tilde{z}) = \tilde{z} + (p, 0)$ *for all* $\tilde{z} \in \pi^{-1}(z)$*.*
- h *satisfies a 'strong intersection property' if for every open set* $U \subset S^1 \times \mathbb{R}$*, homeomorphic to the whole cylinder, such that* $S^1 \times]-\infty, a] \subset U \subset S^1 \times]-\infty, b]$*, for some real numbers* $a < b$ *we have* $h(\partial U) \cap U^c \neq \emptyset$ *and* $h(\partial U) \cap U \neq \emptyset$*. In a certain sense, this is a generalization of the 'usual intersection property', which says that the intersection of every homotopically non-trivial simple closed curve of the cylinder with its image under h is non-empty. In this strong version we demand that the above is true not only for curves but for all compact connected sets that separate the cylinder, and the intersection must be topologically transverse.*

Our main results consist of some sets of hypotheses, together with the ITC, that imply the existence of periodic orbits for h with all possible rotation numbers and with some additional properties. Let us start with the easiest case.

Theorem 1. *Let* $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ *be a homeomorphism homotopic to the identity and let* $\tilde{h}$ *be a lift of h to the plane. Suppose h satisfies the ITC and there exist points* $P, Q \in S^1 \times \mathbb{R}$ *such that*

$$p_2 \circ h^n(Q) \to \pm\infty, \qquad \text{as } n \to \pm\infty,$$
$$p_2 \circ h^n(P) \to \mp\infty, \qquad \text{as } n \to \pm\infty,$$

then for all rational numbers p/q*, h has at least two periodic orbits with rotation number* p/q*. If these orbits are finite, then at least one has positive topological index, and as a consequence, one has negative topological index.*

If we do not demand the existence of points P and Q as above, but instead demand the strong intersection property, then we get the same conclusion essentially because the strong intersection property implies the existence of orbits coming from arbitrarily low regions of the cylinder going to arbitrarily high regions and vice versa.

Theorem 2. *Let* $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ *be a homeomorphism homotopic to the identity and let* $\tilde{h}$ *be a lift of h to the plane. Suppose h satisfies the ITC and the strong intersection property. Then, for all rational numbers* p/q*, h has at least two periodic orbits with rotation number* p/q*. If these orbits are finite, then at least one has positive topological index and, as a consequence, one has negative topological index.*

The next result, together with the one above, is the inspiration for the main theorem of the paper.

Theorem 3. *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be a homeomorphism homotopic to the identity and let $\tilde{h}$ be a lift of h to the plane. Suppose h satisfies the ITC and the usual intersection property. Then, for all rational numbers p/q, h has at least one periodic orbit with rotation number p/q.*

Assuming only the usual intersection property, it is not possible to guarantee the existence of non-degenerate periodic orbits. In fact, in [11] there is an example of a homeomorphism of the closed annulus which satisfies the boundary twist condition, the usual intersection property and has only one fixed point. Carter's theorem implies that for some homotopically non-trivial simple closed curve its intersection with its image is not topologically transverse, but this is not demanded in the usual intersection property. Clearly, this example can be extended to the cylinder in an obvious way satisfying the (ITC) and the usual intersection property; and this is a reason why theorem 3 cannot be strengthened in this general setting.

A natural question then would be: is there any set of hypothesis between the usual intersection property and the strong which already implies the existence of non-degenerate periodic orbits?

We remark that we do not answer the above question with great generality, but we prove that if h is an analytic area-preserving diffeomorphism of the cylinder, then the usual intersection property is almost enough.

Theorem 4. *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be an analytic area-preserving diffeomorphism homotopic to the identity and let $\tilde{h}$ be a lift of h to the plane. Suppose h satisfies the ITC, the usual intersection property and for all $M > 0$, there exist $P \in S^1 \times \mathbb{R}$ and $n > 0$ such that $p_2(P) > M$ and $p_2 \circ h^n(P) < -M$. Then, for all rational numbers p/q, h has at least two periodic orbits with rotation number p/q and there always exists at least one isolated orbit with topological index equal 1.*

Remark.

- We can substitute the existence of a point P as in the theorem (and in the lemma below) by the following weaker hypothesis: for all homotopically non-trivial simple closed curves, $\gamma \subset S^1 \times \mathbb{R}$, there exists $z \in \gamma$ which is not h-periodic. In other words, $h(\gamma)$ may be equal to γ, but γ is not pointwise fixed by some power of h.

Clearly, it would be interesting to know if we can relax the analyticity condition and still guarantee the existence of more than one periodic orbit, at least two of them being topologically non-degenerate if they are finite.

The following lemma is used in the proof of theorem 4, but it is interesting by itself.

Lemma 1. *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be an analytic area-preserving diffeomorphism homotopic to the identity and let $\tilde{h}$ be a lift of h to the plane. Suppose h satisfies the ITC and for all $M > 0$, there exists $P \in S^1 \times \mathbb{R}$ and $n > 0$ such that $p_2(P) > M$ and $p_2 \circ h^n(P) < -M$. If for some rational number p/q, h has infinitely many periodic orbits with rotation number p/q, then at least one of them is isolated and has topological index equal to 1.*

The next corollary is intimately related to Carter's theorem, see [11].

Corollary 1. *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be an analytic area-preserving diffeomorphism homotopic to the identity and let $\tilde{h}$ be a lift of h to the plane. Suppose h satisfies the ITC and for some rational p/q, h has at most a finite number of periodic orbits with rotation number p/q, all of them with zero topological index. Then, there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $h(\gamma) \cap \gamma = \emptyset$.*

To prove this result, we just have to note that if for some rational p/q, h has a finite number of periodic orbits with rotation number p/q, all of them with zero topological index, then from theorem 4 and the remark right after it, h does not satisfy the usual intersection property otherwise there would be at least one isolated p/q-periodic point with index 1. So, for some homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$, we have $h(\gamma) \cap \gamma = \emptyset$.

The proofs of all the previous results depend heavily on the following lemma, which is a direct consequence of Kerekjarto's [22] and Guillou's [21] method of constructing Brouwer lines for fixed point free orientation preserving homeomorphisms of the plane which are lifts of cylinder homeomorphisms.

Lemma 2. *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be a homeomorphism homotopic to the identity and let $\tilde{h}$ be a lift of h to the plane. Suppose h satisfies the ITC and $\tilde{h}$ is fixed point free. Then, there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $h(\gamma) \cap \gamma = \emptyset$.*

In the proof of this lemma, we obtain some precise information on the localization of γ in the cylinder. This will be important in our proofs.

The next results we present are corollaries of the previous theorems, suggested by the referees.

Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be a homeomorphism homotopic to the identity and $\tilde{h}$ be a lift of h to the plane. Suppose that

$$\tilde{h}(\tilde{x}+1, \tilde{y}) = \tilde{h}(\tilde{x}, \tilde{y}) + (1, 0),$$
$$\tilde{h}(\tilde{x}, \tilde{y}+1) = \tilde{h}(\tilde{x}, \tilde{y}) + (k, 1),$$

for some integer $k \geqslant 1$. Clearly, h satisfies the ITC and $\tilde{h}$ induces a homeomorphism of the torus, $\bar{h} : \mathrm{T}^2 \to \mathrm{T}^2$, which is homotopic to the Dehn twist $(\bar{x}, \bar{y}) \to (\bar{x} + k\bar{y} \bmod 1, \bar{y} \bmod 1)$. The next results, in a certain sense, generalize some theorems that were proved in [1] for twist mappings. Here the twist property is not necessary anymore.

Theorem 5. *Let $(\tilde{h}, h, \bar{h})$ be as above. Suppose $\tilde{h}$ is area-preserving. Then, given $\epsilon > 0$, there exists $0 \leqslant \alpha < \epsilon$ such that the torus mapping induced by $\tilde{h} + (0, \alpha)$ has periodic points of arbitrarily large periods.*

Corollary 2. *The subset of area-preserving $C^{\infty(\varpi)}$-diffeomorphisms of the torus which have periodic orbits is dense in the $C^{\infty(\varpi)}$-topology in the set of area-preserving $C^{\infty(\varpi)}$-diffeomorphisms of the torus.*

Proof. According to the Conley–Zehnder theorem, the above is true in the homotopy to the identity case. See [13] and p 121 of [24].

If an orientation preserving homeomorphism of the torus is homotopic neither to the identity nor to a conjugate of a Dehn twist by an element of $GL(2, \mathbb{Z})$, then by Nielsen theory it must have a periodic point.

Now consider mappings $\bar{h} : \mathrm{T}^2 \to \mathrm{T}^2$ which are homotopic to Dehn twists. As we obtain periodic orbits in the above theorem by a perturbation which is just a vertical translation, this case is also okay.

If $\bar{h}$ is orientation reversing, then $\bar{h}^2$ is orientation preserving and everything works as above, because by translations applied to $\tilde{h}$ we can make the rotation vector (or number) of the Lebesgue measure by $\bar{h}^2$ become rational.

So the corollary is proved. ■

The next result is a generalization of theorem 5 of [1]. Before presenting it, we need some definitions (a triplet $(\tilde{h}, h, \bar{h})$ is supposed to be fixed):

- Given a probability $\bar{h}$-invariant measure μ on the torus T^2, we define its vertical rotation number $\rho_{\mathrm{V}}(\mu)$ as follows:

$$\rho_{\mathrm{V}}(\mu) = \int_{\mathrm{T}^2} \phi(\bar{z}) \mathrm{d}\mu,$$

where $\phi : \mathrm{T}^2 \to \mathbb{R}$ is given by $\phi(\bar{z}) = p_2 \circ h(z) - p_2(z)$ (z is any point of the cylinder that projects on $\bar{z}$).
- Given an $\bar{h}$-periodic point $\bar{z}$, we define its vertical rotation number as follows: if $z \in S^1 \times \mathbb{R}$ is a point that projects on $\bar{z}$, then there is a pair $(q, p) \in \mathbb{N}^* \times \mathbb{Z}$ such that $h^q(z) = z + (0, p)$. So define $\rho_{\mathrm{V}}(\bar{z}) = p/q$.

Theorem 6. *Let $(\tilde{h}, h, \bar{h})$ be as above. Then there exists an interval $[\rho^-, \rho^+]$, such that for all rational numbers $p/q \in (\rho^-, \rho^+)$, $\bar{h}$ has a periodic orbit with vertical rotation number p/q and the vertical rotation number of every probability $\bar{h}$-invariant measure belongs to $[\rho^-, \rho^+]$.*

Remarks.

(1) It may happen that $\rho^- = \rho^+$. In this case, the theorem is meaningless.

(2) As was done in [1], after the existence of periodic orbits with all possible vertical rotation numbers was proved, we can follow the method used there, which relies on the Thurston theory of classification of homeomorphisms of surfaces, in order to obtain that for all non-rational $\omega \in (\rho^-, \rho^+)$ there exists a compact $\bar{h}$-invariant set Q_ω such that for all $\bar{z} \in Q_\omega$,

$$\rho_{\mathrm{V}}(\bar{z}) \overset{\text{def.}}{=} \lim_{n\to\infty} \frac{p_2 \circ h^n(z) - p_2(z)}{n} = \omega,$$

where $z \in S^1 \times \mathbb{R}$ is any point that projects on $\bar{z}$.

(3) The only way such an $\bar{h}$ has no periodic point is the following: the vertical rotation number exists at every point and has the same value. As was pointed out in [3], even when the vertical rotation number is rational at every point, $\bar{h}$ may not have periodic points. So if we want to prove an analogue of theorem 5 and its corollary WITHOUT the area-preservation property, we have to show that arbitrarily small perturbations applied to $\bar{h}$ can make $\rho^- < \rho^+$. Essentially this was done in [3], but there $\bar{h}$ was supposed to be a twist mapping.

This paper is organized as follows. In the next section we present a brief summary of the results used in the paper, and in the third section we prove our results.

2. Basic tools

In this part of the paper we state some results we use and present references.

2.1. On fixed point free homeomorphisms of the plane

Let $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed point free orientation preserving homeomorphism of the plane.

2.1.1. First definitions.

(1) $\alpha \subset \mathbb{R}^2$ is a translation arc, if α is the image of a continuous curve, whose endpoints are $\tilde{z} \in \mathbb{R}^2$ and $\tilde{h}(\tilde{z})$ and $\alpha \cap \tilde{h}(\alpha) = \tilde{h}(\tilde{z})$.
(2) A subset $E \subset \mathbb{R}^2$ is called free, if and only if $E \cap \tilde{h}(E) = \emptyset$.
(3) A Brouwer line $L \subset \mathbb{R}^2$ is the image of a proper (closed) immersion $l : \mathbb{R} \to \mathbb{R}^2$, which satisfies: $L = l(\mathbb{R})$, divides the plane into two connected components and $\tilde{h}(L)$ belongs to one of these components while $\tilde{h}^{-1}(L)$ belongs to the other one.
(4) Let $\alpha \subset \mathbb{R}^2$ be a simple arc of endpoints $\tilde{z}$ and $\tilde{w}$. We say that α abuts on its direct (or inverse) image if $\tilde{z} \notin \tilde{h}^{-1}(\alpha) \cup \tilde{h}(\alpha)$ and $\overset{\circ}{\alpha} \cap \tilde{h}(\alpha) = \emptyset$ but $\tilde{w} \in \tilde{h}(\alpha)$ (or $\overset{\circ}{\alpha} \cap \tilde{h}^{-1}(\alpha) = \emptyset$ but $\tilde{w} \in \tilde{h}^{-1}(\alpha)$). If $\tilde{k} = \tilde{h}^{-1}(\tilde{w})$ (or $\tilde{k} = \tilde{h}(\tilde{w})$), then the sub-arc $\tilde{k}\tilde{w}$ of α is a translation arc.

The next two results are both due to Brouwer, but the construction that appeared in the proofs given by Kerekjarto [22] and Guillou [21] (for the plane translation theorem) is the basis of the proof of lemma 2.

2.1.2. Brouwer's main results

Lemma 3 (Brouwer lemma on translation arcs). *Let $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed point free orientation preserving homeomorphism of the plane. Given $\tilde{z} \in \mathbb{R}^2$, there exists a translation arc α with $\tilde{z} \in \overset{\circ}{\alpha}$ and $\alpha \cap \tilde{h}^n(\alpha) = \emptyset$ for all integers $n \notin \{1, -1\}$.*

For a good proof of the above result, see for instance [10]. It has the following important corollary.

Corollary 3. *Let $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed point free orientation preserving homeomorphism of the plane. Given a free path connected subset $E \subset \mathbb{R}^2$, then $E \cap \tilde{h}^n(E) = \emptyset$, for all $n \in \mathbb{Z}^*$.*

Theorem 7 (Brouwer plane translation theorem). *Let $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ be a fixed point free orientation preserving homeomorphism of the plane. Given $\tilde{z} \in \mathbb{R}^2$, there exists a Brouwer line L passing through $\tilde{z}$.*

2.1.3. Brick decompositions of the plane. In this paper, we define a brick decomposition of the plane as follows:

$$\mathbb{R}^2 = \bigcup_{n=1}^{\infty} D_i,$$

where each $D_i \in$ Brick_Decomposition is the closure of a connected open set, such that ∂D_i is a polygonal simple curve and $\text{interior}(D_i) \cap \text{interior}(D_j) = \emptyset$, for $i \neq j$. Moreover, the decomposition is locally finite, that is, $\bigcup_{n=1}^{\infty} \partial D_i$ is a graph whose vertices have three edges adjacent to them and the number of elements of the decomposition contained in any compact subset of the plane is finite.

We say that the brick decomposition is free if all its bricks are free, that is, $\tilde{h}(D_i) \cap D_i = \emptyset$, for all $i \in \mathbb{N}$. Given two bricks, D and E, we say that there is a chain connecting them if there are bricks

$$D = B_0, B_1, B_2, \ldots, B_{n-1}, B_n = E,$$

such that $\tilde{h}(B_i) \cap B_{i+1} \neq \emptyset$ for $i = 0, 1, \ldots, n-1$. The chain is said to be closed if $D = E$.

The following is a version of a theorem of Franks [17] due to Le Roux (see [25, p 39]).

Lemma 4. *The existence of a closed chain of free closed bricks implies that there exists a simple closed curve $\gamma \subset \mathbb{R}^2$, such that*

$$\mathrm{index}(\gamma, \tilde{h}) = \mathrm{degree}\left(\gamma, \frac{\tilde{h}(\tilde{z}) - \tilde{z}}{\|\tilde{h}(\tilde{z}) - \tilde{z}\|}\right) = 1.$$

This result is a clever application of Brouwer's lemma on translation arcs.

2.2. On homeomorphisms of the cylinder that have a lift without fixed points

Here we present the statement of theorem A of [21]. In the proof of lemma 2 we use many arguments contained in the proof of this theorem; some of them are due to Guillou and some, as is pointed out in [21], can already be found in Kerekjarto's work.

Theorem 8 (A). *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be homotopic to the identity homeomorphism of the cylinder, which has a lift $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}^2$ without fixed points. Then at least one of the following possibilities holds:*

(a) there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$, such that $h(\gamma) \cap \gamma = \emptyset$,
(b) $p_2 |_{L_0} = \mathbb{R}$ for some Brouwer line L_0,
(c) if we perform the North–South compactification of the cylinder in order to obtain S^2, then there exists a simple closed curve $\gamma \subset S^2$ which contains only one of the points $\{N, S\}$, let us say, N, in a way that the disc D bounded by γ which does not contain S satisfies:

$$\hat{h}(D) \subset D \text{ or } \hat{h}^{-1}(D) \subset D,$$

where $\hat{h} : S^2 \to S^2$ is the mapping induced by h.

In Alain Sauzet's thesis [30] it was shown that assertion (b) may be stated in a more precise way and condition (c) may be dropped. But in this paper we do not need to use this more precise result.

2.3. On the dynamics near fixed points of analytic diffeomorphisms of the plane

The dynamics near singularities of analytic vector fields in the plane is very well understood (see for instance [14]). It can be proved that if the singularity is not a focus or a centre, then the dynamics can be obtained from a finite number of sectors glued in an adequate way. Topologically, these sectors can be classified into four types: elliptic, hyperbolic, expanding and attracting. Dumortier *et al* studied this problem for planar diffeomorphisms near fixed points in [15].

The situation we want to understand in this section is the following: is there a topological picture of the dynamics near an index zero isolated fixed point of an analytic area-preserving diffeomorphism of the plane?

It turns out that the area-preservation together with the zero index hypotheses imply that the eigenvalues of the derivative of the diffeomorphism at the fixed point are both equal to 1. The area-preservation implies that in a sufficiently small neighbourhood of the fixed point, the diffeomorphism is the time of one mapping of a formal vector field (defined by a formal series), see theorem 1.1 p 11 of [15] which comes from [28, 32]. So we are able to apply results from [15], which say that, at least in a topological sense, the dynamics near the fixed point can be obtained as in the vector field setting gluing a finite number of sectors. As we suppose that the area is preserved by diffeomorphism, there cannot be elliptic, expanding and attracting sectors. As the topological index of the fixed point is zero, it cannot be a centre and so there must be exactly two hyperbolic sectors and the dynamics is topological as in figure 1.

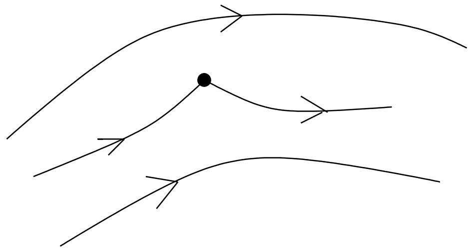

Figure 1. Diagram showing the dynamics near an isolated fixed point of null topological index for an analytic area-preserving diffeomorphism.

3. Proofs

In this section we are going to prove the results stated in the introduction, as well as some auxiliary ones.

3.1. Proof of lemma 2

As h is homotopic to the identity and satisfies the ITC, $p_1 \circ \tilde{h}(\tilde{x}, \tilde{y}) - \tilde{x}$ is periodic in $\tilde{x}$, and there exists $M_1 > 0$ such that if $(\tilde{x}, \tilde{y}) \notin \mathbb{R} \times [-M_1, M_1]$, then

$$\begin{aligned} &|p_1 \circ \tilde{h}(\tilde{x}, \tilde{y}) - \tilde{x}| > 1, \\ &|p_1 \circ \tilde{h}^{-1}(\tilde{x}, \tilde{y}) - \tilde{x}| > 1 \end{aligned} \tag{2}$$

and for some $M_2 > 4M_1$

$$\begin{aligned} &\tilde{h}(\mathbb{R} \times [-M_1, M_1]) \subset \mathbb{R} \times [-M_2/2, M_2/2], \\ &\tilde{h}^{-1}(\mathbb{R} \times [-M_1, M_1]) \subset \mathbb{R} \times [-M_2/2, M_2/2]. \end{aligned} \tag{3}$$

As $\tilde{h}$ is fixed point free, let us follow Kerekjarto's and Guillou's method of constructing Brouwer lines, and start with an horizontal translation arc α_0 contained in $\{\tilde{y} = 0\}$. If such an arc does not exist, then the horizontal line $\{\tilde{y} = 0\}$ is free and so the cylinder curve $S^1 \times \{0\}$ is also free under h, and the lemma is proved. So we can suppose that there exists an α_0 as above. Now consider vertical segments starting at α_0 and abutting on their images, one at each side of α_0. Continue the process as in [21,22]. We obtain a Brouwer line L_0, which intersects α_0 at a free horizontal segment and consists of horizontal and vertical pieces.

Let us write $L_0 = L_0^1 \cup L_0^2$, where L_0^1 and L_0^2 are both closed, and intersect only at some point $\tilde{z}_0 \in \alpha_0$. If at least one element of $\{L_0^1, L_0^2\}$ is contained in some bounded horizontal strip, from the way L_0 is constructed (see [21]), we get that this element induces a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ which is free under h (this argument appears in the proof of theorem A of [21]) and the lemma is proved.

In this way, suppose L_0^1 is not contained in $\mathbb{R} \times [-M_2, M_2]$. Then it reaches one of the boundaries, let us say $\mathbb{R} \times \{M_2\}$. So start at $\tilde{z}_0 \in \alpha_0$ and follow L_0^1 until it reaches $\mathbb{R} \times \{M_2\}$ for the first time and denote this point by $\tilde{z}_1$. From the way L_0 is constructed, $\tilde{z}_1$ is contained in some free vertical segment. From the choice of M_1 and M_2 (see (2) and (3)), and the method developed by Kerekjarto and Guillou, we get that this vertical segment is unbounded, i.e. L_0^1 goes to infinity at some fixed vertical direction. This happens because in Guillou's construction of the Brouwer lines, each linear segment is continued until it reaches its direct or inverse image. But from the choice of M_1 and M_2, a free vertical segment which is below

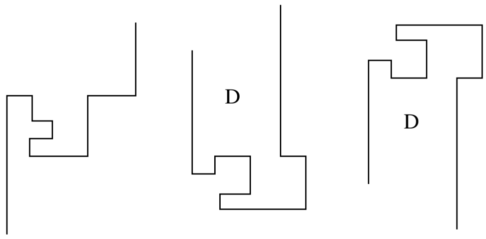

Figure 2. Diagram showing the possibilities for the curve L_0 in the case it is not contained in a bounded horizontal strip.

$\mathbb{R} \times \{M_2\}$ and ends at some point in $\mathbb{R} \times \{M_2\}$ can be continued to $+\infty$ keeping the property of being free.

The same happens for L_0^2. So if both L_0^1 and L_0^2 are not contained in $\mathbb{R} \times [-M_2, M_2]$ we get one of the pictures in figure 2. If $p_2 |_{L_0} = \mathbb{R}$, then we get a contradiction with the ITC, because L_0 is bounded in the horizontal direction. If $p_2 |_{L_0}$ is bounded from above or below, then we have two cases:

1. $L_0 \cap L_0 + (1, 0) \neq \emptyset$. As was shown in the proof of theorem A of [21], in this case there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ which is free under h. A lift of this curve to the plane is constructed using pieces of integer translations of L_0. Clearly, $\gamma \subset S^1 \times [-M_2, M_2]$, because $L_0 \cap (\mathbb{R} \times [-M_2, M_2])^c$ is composed of two vertical infinite semi-lines.
2. $L_0 \cap L_0 + (1, 0) = \emptyset$. In this case, corollary 3 implies that $L_0 \cap L_0 + (n, 0) = \emptyset$, for all $n \in \mathbb{Z}^*$. So we are in case (c) of theorem A. If D is one of the regions denoted in figure 2, one of the following relations should be true:

$$h(D) \subset D \qquad \text{or} \qquad h^{-1}(D) \subset D.$$

And this is a contradiction with the ITC. So the lemma is proved.

3.2. *Proof of theorem 1*

Suppose that for some rational number p/q, $(h, \tilde{h})$ has only finitely many periodic points of this rotation number, all of them with zero index. This implies that if we define $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\tilde{g}(\bullet) = \tilde{h}^q(\bullet) - (p, 0),$$

then $\tilde{g}$ has finitely many fixed points in each fundamental domain of the cylinder, all of them with zero topological index and it induces a homeomorphism of the cylinder $g = h^q$. Let $\{P_1, P_2, \ldots, P_N\} \subset S^1 \times \mathbb{R}$ be $\pi(\text{Fix}(\tilde{g}))$. As h is homotopic to the identity and satisfies the ITC, g is also homotopic to the identity and also satisfies the ITC.

From the choice of the points P and Q, there exists $\delta > 0$, such that for all integers n

$$d(g^n(P), \{P_1, P_2, \ldots, P_N\}) > 2\delta,$$
$$d(g^n(Q), \{P_1, P_2, \ldots, P_N\}) > 2\delta,$$
$$B_{2\delta}(P_i) \cap B_{2\delta}(P_j) = \emptyset, \qquad \text{if } i \neq j.$$

So if we perturb g inside $\bigcup_{i=1}^N B_\delta(P_i)$ in order to destroy the fixed points (see theorem 3.4 of [7] and theorem 2.1 of [31]), we get a mapping g' which is equal to g outside $\bigcup_{i=1}^N B_\delta(P_i)$, lifts to a homeomorphism $\tilde{g}'$ of the plane without fixed points and satisfies the ITC. So by lemma 2, there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $g'(\gamma) \cap \gamma = \emptyset$. But this contradicts the fact that for all integers n,

$$g^n(P) = (g')^n(P),$$
$$g^n(Q) = (g')^n(Q).$$

So the theorem is proved.

3.3. On the usual intersection property

Before continuing, we are going to present a lemma from [8], which will be used in the proofs.

Lemma 5. *Let $h : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be a homeomorphism homotopic to the identity which satisfies the usual intersection property, that is, for every homotopically non-trivial simple closed curve $\gamma \subset S^1 \times$ IR, $h(\gamma) \cap \gamma \neq \emptyset$. Then for all $n \in \mathbb{N}$, h^n also satisfies the usual intersection property.*

Proof. Suppose that for some $n \in \mathbb{N}$, h^n does not satisfy the usual intersection property. Then there exists a homotopically non-trivial simple closed curve $\alpha \subset S^1 \times \mathbb{R}$, such that $h^n(\alpha) \cap \alpha = \emptyset$. Let $\alpha^c = \alpha^+ \cup \alpha^-$, where $\alpha^{+(-)}$ is the upper (lower) connected component of α^c. Suppose without loss of generality that $h^n(\alpha) \subset \alpha^-$. Now let us perform the North–South compactification of the cylinder, in order to obtain S^2. The mapping h induces a mapping $\hat{h} : S^2 \to S^2$ such that $\hat{h}(N) = N$ and $\hat{h}(S) = S$. If we denote by $\hat{\alpha} \subset S^2$ the simple closed curve that corresponds to α and by $\hat{\alpha}^{-(+)}$ the disc bounded by $\hat{\alpha}$ that contains $S(N)$, we get that $\hat{h}^n(\hat{\alpha}^-) \subset \text{interior}(\hat{\alpha}^-)$. Following [12] and [18], let $A = \bigcup_{i=1}^{\infty} \hat{h}^{i.n}(\hat{\alpha}^-)$ and $A^* = \bigcap_{i=1}^{\infty} \hat{h}^{-i.n}(\hat{\alpha}^+)$. So (A, A^*) is an attractor–repellor pair for $\hat{h}^n$. If we denote $B = A \cup \hat{h}(A) \cup \cdots \cup \hat{h}^{n-1}(A)$ and $B^* = A^* \cup \hat{h}(A^*) \cup \cdots \cup \hat{h}^{n-1}(A^*)$, then (B, B^*) is an attractor–repellor pair for $\hat{h}$. Clearly $S \in B$ and $N \in B^*$.

Let $\psi : S^2 \to [0, 1]$ be a Liapunov function associated with the pair (B, B^*). This means that $\psi^{-1}(0) = B$, $\psi^{-1}(1) = B^*$ and if $z \in S^2 \backslash (B \cup B^*)$, then $\psi(\hat{h}(z)) < \psi(z)$. Following [18], we can perturb ψ a little bit in order to get a C^∞ Liapunov function ψ_∞, which by Sard's theorem has a regular value $a \in]0, 1[$. As $S \in B$ and $N \in B^*$, $\psi_\infty^{-1}(a)$ contains a simple closed curve $\hat{\beta} \subset S^2$ with $(S \cup N) \cap \hat{\beta} = \emptyset$ and $\hat{\beta}^c = D_S \cup D_N$, where $S \in D_S$ and $N \in D_N$. From the choice of a, $\hat{h}(\hat{\beta}) \cap \hat{\beta} = \emptyset$. Finally, there exists a homotopically non-trivial simple closed curve $\beta \subset S^1 \times \mathbb{R}$, which corresponds to $\hat{\beta}$ and thus satisfies $h(\beta) \cap \beta = \emptyset$, a contradiction with the usual intersection property.

3.4. Proof of theorem 2

Suppose that for some rational number p/q, h has a finite number of periodic points of rotation number equal to p/q, all with zero topological index. Let $\tilde{g}(\tilde{z}) = \tilde{h}^q(\tilde{z}) - (p, 0)$, $\tilde{z} \in \mathbb{R}^2$ and let $g : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be the cylinder homeomorphism induced by $\tilde{g}$. So g has a finite number of fixed points which correspond to fixed points of $\tilde{g}$, all of zero index. As h satisfies the strong intersection property, the following proposition holds.

Proposition 1. *For every $M > 0$, there exists $z \in S^1 \times \mathbb{R}$ such that $p_2(z) < -M$ and $p_2 \circ g^n(z) > M$, for some $n \in \mathbb{N}$. In the same way, there exists $w \in S^1 \times \mathbb{R}$ such that $p_2(w) > M$ and $p_2 \circ g^n(w) < -M$, for some $n \in \mathbb{N}$.*

Proof. Let $V_C = \{z \in S^1 \times \mathbb{R} : p_2(z) < C\}$. Suppose that for some $k_1, k_2 > 0$,

$$V = \bigcup_{i=0}^{\infty} h^i(V_{-k_1}) \subset V_{k_2}. \tag{4}$$

Clearly $h(V) \subset V$ and V is an open connected set that by (4) satisfies the following: one connected component of V^c denoted by W contains $V^c_{k_2}$. If we define $V_F = W^c$, we get that V_F is an open connected set, homeomorphic to the cylinder, which satisfies $h(V_F) \subset V_F$. And this contradicts the strong intersection property. So, for all $k_1, k_2 > 0$, there exists $z \in S^1 \times \mathbb{R}$ such that $p_2(z) < -k_1$ and $p_2 \circ h^l(z) > k_2$, for some $l \in \mathbb{N}$. In this way, given $M > 0$, let $k_2 > M$ be such that $h^i(S^1 \times]k_2, +\infty[) \subset S^1 \times]M+1, +\infty[$ for all $i \in \{0, 1, 2, \ldots, q-1\}$. By the previous argument, there exists $z \in S^1 \times \mathbb{R}$ such that $p_2(z) < -M$ and $p_2 \circ h^l(z) > k_2$, for some $l \in \mathbb{N}$. Now we keep iterating the orbit of z until the first multiple of q. From the choice of k_2, $p_2 \circ h^{l+r}(z) > M$, where $0 \leqslant r \leqslant q-1$ is an integer such that $l + r = n \cdot q$ for some $n \in \mathbb{N}$. The same argument works for the second part of the proposition. ∎

Now let $M_1 > 0$ be such that if $(\tilde{x}, \tilde{y}) \notin \mathbb{R} \times [-M_1, M_1]$, then

$$|p_1 \circ \tilde{g}(\tilde{x}, \tilde{y}) - \tilde{x}| > 1,$$

$$|p_1 \circ \tilde{g}^{-1}(\tilde{x}, \tilde{y}) - \tilde{x}| > 1$$

and let $M_2 > 4M_1$ be such that

$$\tilde{g}(\mathbb{R} \times [-M_1, M_1]) \subset \mathbb{R} \times [-M_2/2, M_2/2],$$

$$\tilde{g}^{-1}(\mathbb{R} \times [-M_1, M_1]) \subset \mathbb{R} \times [-M_2/2, M_2/2].$$

Clearly, all the $\tilde{g}$-fixed points belong to $\mathbb{R} \times [-M_1, M_1]$. Let $z, w \in S^1 \times \mathbb{R}$ be such that

$$p_2(z) < -2M_2 \quad \text{and} \quad p_2 \circ g^{n_1}(z) > 2M_2,$$

$$p_2(w) > 2M_2 \quad \text{and} \quad p_2 \circ g^{n_2}(w) < -2M_2,$$

for some $n_1, n_2 \in \mathbb{N}$ (see proposition 1). Let

$$\text{Fix} = \{z \in S^1 \times \mathbb{R} : \tilde{g}(\tilde{z}) = \tilde{z}, \text{ for any } \tilde{z} \in \pi^{-1}(z)\}.$$

From the theorem hypotheses #Fix $< \infty$ and all the points in Fix have zero topological index. Let $\delta > 0$ be sufficiently small such that $B_{2\delta}(\text{Fix})$ is the disjoint union of a finite number of balls and

$$B_{2\delta}(\text{Fix}) \cap \{z, g(z), \ldots, g^{n_1}(z), w, g(w), \ldots, g^{n_2}(w)\} = \emptyset.$$

Let $g^* : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be such that

$$g^* = g \text{ in } B_\delta(\text{Fix})^c \quad \text{and} \quad \tilde{g}^* \text{ is fixed point free.}$$

As in theorem 1, in order to construct such a g^*, see theorem 3.4 of [7] and theorem 2.1 of [31].

Now we follow Guillou's and Kerekjarto's construction of a Brouwer line for $\tilde{g}^*$ starting at some free horizontal segment contained in the $\tilde{x}$-axis. From the choice of M_1 and M_2, this Brouwer line, as in lemma 2, implies the existence of a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times [-M_2, M_2]$ which is free under g^*. But this is in contradiction to the orbits of z and w. So the theorem is proved.

3.5. *Proof of theorem 3*

This theorem is a simple corollary of lemmas 2 and 5.

3.6. Proofs of theorem 4 and lemma 1

As in the previous proofs, let p/q be a rational number and suppose that $\tilde{g}(\bullet) = \tilde{h}^q(\bullet) - (p, 0)$ is a fixed point free homeomorphism of the plane. This implies by lemma 2 that there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times \mathbb{R}$ such that $g(\gamma) \cap \gamma = \emptyset$. Lemma 5 tells us that the same must hold for h, in other words, there exists a homotopically non-trivial simple closed curve $\beta \subset S^1 \times \mathbb{R}$ such that $h(\beta) \cap \beta = \emptyset$. But this contradicts the fact that h satisfies the usual intersection property. So $\tilde{g}$ has fixed points.

Now we are going to prove lemma 1.

Proof of lemma 1. Let

$$\text{Fix} = \{z \in S^1 \times \mathbb{R} : \tilde{g}(\tilde{z}) = \tilde{z}, \text{ for any } \tilde{z} \in \pi^{-1}(z)\}$$

and suppose that #Fix $= \infty$. As h satisfies the ITC, there exists $M_1 > 0$ such that if $(\tilde{x}, \tilde{y}) \notin \mathbb{R} \times [-M_1, M_1]$, then

$$|p_1 \circ \tilde{g}(\tilde{x}, \tilde{y}) - \tilde{x}| > 1.$$

So Fix $\subset S^1 \times [-M_1, M_1]$ is a compact set. Let $\tilde{G} : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$\tilde{G}(\tilde{x}, \tilde{y}) = (p_1 \circ \tilde{g}(\tilde{x}, \tilde{y}) - \tilde{x})^2 + (p_2 \circ \tilde{g}(\tilde{x}, \tilde{y}) - \tilde{y})^2.$$

As h and thus g are homotopic to the identity, we get that

$$\tilde{G}(\tilde{x} + l, \tilde{y}) = \tilde{G}(\tilde{x}, \tilde{y}), \qquad \text{for all } l \in \mathbb{Z},$$

so $\tilde{G}$ induces $G : S^1 \times \mathbb{R} \to \mathbb{R}$, such that $G^{-1}(0) = \text{Fix}$. As G is an analytic function, #Fix $= \infty$ and Fix is compact (see [2, 26]), there exists a simple closed curve $\alpha \subset S^1 \times \mathbb{R}$ such that $G(\alpha) = 0 \Leftrightarrow \alpha \subset \text{Fix}$. The existence of the orbit of P implies that α must be homotopically trivial, otherwise it would divide the cylinder into two g-invariant sets both homeomorphic to the cylinder itself, something that is in contradiction to the existence of the orbit of P (this is the only place in the proofs of lemma 1 and theorem 4 where the existence of the orbit of such a point P is used).

Let $\tilde{\alpha}$ be a lift of α to the plane. As $\tilde{\alpha}$ is a simple closed curve, $\tilde{g}$-invariant, it is the boundary of a closed disc $\tilde{D}$, which is also $\tilde{g}$-invariant. If $\tilde{g}$ has infinitely many fixed points inside $\tilde{D}$, as before there must be another simple closed curve $\tilde{\alpha}_1 \subset \text{closure}(\tilde{D})$, such that $\tilde{g}$ fixes $\tilde{\alpha}_1$ pointwise. Let $\tilde{D}_1$ be the disc bounded by $\tilde{\alpha}_1$. If we continue this construction, as $\tilde{g}$ is analytic, we get that it must finish (see [26]) at some finite step; in other words, there exists a disc $\tilde{D}_n \subset \text{closure}(\tilde{D}_{n-1}) \subset \cdots \subset \text{closure}(\tilde{D})$ such that $\tilde{g}$ fixes the boundary of $\tilde{D}_n$ pointwise and $\tilde{g}$ has only finitely many fixed points in the interior of $\tilde{D}_n$. As $\tilde{g}$ is area-preserving, the Poincaré recurrence theorem together with the Brouwer lemma on translation arcs imply that $\tilde{g}$ must have at least one fixed point of index 1 in the interior of $\tilde{D}_n$ because an area-preserving mapping cannot have periodic points with index greater than one [23]. And the lemma is proved. ■

Now let us continue with the proof of the main theorem.

By lemma 1, we can suppose that Fix $= \{P_1, P_2, \ldots, P_N\}$ is a finite set, all points with zero topological index, otherwise the theorem is proved. As in the previous results, let $M_1 > 0$ be such that if $(\tilde{x}, \tilde{y}) \notin \mathbb{R} \times [-M_1, M_1]$, then

$$\begin{aligned} &|p_1 \circ \tilde{g}(\tilde{x}, \tilde{y}) - \tilde{x}| > 2, \\ &|p_1 \circ \tilde{g}^{-1}(\tilde{x}, \tilde{y}) - \tilde{x}| > 2. \end{aligned} \tag{5}$$

Clearly, all the $\tilde{g}$-fixed points belong to $\mathbb{R} \times [-M_1, M_1]$.

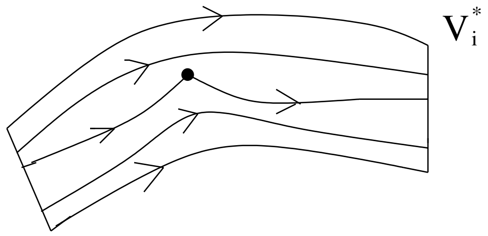

Figure 3. Diagram showing a flow neighbourhood V_i^*.

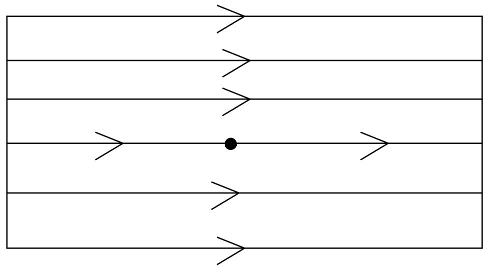

Figure 4. Diagram showing the rectificated dynamics inside V_i^*.

Now choose $0 < \theta^* < 1/4$ such that the following conditions hold:

1. $B_{\theta^*}(\tilde{P}_i) \cap B_{\theta^*}(\tilde{P}_i + (l, 0)) = \emptyset$, for any $\tilde{P}_i \in \pi^{-1}(P_i)$, $1 \leqslant i \leqslant N$ and any $l \in \mathbb{Z}^*$ (this holds for all $0 < \theta^* < 1/4$).
2. $B_{\theta^*}(\tilde{P}_i) \cap B_{\theta^*}(\tilde{P}_j) = \emptyset$ for any $\tilde{P}_i \in \pi^{-1}(P_i)$ and $\tilde{P}_j \in \pi^{-1}(P_j)$ with $1 \leqslant i, j \leqslant N$ and $i \neq j$.
3. $(\bigcup_{i=1}^N B_{\theta^*}(P_i)) \subset S^1 \times [-M_1, M_1]$.

For each $i \in \{1, 2, \ldots, N\}$, let $V_i^* \subset B_{\theta^*}(P_i)$ be a rectangular like flow neighbourhood of P_i, in a way that the boundary of V_i^* consists of four arcs: two of them are transversal to the invariant lines near P_i and thus are free under g, and two of them are contained in two distinct invariant lines as in figure 3.

The next step is to apply a coordinate change in g, supported in $\bigcup_{i=1}^N B_{\theta^*}(P_i)$, that will rectify the dynamics inside each V_i^*.

More precisely, let $m : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ be a homeomorphism such that:

- $m\,|_{\bigcup_{i=1}^N B_{\theta^*}(P_i))^c} = \mathrm{id}$.
- m applied to V_i^* gives an Euclidean rectangle of sides parallel to the x and y axes in a way that the image under m of each invariant line inside V_i^* becomes horizontal, parallel to the x-axis.

Now let us consider the mapping $g' = m \circ g \circ m^{-1}$. As m is supported in the compact set $(\bigcup_{i=1}^N B_{\theta^*}(P_i))$, we have:

- g' satisfies the ITC and the usual intersection property,
- g' has a finite number of fixed points that correspond to fixed points of $\tilde{g}' = \tilde{m} \circ \tilde{g} \circ \tilde{m}^{-1}$, all of them with zero topological index, and by construction the dynamics near each of these fixed points is as in figure 4.

In the following we denote g' simply by g and V_i^* will be an Euclidean rectangular neighbourhood of each P_i of sides parallel to the x and y axes, where the dynamics is simply

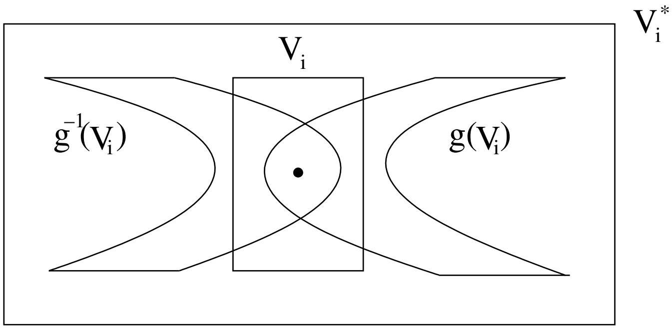

Figure 5. Diagram showing the neighbourhood $V_i \subset V_i^*$ and its forward and backward images.

an horizontal translation, except for the fixed point itself. Again, see figure 4. Let us analyse the dynamics inside each V_i^* in more detail.

Claim 1. There exists $P_i \in V_i \subset V_i^*$, where V_i is a rectangle of sides parallel to the sides of V_i^* such that the following conditions hold:

(i) $g(V_i) \cup g^{-1}(V_i) \subset \text{interior}(V_i^*)$,
(ii) the four sides of V_i are free under g.

Proof. For all sufficiently small $V_{i,}$ (i) holds. Let us look at (ii). The vertical sides of V_i are always free because they cross each g-invariant line only once. On the other hand, if the width of V_i is sufficiently small then the horizontal sides are also free, see figure 5. ■

Now we are going to construct a $\tilde{g}$-'semi-free' brick decomposition of the plane, $\mathcal{B}_{\text{sf}}$, which is invariant under integer horizontal translations $(x, y) \to (x+1, y)$, in the following way:

(1) for $k \in \mathbb{Z}$, $[k, k+1] \times]-\infty, -M_1]$ is a brick (called lower brick),
(2) for $k \in \mathbb{Z}$, $[k, k+1] \times [M_1, +\infty[$ is a brick (called upper brick),
(3) each connected rectangle of $\pi^{-1}(\bigcup_{i=1}^N V_i)$ is a brick,
(4) the rest of the plane is obtained as the union of bricks, which are closed topological free discs, whose boundaries are polygonal simple closed curves,
(5) As we said in 2.1.3, the decomposition is locally finite, that is, $\cup\partial(\text{bricks})$ is a graph whose vertices have three edges adjacent to them and the number of bricks of the decomposition contained in any compact subset of the plane is finite.

Clearly, such a construction is possible. The name 'semi-free' comes from the fact that any element from $\pi^{-1}(\bigcup_{i=1}^N V_i)$ is not free under $\tilde{g}$ and these are the only non-free elements.

Suppose that there is no chain of bricks starting at a lower brick and ending at a upper brick. This means that if we define the accessible set by lower bricks,

$$\tilde{A}c_- = \{D \in \mathcal{B}_{\text{sf}} : \text{ there is a chain from a brick } [k, k+1] \times]-\infty, -M_1] \text{ to } D\},$$

for some $k \in \mathbb{Z}$ which is not fixed, then:

(1) $\mathbb{R} \times]-\infty, -M_1] \subset \tilde{A}c_- \subset \mathbb{R} \times]-\infty, M_1]$.
(2) $\tilde{A}c_-$ is invariant under integer horizontal translations.
(3) $\tilde{g}(\tilde{A}c_-) \subset \text{interior}(\tilde{A}c_-)$, see for instance [25].

Assertions (1) and (2) follow from the infinity twist condition (ITC) and the fact that g is homotopic to the identity, because there exists $K_{\text{lef}} \in \mathbb{N}^*$ such that for $k \in \mathbb{Z}$, $\tilde{g}([k, k+1] \times]-\infty, -M_1]) \cap [s, s+1] \times]-\infty, -M_1] \neq \emptyset$ for all $s \leqslant k - K_{\text{lef}}$.

As all bricks are closed, to prove (3), first note that the definition of $\tilde{A}c_-$ implies that $\tilde{g}(\tilde{A}c_-) \subset \tilde{A}c_-$ and $\partial\tilde{A}c_-$ is contained in the $\cup\partial(\text{bricks})$. Now take a point $\tilde{z} \in \partial\tilde{A}c_-$, which belongs to a brick $D \in \tilde{A}c_-$ and suppose that $\tilde{g}(\tilde{z}) \in \partial\tilde{A}c_-$. This means that $\tilde{g}(\tilde{z})$ belongs to a brick $E \notin \tilde{A}c_-$, whose boundary intersects $\partial\tilde{A}c_-$. But this is a contradiction, since $E \notin \tilde{A}c_-$ and $\tilde{g}(D) \cap E \neq \emptyset$, for some $D \in \tilde{A}c_-$. So, $\tilde{g}(\partial\tilde{A}c_-) \subset \text{interior}(\tilde{A}c_-)$.

As already stated, the boundary of $\tilde{A}c_-$ is contained in the union of the boundaries of the bricks, so $\partial\tilde{A}c_-$ is a topological manifold of dimension 1 (see also [20]) and if we denote $Ac_- = \pi(\tilde{A}c_-)$ we get from properties (1), (2) and (3) above that $S^1 \times]-\infty, -M_1] \subset Ac_- \subset S^1 \times]-\infty, M_1]$ and $g(Ac_-) \subset \text{interior}(Ac_-)$, which implies that there exists a homotopically non-trivial simple closed curve $\gamma \subset S^1 \times [-M_1, M_1]$ which is free under g. But this contradicts the usual intersection property.

So there are brick chains starting at lower bricks and ending at upper bricks. By an analogous argument, there are brick chains starting at upper bricks and ending at lower bricks.

In order to prove the existence of a closed brick chain, let us consider a lower brick $[k_l, k_l+1] \times]-\infty, -M_1]$ which can be connected by a chain of bricks to an upper brick $[k_u, k_u+1] \times [M_1, +\infty[$, for some $k_l, k_u \in \mathbb{Z}$ and take also a brick chain from some upper brick $[k_{u'}, k_{u'}+1] \times [M_1, +\infty[$ to another lower brick $[k_{l'}, k_{l'}+1] \times]-\infty, -M_1]$, for some $k_{u'}, k_{l'} \in \mathbb{Z}$. Let us define $n_+ = k_u - k_l$ and $n_- = k_{l'} - k_{u'}$. As everything commutes with integer horizontal translations, n_+ and n_- do not depend on the particular choice of k_l and $k_{u'}$. So if $(n_+) + (n_-) = 0$, then it is complete. If not, then proceed as follows.

From the ITC, there are brick chains starting at a brick $[k_{l0}, k_{l0}+1] \times]-\infty, -M_1]$ and ending at a brick $[k_{l1}, k_{l1}+1] \times]-\infty, -M_1]$, for some integers $k_{l1} < k_{l0}$ such that $-n_l = -(k_{l1} - k_{l0}) > 0$ is a prime number. In the same way, for upper bricks, there are brick chains starting at a brick $[k_{u0}, k_{u0}+1] \times [M_1, +\infty[$ and ending at a brick $[k_{u1}, k_{u1}+1] \times [M_1, +\infty[$, for some integers $k_{u1} > k_{u0}$. As above, let $n_r = k_{u1} - k_{u0} > 0$ (in both cases, we just have to iterate the first brick once). Again, as everything commutes with integer horizontal translations, n_l and n_r do not depend on the particular choice of k_{l0} and k_{u0}.

Now let us consider the following Diophantine equation to be solved in natural numbers a and b:

$$(n_+) + a.n_r + (n_-) + b.n_l = 0 \;\Leftrightarrow a.n_r - b.(-n_l) = d \overset{\text{def.}}{=} -[(n_+) + (n_-)]. \tag{6}$$

As $-n_l$ is a prime number, there is an integer solution (a_0, b_0) of (6) and thus for some sufficiently large $i \in \mathbb{N}$, $a = a_0 - i.n_l > 0$ and $b = b_0 + i.n_r > 0$ are natural solutions of (6).

So if we concatenate a lower–upper brick chain, a copies of an upper–upper, an upper–lower and b copies of a lower–lower brick chain, we get a closed one, denoted by $B_{\text{closed}} = \{D_0, D_1, \ldots, D_n = D_0\}$, for some positive integer n. But we still have to work a little more before we can apply lemma 4, because some bricks contained in B_{closed} may belong to $\pi^{-1}(\bigcup^N{}_{i=1} V_i)$ and thus may not be free under $\tilde{g}$.

Fortunately, this is an easy difficulty to overlap. Suppose that some $D_{i_0} \in \pi^{-1}(\bigcup_{i=1}^N V_i)$. From the choice of the V_i', we get that neither D_{i_0-1} nor D_{i_0+1} belong to $\pi^{-1}(\bigcup_{i=1}^N V_i)$. So they are both free under $\tilde{g}$.

Let $\tilde{z}_{i_0} \in \tilde{g}(D_{i_0-1}) \cap D_{i_0}$ and $\tilde{w}_{i_0} \in D_{i_0} \cap \tilde{g}^{-1}(D_{i_0+1})$ be two points which are not in the same horizontal segment. If they cannot be chosen to satisfy this condition, then they must belong to the same horizontal side of D_{i_0}, which by claim 1 is free. In this case, let $D_{i_0}' \subset D_{i_0}$ be a closed rectangle, of sides parallels to the sides of D_{i_0} which contains $\tilde{z}_{i_0}$ and $\tilde{w}_{i_0}$ and is $\tilde{g}$-free.

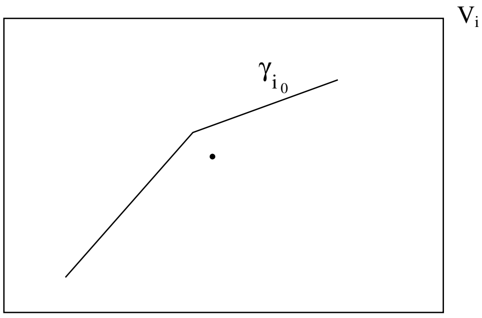

Figure 6. Diagram showing how to connect two points by a free curve inside V_i.

If $\tilde{z}_{i_0}$ and $\tilde{w}_{i_0}$ are not in the same horizontal segment, then as the dynamics inside the $V_i's$ is given by figure 4, $\tilde{z}_{i_0}$ and $\tilde{w}_{i_0}$ can be connected by a FREE polygonal arc $\gamma_{i_0} \subset D_{i_0}$. See figure 6. So, we can choose a $\tilde{g}$-free closed neighbourhood D'_{i_0} of γ_{i_0} such that $D'_{i_0} \subset D_{i_0}$.

To conclude, consider the closed brick chain B_{closed} and substitute D_{i_0} by D'_{i_0}. Perform the same construction for every brick $D_j \in B_{\text{closed}}$ which belongs to $\pi^{-1}(\bigcup_{i=1}^{N} V_i)$. After a finite number of steps, we arrive at a $\tilde{g}$-free closed brick chain and thus we can apply lemma 4 and arrive at a contradiction, because we are supposing that in each fundamental domain of the cylinder, there is a finite number of $\tilde{g}$-fixed points, all with zero topological index, and the lemma gives us a simple closed curve with index one.

So some $\tilde{g}$-fixed point has non zero index, which implies that one of them has positive index. As h is area-preserving, the only positive index allowed is one (see [23]), and our main theorem is proved.

3.7. Proof of theorem 5

Let $\phi : \mathrm{T}^2 \to \mathbb{R}$ be given by $\phi(\bar{z}) = p_2 \circ h(z) - p_2(z)$ (z is any point of the cylinder that projects on $\bar{z}$). The vertical rotation number of the Lebesgue measure, which is $\bar{h}$ invariant, is defined as

$$\rho_{\mathrm{V}}(\mathrm{Leb}, \bar{h}) = \int_{\mathrm{T}^2} \phi(\bar{z}) \mathrm{d}\bar{z} = \omega,$$

for some $\omega \in \mathbb{R}$. Our argument will be divided into two cases:

(1) $\omega = p/q$ is a rational number. In this case, we are going to prove that $h^q(\bullet) - (0, p)$ has fixed points, and so $\bar{h}$ has periodic points with vertical rotation number p/q. As h preserves area, so does $g(\bullet) = h^q(\bullet) - (0, p)$. Denote by $\bar{g}$ the mapping on the torus induced by g. The mapping $\bar{g}$ preserves area and

$$\rho_{\mathrm{V}}(\mathrm{Leb}, \bar{g}) = \int_{\mathrm{T}^2} \varphi(\bar{z}) \mathrm{d}\bar{z} = 0, \tag{7}$$

where $\varphi : \mathrm{T}^2 \to \mathbb{R}$ is given by $\varphi(\bar{z}) = p_2 \circ g(z) - p_2(z)$ (z is any point of the cylinder that projects on $\bar{z}$). Expression (7) implies that for every simple closed curve $\gamma \subset S^1 \times \mathbb{R}$, which is homotopically non-trivial, the area of the region above γ and below $g(\gamma)$ is equal to the area below γ and above $g(\gamma)$. In other words, g has zero flux. So, g satisfies the usual intersection property and thus by theorem 3 it has a fixed point. Moreover, it has periodic points of arbitrarily large periods. (For instance, consider the periodic orbits for g which have rotation numbers equal to $1/n$, for arbitrarily large $n > 0$.)

(2) ω is an irrational number. In this case, consider the following perturbation of h:

$$h_\alpha(x, y) = h(x, y) + (0, \alpha).$$

Clearly,

$$\rho_V(\text{Leb}, \bar{h}_\alpha) = \int_{T^2} \phi_\alpha(\bar{z}) d\bar{z} = \rho_V(\text{Leb}, \bar{h}) + \alpha = \omega + \alpha.$$

So, given $\epsilon > 0$, it is possible to choose $0 < \alpha < \epsilon$ in a way that $\omega + \alpha$ is a rational number. By case (1) above, $\bar{h}_\alpha$ will have periodic points with arbitrarily large periods. And the proof is complete.

3.8. *Proof of theorem 6*

Given a triplet $(\tilde{h}, h, \bar{h})$, following Misiurewicz and Ziemian [27], we define the vertical rotation set of h as follows:

$$\rho_V(h) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{n \geqslant i} \left\{ \frac{p_2 \circ h^n(z) - p_2(z)}{n} : z \in S^1 \times \mathbb{R} \right\}}, \tag{8}$$

that is, $\sigma \in \rho_V(h)$, if and only if there are sequences $z_i \in S^1 \times \mathbb{R}$ and $n_i \to \infty$, such that

$$\lim_{i \to \infty} \frac{p_2 \circ h^{n_i}(z_i) - p_2(z_i)}{n_i} = \sigma.$$

If we denote $\rho^- = \inf \rho_V(h)$ and $\rho^+ = \sup \rho_V(h)$, theorem 2.4 of [27] gives two ergodic $\bar{h}$-invariant measures μ_- and μ_+ with vertical rotation numbers ρ^- and ρ^+, respectively.

Therefore, from the Birkhoff ergodic theorem, there are points $\bar{z}^+$ and $\bar{z}^-$ with $\rho_V(\bar{z}^+) = \rho^+$ and $\rho_V(\bar{z}^-) = \rho^-$. So suppose $\rho^- < p/q < \rho^+$ and let $g(\bullet) = h^q(\bullet) - (0, p)$. It is easy to see that g satisfies the hypotheses of theorem 1, so it has a fixed point and the theorem is thus proved.

Acknowledgments

SAZ is grateful to Professor L Guillou for giving him a copy of [21] and for all his attention and comments, to F Le Roux for his suggestions on the subject of this paper and to the referees for their suggestions on how to greatly simplify the proof of theorem 4, using brick decompositions, and for pointing out some interesting consequences of the theorems proved. This work is partially supported by CNPq, grant: 301485/03-8.

References

[1] Addas-Zanata S 2002 On the existence of a new type of periodic and quasi-periodic orbits for twist maps of the torus *Nonlinearity* **15** 1399–416

[2] Addas-Zanata S and Grotta-Ragazzo C 2002 On the stability of some periodic orbits of a new type for twist maps *Nonlinearity* **15** 1385–97

[3] Addas-Zanata S 2005 On properties of the vertical rotation interval of twist mappings *Ergod. Theory Dynam. Syst.* **25** 641–60

[4] Alpern S and Prasad V 1993 Combinatorial proofs of the Conley–Zehnder–Franks theorem on a fixed point for torus homeomorphisms *Adv. Math.* **99** 238–47

[5] Birkhoff G 1913 Proof of Poincaré's geometric theorem *Trans. Am. Math. Soc.* **14** 14–22

[6] Birkhoff G 1925 An extension of Poincaré's last geometric theorem *Acta Math.* **47** 297–311

[7] Bonino M 2001 A dynamical property for planar homeomorphisms and an application to the problem of canonical position around an isolated fixed point *Topology* **40** 1241–57

[8] Boyland P 1988 Rotation sets and Morse decompositions in twist maps *Ergod. Theory Dynam. Syst.* **8*** 33–61

[9] Brown M and Neumann W 1977 Proof of the Poincaré–Birkhoff fixed point theorem *Michigan Math. J.* **24** 21–31

[10] Brown M 1984 A new proof of Brouwer's lemma on translation arcs *Houston J. Math.* **10** 35–41

[11] Carter P 1982 An improvement of the Poincaré–Birkhoff fixed point theorem *Trans. Am. Math. Soc.* **269** 285–99

[12] Conley C 1978 Isolated invariant sets and Morse index *Am. Math. Soc. CBMS* **38**

[13] Conley C and Zehnder E 1983 The Birkhoff–Lewis fixed point theorem and a conjecture of V I Arnold *Invent. Math.* **73** 33–49

[14] Dumortier F 1977 Singularities of vector fields on the plane *J. Diff. Eqns* **23** 53–106

[15] Dumortier F., Rodrigues P and Roussarie R 1981 *Germs of Diffeomorphisms in the Plane* (*Lecture Notes in Mathematics* vol 902)

[16] Franks J 1988 Recurrence and fixed points of surface homeomorphisms *Ergod. Theory Dynam. Syst.* **8*** 99–107

[17] Franks J 1988 Generalizations of the Poincaré–Birkhoff theorem *Ann. Math.* **128** 139–51

[18] Franks J 1987 A variation on the Poincaré–Birkhoff theorem *Contemp. Math.* **81** 111–17

[19] Guillou L 1994 Théorème de translation plane de Brouwer et généralisations du théorème de Poincaré–Birkhoff *Topology* **33** 331–51

[20] Guillou L 1997 A simple proof of P Carter's theorem *Proc. Am. Math. Soc.* **125** 1555–9

[21] Guillou L Manuscript

[22] Kerekjarto B 1928–29 The plane translation theorem of Brouwer and the last geometric theorem of Poincaré *Acta Sci. Math. (Szeged.)* **4** 86–102

[23] Le Calvez P 1999 Une propriété dynamique des homéomorphismes du plan au voisinage d'un point fixe d'indice >1 *Topology* **38** 23–35

[24] Le Calvez P 1999 Décomposition des difféomorphismes du tore en applications déviant la verticale *Mém. Soc. Math. Fr. (N.S.)* **79** 146

[25] Le Roux F 2004 Homéomorphismes de surfaces: théorèmes de la fleur de Leau-Fatou et de la variété stable *Astérisque* **292** 210

[26] Milnor J 1968 *Singular Points of Complex Hypersurfaces* (*Annals of Mathematics Studies* vol 61) (Princeton, NJ: Princeton University Press)

[27] Misiurewicz M and Ziemian K 1989 Rotation sets for maps of tori *J. Lond. Math. Soc.* **40** 490–506

[28] Moser J 1968 Lectures on Hamiltonian systems *Mem. Am. Math. Soc.* **81** 60

[29] Poincaré H 1912 Sur un théorème de géométrie *Rend. Circ. Math. Palermo* **33** 375–407

[30] Sauzet A 2001 *Thesis* University of Paris 13

[31] Slaminka E 1993 Removing index 0 fixed points for area preserving maps of two-manifolds *Trans. Am. Math. Soc.* **340** 429–44

[32] Takens F 1974 Forced oscillations and bifurcations *Commun. Math. Inst. Rijksuniversiteit Utrecht* **3** 1–59

[33] Winkelnkemper H 1988 A generalisation of the Poincaré–Birkhoff theorem *Proc. Am. Math. Soc.* **102** 1028–30