ON STABLE AND UNSTABLE BEHAVIOUR OF CERTAIN ROTATION SEGMENTS

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ABSTRACT. In this paper, we study non-wandering homeomorphisms of the two torus homotopic to the identity, whose rotation sets are non-trivial segments from $(0,0)$ to some totally irrational point (α, β). We show that for any $r \geqslant 1$, this rotation set only appears for C^r diffeomorphisms satisfying some degenerate conditions. And when such a rotation set does appear, assuming several natural conditions that are generically satisfied in the area-preserving world, we give a clearer description of its rotational behaviour. More precisely, the dynamics admits bounded deviation along the direction $-(\alpha, \beta)$ in the lift, and the rotation set is locked inside an arbitrarily small cone with respect to small C^0-perturbations of the dynamics. On the other hand, for any non-wandering homeomorphism f with this kind of rotation set, we also present a perturbation scheme in order for the rotation set to be eaten by rotation set of some nearby dynamics, in the sense that the later set has non-empty interior and contains the former one. These two flavors interplay and share the common goal of understanding the stability/instability properties of this kind of rotation set.

CONTENTS

1. INTRODUCTION

The notion of a rotation number was introduced by Poincaré in order to gather information on the average "rotational" linear speed of a dynamical system. Rotation theory is well understood only for circle homeomorphisms or endomorphisms, and it is still a great source of problems in two dimensional manifolds (annulus, two torus, higher genus surfaces, and so on). Focusing only on the two torus, there are two most important and related topics, namely, the shape of the rotation set and how it changes depending on the dynamics. In this paper, we will work on both topics, based on one specific type of rotation set. On the one hand,

we try to understand the variation of a rotation set depending on the homeomorphism, under different regularities. On the other hand, we study how a certain shape of the rotation set restricts the dynamics.

In general, with respect to the Hausdorff topology, the rotation set varies upper-semicontinuously. It is also known that if the rotation set of some f has non-empty interior, then it is in fact continuous at f (see [31] and [32]). Moreover, for a C^0-open and dense subset of homeomorphisms, the rotation set is stable (i.e., it remains unchanged under small perturbations, see [36] and [16]). Note that this is true both in the set of all homeomorphisms, and in the set of area-preserving ones.

Below, in order to state our main results, we will use some notations that are mostly standard, and postpone their precise definitions to Section 2.

Our objective is to look at an interesting situation, where the rotation set is a segment connecting $(0,0)$ to some totally irrational point (α, β). We want to understand how it can be changed under sufficiently small C^0 perturbations, and what properties should such a homeomorphism satisfy. An Example with such a rotation set was first described in [17] by Handel, who attributed it to Katok. It is produced by a slowing-down procedure from a constant speed irrational flow. A smooth area-preserving example was obtained by Addas-Zanata and Tal in [5]. See also the paper [26] by Kwapisz and Mathison which shows some special ergodic properties for an explicit example.

In the C^0 category, the more precise task for us is to study when and how the rotation set of a dynamical system is ready to grow. One of the first results on this subject appeared in [1], where it was proved that if the rotation set $\rho(\widetilde{f})$ contains a non-rational vector (α, β) as an extremal point, then for any supporting line r (i.e., a straight line containing (α, β), such that the rotation set avoids one connected component of its complement, denoted O), by certain arbitrarily small C^0-perturbations, the new rotation set will intersect O (See also [15] for a C^1 version of this theorem). In this paper, we are able to go one step further in this direction.

Theorem 1.1. *Let $\widetilde{f} \in \widetilde{Homeo}_{0,nw}(\mathbb{T}^2)$, whose rotation set $\rho(\widetilde{f})$ is a segment from $(0,0)$ to some totally irrational point (α, β). For any $\varepsilon > 0$, there exists $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$ with $dist_{C^0}(\widetilde{g}, \widetilde{f}) < \varepsilon$, such that, $\rho(\widetilde{g})$ has non-empty interior, and $\rho(\widetilde{f}) \backslash \{(0,0)\} \subset Int(\rho(\widetilde{g}))$.*

Remark 1.1. *If the map $\widetilde{f}$ from the above theorem preserves area, then the perturbed map $\widetilde{g}$ can also be chosen in the area-preserving world. This is a well-known fact (follows from Lemma 3.3), but it deserves to be mentioned, as it is one of the only cases known to us on how to perturb a non-wandering homeomorphism and remain non-wandering.*

It is interesting to ask if the same statement is also true in C^1 topology. If we go on to consider C^r diffeomorphisms, $r \geqslant 1$, clearer descriptions should be expected. In particular, we proved the following result, which suggests the non-genericity of the set of non-wandering diffeomorphisms with this special kind of rotation set. We say "suggest", because it is not known how to perturb a non-wandering diffeomorphisms and remain non-wandering (unless of course, in some particular cases, like area-preserving maps).

Theorem 1.2. *Let $\widetilde{f} \in \widetilde{Diff}_{0,nw}(\mathbb{T}^2)$. Assume that for every integer $n > 0$, the linear part Df^n computed at each n-periodic point does not have 1 as an eigenvalue and there are no saddle-connections. Then, the rotation set $\rho(\widetilde{f})$ is not a segment from $(0,0)$ to some totally irrational point (α, β).*

Note that the above conditions are satisfied by generic area-preserving C^r diffeomorphisms, for all $r \geqslant 1$. Naturally, the next task is the following. How can we understand a typical non-wandering diffeomorphism f which does admit such a rotation set? This seems to be hard, because very little is known on the set of non-wandering homeomorphisms or diffeomorphisms of a surface. As we said above, unlike in the area-preserving case, there does not exist a method available to make a perturbation within the set of non-wandering homeomorphisms.

Nevertheless, in the area-preserving setting, if one wants to obtain more information on the non-generic diffeomorphisms, traditionally, one works with generic families. This inspires us to formulate nice conditions, which hold true in a broader set of diffeomorphisms. This approach eventually helps us to detect properties, which general non-wandering homeomorphisms might satisfy. Along this direction, we obtain the following two results.

Theorem 1.3. *Suppose $f \in \text{Diff}_{0,nw}(\mathbb{T}^2)$ satisfies certain natural conditions, see Definition (6). Suppose also that $\rho(\widetilde{f})$ for some lift $\widetilde{f}$ is a segment from $(0,0)$ to a totally irrational point (α, β). Then, f has (finitely many) fixed points (and no periodic point which is not fixed), all with 0 topological index, and the local dynamics around them is obtained by gluing exactly two hyperbolic sectors. The stable branch of any of these fixed points does not intersect the unstable branch of any other point. Moreover, in the plane, for each fixed point, its unstable and stable branches are bounded in the direction orthogonal to (α, β) and the unstable (resp. stable) branch goes to infinity following the vector (α, β) (resp. $-(\alpha, \beta)$). Finally, any stable or unstable branch of a fixed point is dense in the torus.*

Theorem 1.4. *Under the same hypotheses of the above Theorem, for any non-zero integer vector (a,b), a and b coprimes, there exists a simple closed curve θ in $\mathbb{T}^2$, with homological direction (a,b), such that any connected component of the lift of θ to the plane is a Brouwer line for $\widetilde{f}$. Moreover, for any straight line γ containing $(0,0)$ and avoiding (α, β), there exists $\varepsilon > 0$, such that, for any $\widetilde{g} \in \widetilde{\text{Homeo}}_0(\mathbb{T}^2)$, with $d_{C^0}(\widetilde{f}, \widetilde{g}) < \varepsilon$, $\rho(\widetilde{g})$ is contained in the union of γ and one of the connected components of its complement, the one which contains $\rho(\widetilde{f}) \backslash \{(0,0)\}$.*

The next Corollary is a direct consequence of Theorem 1.4. Nevertheless, we will actually prove the Corollary before the theorem (see Lemma 6.2).

Corollary 1.5. *Let f satisfy the conditions in Theorem 1.4. Then it has bounded deviation along the direction $-(\alpha, \beta)$.*

We also consider the unbounded deviation along the direction (α, β). The next theorem requires one more condition, the existence of an invariant foliation, which is clearly satisfied in the particular case when f is the time-one map of some flow. See its more precise statement in Section 7.

Theorem 1.6. *Consider any $\widetilde{f}$ as in Theorem 1.4. If f preserves a C^0 foliation of $\mathbb{T}^2$, then $\widetilde{f}$ has unbounded deviation along (α, β).*

This last result leads to the following interesting question.

Question 1. *For $\widetilde{f}$ which lifts $f \in \text{Homeo}_{0,nw}(\mathbb{T}^2)$, suppose $\rho(\widetilde{f})$ is the line segment from $(0,0)$ to a totally irrational (α, β). Is it true that $\widetilde{f}$ has unbounded deviation along the direction (α, β)?*

To conclude, let us briefly describe the organisation of this paper and the main ideas used in the proofs.

In Section 2, we will summarise some notations and previous results that will be used along the text.

In Section 3, we introduce a perturbation technique, which is very useful under the condition of unbounded deviations along some direction. The purpose is to find ε-pseudo periodic orbits, which can be "closed" in order to become periodic orbits, so with rational rotation vector. The difficulty is that, a priory, the original method in [1] does not give enough information to locate the position of these rational numbers, except that they are outside the rotation set $\rho(\tilde{f})$.

In Section 4, we focus on the case when the map has bounded deviations. We apply a result proved by Jäger (see [18]) in order to obtain a semi-conjugacy between the restriction of the dynamics to a certain minimal set and the rigid torus rotation. Then, we prove that whenever one can perturb the rigid rotation, we can also perturb the original homeomorphism. Combining both results from Section 3 and Section 4, we complete the proof of Theorem 1.1.

From Section 5, we start working with diffeomorphisms. There, we prove Theorem 1.2. There are three main ingredients in this proof. The first one belongs to the theory of prime ends rotation numbers. The second one is the so called *bounded disk lemma*, firstly proved by Koropecki and Tal in [25]. The third one consists of certain properties of invariant branches at hyperbolic periodic saddles, mostly from Fernando Oliveira's paper [34].

In Section 6, we work with diffeomorphisms satisfying certain conditions, which in the area-preserving case, are generic in the complement of the set of maps which satisfy the hypotheses of theorem 1.2. See for instance [11] and [10]. First, we collect several results describing dynamical properties of maps which satisfy the conditions in Definition (6). These results together imply Theorem 1.3 and are also an important part in the proof of existence of the Brouwer lines (Theorem 1.4).

In Section 7, we continue to study diffeomorphisms satisfying the conditions from Section 6 and prove Theorem 1.6.

Notational Remark. There will be a small abuse of notation among the text below. For example, when we introduce integers, positive constants along the arguments, choices will be made differently in different subsections, sometimes with the same name. However, they will be consistent within one single subsection.

2. PRELIMINARIES AND PREVIOUS RESULTS

The main purpose of this section is to fix notations and to recall some previous results for later use. In some cases, the formulation contains some minor variations from the reference, and we will give some short proofs only stressing the differences. We will also show some elementary lemmas as well.

Note that some of the notations were already used in the statements of the theorems in the introduction.

2.1. **Planar Topology and Dynamics.**

For any planar subset M, denote by $\text{Inter}(M)$ the interior of M, and by ∂M the boundary of M. The following property in planar topology will be used. We say a planar set F separates the points x and y if they are in different connected components of F^c.

Lemma 2.1. *[Theorem 14.3 in Chapter V of* [33]*]. Let F be a closed subset of the plane $\mathbb{R}^2$, separating two points x and y. Then some connected component of F also separates x and y.*

Let M denote a metric space, and consider a homeomorphism $f : M \to M$. For any starting point x_0, we often use subscript to denote the f-iterates of x_0, i.e., $x_n = f^n(x_0)$. For $\varepsilon > 0$, we call an ε-pseudo periodic orbit (with period n), for a finite sequence of points $\{x^{(0)}, x^{(1)}, \cdots, x^{(n-1)}\}$, with the following properties.

$$(2.1) \qquad \text{dist}(f(x^{(i)}), x^{(i+1)}) < \varepsilon, \text{ for any } i = 0, \cdots, n-2, \text{ and,}$$

$$(2.2) \qquad \text{dist}(f(x^{(n-1)}), x^{(0)}) < \varepsilon.$$

Moreover, any point in an ε-pseudo periodic orbit as above, will be referred as an ε-pseudo periodic point.

We say a point p is f-recurrent, if there exists $n_k \to \infty$, such that $f^{n_k}(p) \to p$. The following is a standard fact for all non-wandering dynamical systems on compact metric spaces M. We provide a proof for completeness.

Lemma 2.2. *Suppose $f : M \to M$ is non-wandering. Then, the set of f-recurrent points, denoted by $R(f) \subset M$, is dense.*

Proof. Pick any open disk $B = B_0$. It suffices to show $B \bigcap R(f) \neq \varnothing$. Since f is non-wandering, there is some n_1 such that $f^{-n_1}(B_0) \bigcap B_0 \neq \varnothing$. Then we can choose some small closed disk B_1, with radius smaller than 1, such that $B_1 \subset f^{-n_1}(B_0) \bigcap B_0$. Note that every point in B_1 will return to B_0 at iterate n_1.

Inductively, suppose we have found increasing integers $n_1 < \cdots < n_k$, and closed disks $\{B_i\}_{i=1}^{k}$, such that for all $i = 1, \cdots, k$, B_i has diameter smaller than $\frac{1}{i}$, such that $B_i \subset \text{Inter}(B_{i-1} \bigcap f^{-n_i}(B_{i-1}))$. Then, we can choose some $n_{k+1} > n_k$ such that $B_k \bigcap f^{-n_{k+1}}(B_k) \neq \varnothing$, and some closed disk B_{k+1} with radius smaller than $\frac{1}{k+1}$, such that $B_{k+1} \subset \text{Inter}(B_k \bigcap f^{-n_{k+1}}(B_k))$.

Now for all $k \geqslant 1$, B_k consists of points that will return to B_{k-1} at time n_k. Now $\bigcap_{k \geqslant 1} B_k$ is a singleton, say, $\{x^*\}$. It follows that $f^{n_k}(x^*) \to x^*$. $\qquad \square$

Denote by $\mathbb{T}^2$ the two-dimensional "flat" torus, whose universal covering space is $\mathbb{R}^2$, and let $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ be the natural projection. Let $\text{Homeo}_0(\mathbb{T}^2)$ denote the set of homeomorphisms of $\mathbb{T}^2$ homotopic to the identity. Then, denote by $\text{Homeo}_{0,\text{nw}}(\mathbb{T}^2)$ and $\text{Homeo}_{0,\text{Leb}}(\mathbb{T}^2)$, the set of non-wandering and area-preserving homeomorphisms, respectively. Note that both are subsets of $\text{Homeo}_0(\mathbb{T}^2)$. We also denote by $\widetilde{\text{Homeo}}_0(\mathbb{T}^2)$ (respectively, $\widetilde{\text{Homeo}}_{0,nw}(\mathbb{T}^2)$) the set of lifts of homeomorphisms from $\text{Homeo}_0(\mathbb{T}^2)$ (respectively, $\text{Homeo}_{0,nw}(\mathbb{T}^2)$) to the plane. Similarly, for $r \geqslant 1$, or $r = \infty$, denote by $\text{Diff}_{r,\text{nw}}(\mathbb{T}^2)$ (respectively $\text{Diff}_{r,\text{Leb}}(\mathbb{T}^2)$) the set of C^r diffeomorphisms of $\mathbb{T}^2$, which are non-wandering (respectively, area-preserving) and homotopic to the identity. Also, the sets of their lifts are denoted by $\widetilde{\text{Diff}}_{r,\text{nw}}(\mathbb{T}^2)$ and $\widetilde{\text{Diff}}_{r,\text{Leb}}(\mathbb{T}^2)$, respectively.

Usually, the choice of a lift is not relevant for our purposes, as can be seen in the next subsection.

We say a subset $K \subset \mathbb{T}^2$ is *essential* if there exists a non-trivial homotopy class, such that for any representative loop β of it, $\beta \cap K \neq \emptyset$. A subset K is called *inessential* if it is contained in a topological disk in $\mathbb{T}^2$. In this case, the complement of it is called *fully essential*. We will need the following result. For more details about the above notations, see [23] and [25].

Lemma 2.3. *[Theorem 6 in [23]] Let $f \in Homeo_{0,nw}(\mathbb{T}^2)$ and suppose that the set of fixed points is inessential. Then there exists $M > 0$, such that, for any f-invariant topological open disk D, each connected component of $\pi^{-1}(D)$ has diameter bounded from above by M.*

2.2. **Misiurewicz-Ziemian Rotation Set.**

Consider $\widetilde{f} \in \widetilde{Homeo_0}(\mathbb{T}^2)$. The foundations of the rotation theory in the torus were mainly developed by Misiurewicz and Ziemian in [31]. There, the following notion of a rotation set $\rho(\widetilde{f})$ appears:

(2.3)
$$\rho(\widetilde{f}) := \{v = \lim_{k \to \infty} \frac{1}{n_k}(\widetilde{f}^{n_k}(\widetilde{z}_k) - \widetilde{z}_k), |n_k \to \infty, \widetilde{z}_k \in \mathbb{R}^2, \text{whenever the limit exists}\}.$$

Remark 2.1. *For any $f \in Homeo_0(\mathbb{T}^2)$, and any f-invariant compact subset $K \subset \mathbb{T}^2$, $\rho(\widetilde{f}, K)$ is the rotation set of the map restricted to the invariant set K, which is defined similarly as in (2.3), where the only difference is that, the points $\widetilde{z}_k$ in the expression are only allowed to be chosen in $\pi^{-1}(K)$.*

One can also define the point-wise rotation vector as follows. For any $z \in \mathbb{T}^2$,

(2.4) $$\rho(\widetilde{f}, z) := \lim_{n \to \infty} \frac{1}{n}(\widetilde{f}^n(\widetilde{z}) - \widetilde{z}), \text{ when the limit exists.}$$

Another important definition is as follows. Consider any f-invariant Borel probability measure μ, and denote by $\rho_\mu(\widetilde{f}) = \int_{\mathbb{T}^2} (\widetilde{f}(\widetilde{x}) - \widetilde{x})d\mu(x)$ the average rotation vector of the measure μ (note that the integrand in this expression is in fact a function on $\mathbb{T}^2$). Define

(2.5) $$\rho_{\text{meas}}(\widetilde{f}) := \{\rho_\mu(\widetilde{f}) | \mu \text{ is a } f\text{-invariant Borel probability measure}\}.$$

The following result gathers many important properties of these notions:

Theorem 2.4. *[See [31] and [32]] For any $\widetilde{f} \in \widetilde{Homeo_0}(\mathbb{T}^2)$, $\rho_{meas}(\widetilde{f})$ equals $\rho(\widetilde{f})$, which is a compact and convex subset of $\mathbb{R}^2$. Moreover, every extremal point of the rotation set can be realized as the average rotation vector of some ergodic measure μ.*

2.3. **Bounded Deviations.**
For any non-trivial vector $w \in \mathbb{R}^2$, denote by pr_w the projection of a vector along the w direction.

(2.6) $$\text{pr}_w : \mathbb{R}^2 \to \mathbb{R}, r \mapsto \langle r, w \rangle.$$

Next, we introduce the important notation of bounded deviations.

Definition 1. *Fix a non-trivial vector $w \in \mathbb{R}^2$. We say that $\widetilde{f}$ has bounded deviation along direction w (from its rotation set $\rho(\widetilde{f})$), if there exists $M > 0$, such that for any $n \geqslant 0$ and any $\widetilde{x} \in \mathbb{R}^2$,*

$$(2.7) \qquad pr_w(\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n\rho(\widetilde{f})) \leqslant M.$$

Remark 2.2. *Note that with respect to this definition, having bounded deviation along w and $-w$ are two different conditions.*

The following statement essentially follows from Gottschalk-Hedlund theorem, see also [18] for a somewhat more elementary proof.

Lemma 2.5 (Proposition A in [18]). *Let $f \in Homeo_0(\mathbb{T}^2)$ preserve a minimal set $K \subset \mathbb{T}^2$. Suppose $\rho(\widetilde{f}, K) = \{(\alpha, \beta)\}$ where (α, β) is totally irrational, and $\widetilde{f}\big|_{\pi^{-1}(K)}$ has bounded deviation along every direction. Then, there exists a continuous surjective map $\phi : K \to \mathbb{T}^2$, homotopic to the inclusion, satisfying that*

$$(2.8) \qquad \phi \circ f\big|_K = R_{(\alpha, \beta)} \circ \phi,$$

where $R_{(\alpha, \beta)}$ is the rigid rotation on $\mathbb{T}^2$.

The following result establishes the bounded deviation property in the perpendicular direction when the rotation set is the special one we are interested in.

Theorem 2.6 ([39]). *Suppose $\widetilde{f} \in \widetilde{Homeo_0}(\mathbb{T}^2)$, and $\rho(\widetilde{f})$ is the segment from $(0, 0)$ to a totally irrational point (α, β). Then $\widetilde{f}$ has bounded deviation along the perpendicular directions $(-\beta, \alpha)$ and $(\beta, -\alpha)$.*

2.4. **Some Fundamental Tools in Topological Dynamics.**

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ denote an orientation-preserving homeomorphism. A properly embedded oriented line $\gamma : \mathbb{R} \to \mathbb{R}^2$ is called a *Brouwer line*, if $f(\gamma(\mathbb{R}))$ and $f^{-1}(\gamma(\mathbb{R}))$ belong to different connected components of the complement of $\gamma(\mathbb{R})$. We also abuse notation by writing $\gamma = \gamma(\mathbb{R})$. We call these components *the right of γ* and *the left of γ*, and denote them by $\mathcal{R}(\gamma)$ and $\mathcal{L}(\gamma)$, respectively.

The following result is usually attributed to Brouwer (see also [**?**]), we refer to [[12], Proposition 1.3] for a very useful generalization. Here we state a weaker version, which is sufficient for our use.

Lemma 2.7. *Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation-preserving homeomorphism. If there exists a topological open disk U, such that $f(U) \bigcap U = \varnothing$, and for some $k \geqslant 2$, $f^k(U) \bigcap U \neq \varnothing$, then, if the fixed points are isolated, f admits a fixed point with positive topological index.*

Definition 2. *Let $f : (U, p) \to (V, p)$ denote a local homeomorphism, where U and V are two open neighbourhoods of an isolated fixed point $p \in \mathbb{R}^2$. Choose a small disk $D \subset U$, whose boundary is $\beta = \partial D$. Define $g : \beta \to S^1$, such that $g(x) = \frac{x - f(x)}{\|x - f(x)\|}$. The topological index of f at the point p is defined as the degree of the map g, denoted as $Index_f(p)$.*

The following is a consequence of the classical result usually referred to as Lefschetz fixed-point formula. (See for example Theorem 8.6.2 in [20]).

Lemma 2.8. *Let $f \in Homeo^0_+(\mathbb{T}^2)$ and assume all fixed points are isolated. Then*

$$(2.9) \qquad\qquad \sum_{p \in Fix(f)} Index_f(p) = 0.$$

2.5. **Generic Conditions.**

Recall $Diff_{r,nw}(\mathbb{T}^2)$ is the set of non-wandering C^r diffeomorphisms of $\mathbb{T}^2$, which are homotopic to the identity, and $Diff_{r,Leb}(\mathbb{T}^2) \subset Diff_{r,nw}(\mathbb{T}^2)$ is the subset of area-preserving ones. Below, we say p is a periodic *saddle-like* point for f if there is $n > 0$, such that p is a f^n-fixed point, and with respect to f^n, the dynamics near p is obtained by gluing a finite number of topological saddle-sectors, see [11]. As usual, we denote by $W^u(p)$ (respectively, $W^s(p)$) the union of p and all the unstable branches at p (respectively, the stable branches at p).

Definition 3. *For any $r \geqslant 1$ or $r = \infty$, define $\mathcal{G}^r \subset Diff_{r,nw}(\mathbb{T}^2)$ as the subset of diffeomorphisms f satisfying the following two conditions:*

    (1) if $p \in \mathbb{T}^2$ is an n-periodic point, then $Df^n(p)$ does not have 1 as an eigenvalue.
    (2) f does not have saddle connections.

Remark 2.3. *By Theorem 3 (iii) and Theorem 9 of [37], for all $r \geqslant 1$, the set $\mathcal{G}^r \bigcap Diff_{r,Leb}(\mathbb{T}^2)$ contains a residual subset of $Diff_{r,Leb}(\mathbb{T}^2)$. Thus the above conditions are generic for area-preserving diffeomorphisms.*

Definition 4. *Let $f : S \to S$ be a C^1 diffeomorphism on a closed surface S. Let p, q be two periodic saddle-like points. We say that $W^u(p)$ and $W^s(q)$ intersect at a point w in a topologically transverse way, if there exists an open disk $B = B(w, \delta)$ with some radius $\delta > 0$ such that (denote by α, respectively β, the connected component of $W^u(p) \bigcap B$, respectively $W^s(q) \bigcap B$, both containing the point w):*

    (1) $B \backslash \alpha = B_1 \sqcup B_2$, which is a disjoint union.
    (2) $\beta \backslash \{w\} = \beta_1 \sqcup \beta_2$, which is a disjoint union and $\beta_1 \subset B_1 \bigcup \alpha$, $\beta_2 \subset B_2 \bigcup \alpha$, with $\beta_1 \not\subset \alpha$ and $\beta_2 \not\subset \alpha$. In other words, β_1 intersects B_1 and β_2 intersects B_2.

Remark 2.4. *The radius of the disk, $\delta > 0$, might have to be taken strictly away from 0, because the connected component of $\alpha \bigcap \beta$ containing w could be a non-trivial arc.*

The following lemma will be useful for obtaining non-contractible periodic orbits, i.e., those orbits with non-zero rotation vectors.

Lemma 2.9. *[Lemma 0 in [3]] Suppose $\widetilde{f} \in \widetilde{Diff}_r(\mathbb{T}^2)$ has a hyperbolic periodic saddle point $\widetilde{q}$, and suppose $W^u(\widetilde{q})$ and $W^s(\widetilde{q}) + (a, b)$ intersect in a topologically transverse way, for some integer vector (a, b). Then there exists some integer $N > 0$ such that the diffeomorphism $\widetilde{g} := \widetilde{f}^N - (a, b)$ admits a fixed point $\widetilde{p}$. Thus, $\rho(\widetilde{f}, \pi(\widetilde{p})) = (\frac{a}{N}, \frac{b}{N})$.*

Remark 2.5. *Following exactly the same proof as in [3], the conclusion is also true when the periodic point $\widetilde{q}$ has topological index 0, and admits exactly one stable and one unstable branch, whose local dynamics is described in Figure 1.*

2.6. **A Broader Class of (Non-generic) Diffeomorphisms.**

The following definition appears as Definition 1.6 on page 12 of [11]. Such a study is based on the important work of Takens (See Theorem 4.6 in [38]). Although all the theory in [11] was stated for C^∞ maps, for the results we consider below, there is no substantial difference in the C^r case.

Definition 5. *Let $f : (U, p) \to (W, p)$ be a local planar C^r-diffeomorphism with an isolated fixed point p. Assume all eigenvalues of $Df(p)$ belong to the unit circle and let S be the semi-simple part of $Df(p)$ in its Jordan normal form. Then up to a C^r-coordinate change, there exists a formal C^r vector field X, invariant under S, such that, the r-jet of f and the r-jet of $S \circ X_1$ coincide at p, where X_1 is the time-1 map of the formal flow generated by X. We say f is of Lojasiewicz type at p, if the following condition holds:*

- *there exists an integer $k \leqslant r$ and constants $C, \delta > 0$, such that, for any z satisfying $\|z - p\| \leqslant \delta$, then*

(2.10)
$$\|X(z)\| \geqslant C\|z - p\|^k.$$

The following result was essentially obtained in Section 2 of [4].

Lemma 2.10. *Assume $f \in Diff_{r,nw}(\mathbb{T}^2)$ has an isolated fixed point p and the topological index of f at p is zero. Also suppose that if both eigenvalues of Df at p lie in the unit circle, then f is of Lojasiewicz type at p. Then, there exists exactly one stable and one unstable branch at p. Moreover, the local dynamics can be precisely described, see Figure 1.*
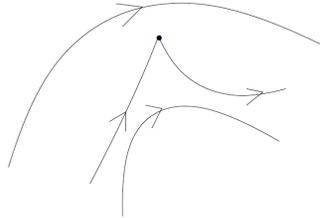
FIGURE 1. The Local Dynamics Around a Fixed Point

Proof. Consider the linear transformation $Df(p)$, which has positive determinant. If 1 is not an eigenvalue of $Df(p)$, then p is either an hyperbolic fixed saddle point, an elliptic fixed point (that is, both eigenvalues are in the unit circle and not real), or -1 is an eigenvalue. In all the above possibilities, p has topological index equal to -1 or 1. As the index at p is zero, the above cases do not happen. If the two eigenvalues of $Df(p)$ are 1 and some $a > 0$ with $a \neq 1$, then as an application of center manifold theory (see [8]), we get that p can be a topological saddle, a topological sink (or source), or a saddle-node. Since by assumption p has topological index 0, it must be a saddle-node. In this case, p has two saddle sectors, and one sector which is either contracting or expanding, a contradiction with the non-wandering condition. For more details, see Proposition 6 from [4], as well as [8].

Thus, $Df(p)$ must have both eigenvalues equal to 1. The rest of the proof follows exactly the same lines from the argument in Section 2 of [4]. □

Definition 6. *For any $r \geqslant 1$ or $r = \infty$, define $\mathcal{K}^r \subset Diff_{r,nw}(\mathbb{T}^2)$ to be the subset of diffeomorphisms f satisfying the following four conditions:*

(1) for all $n > 0$, there are at most finitely many f^n-fixed points.

(2) *for any f^n-fixed point z, of prime period n, if 1 is an eigenvalue of $Df^n(z)$, then it has multiplicity 2 and f^n is of Lojasiewicz type at z. Moreover, in this case the $Index_{f^n}(z)$ is zero, and so Lemma 2.10 implies that the local dynamics near z is given by Figure 1. For families of maps, this situation corresponds to the birth or death of periodic points.*

(3) *for any f^n-fixed point w, of prime period n, if -1 is an eigenvalue of $Df^n(w)$, then $Index_{f^n}(z)$ is 1. For families, this situation corresponds to a period-doubling bifurcation.*

(4) *there are no connections between saddle-like periodic points.*

$\mathcal{K}^r \backslash \mathcal{G}^r$ can be thought as a sort of set of typical diffeomorphisms in the complement of $\mathcal{G}^r$. The definition can also be justified as follows. Let $\mathcal{F}$ denote some one-parameter C^r-generic family of area-preserving diffeomorphisms,

$$(2.11) \qquad \mathcal{F} := \{f_t\}_{t \in [a,b]} \subset \text{Diff}_{r,\text{Leb}}(\mathbb{T}^2), \text{ for some } r \geqslant 1.$$

The following statement is a combination of results from [3] and [4]. The proofs were based on previous results contained in [30] and [11].

Lemma 2.11. *[Section 1.3.3 of* [3] *and Section 2 of* [4]*] For such a generic C^r-family $\mathcal{F}$ as above, all maps f_t belong to $\mathcal{K}^r$; in particular, such a family never has saddle-like connections (tangencies are of course allowed), and with respect to periodic points, the only allowed degeneracies for a certain parameter t are, period-doubling bifurcations (item (3) above) and the one which appears in item (2), which as we already said, is related to the birth or disappearance of periodic points for families of maps.*

The next result gives a perturbation consequence based on the local dynamics near fixed points. Let $Fix(\widetilde{f}) = \{z \in \mathbb{T}^2 : \forall \widetilde{z} \in \pi^{-1}(z), \widetilde{f}(\widetilde{z}) = \widetilde{z}\}$.

Proposition 2.12 (Proposition 9 of [4]). *Suppose $\widetilde{f} \in \widetilde{Homeo}_0(\mathbb{T}^2)$. Assume $Fix(\widetilde{f})$ is finite and for any $z_0 \in Fix(\widetilde{f})$, there exists a local continuous chart $\psi : U \to \mathbb{R}^2$, such that, for any $z \in (U \cap f^{-1}(U)) \backslash \{z_0\}$, $pr_1 \circ \psi \circ f(z) > pr_1 \circ \psi(z)$. Then, there exists $\varepsilon > 0$, such that for any any $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$ with $dist_{C^0}(\widetilde{g}, \widetilde{f}) < \varepsilon$, $(0,0) \notin int(\rho(\widetilde{g}))$.*

2.7. **Prime Ends Rotation Numbers.**

For an open topological disk D contained in some surface, one can attach an artificial circle, called the *prime ends circle*, denoted by $b_{\text{PE}}(D)$. Moreover, the prime ends circle $b_{\text{PE}}(D)$ can be be topologized, such that the union $D \bigcup b_{\text{PE}}(D)$ is homeomorphic to the standard closed unit disk in $\mathbb{R}^2$. We call it the *prime ends disk*, and this procedure is referred to as *prime ends compactification*. This is the beginning of Carathéodory's prime ends theory, and we refer to [29], [28] and [21] for more details.

If f is a homeomorphism on the closure of an open topological disk D into itself, then f extends to the unit disk $D \bigcup b_{\text{PE}}(D)$, so it induces a homeomorphism on the prime ends circle $b_{\text{PE}}(D)$. Then, the dynamics restricted to this circle defines a rotation number, called the *prime ends rotation number*, and denoted by $\rho_{\text{PE}}(f, D)$. The following lemma is a combination of important results from several papers.

Lemma 2.13. *Suppose $\widetilde{h}$ is a lift of some $h \in Diff_{r,nw}(\mathbb{T}^2)$ satisfying the following properties:*

(1) *There are at most finitely many h^n-fixed points for all $n \geqslant 1$;*

(2) *For all $n \geqslant 1$, and any h^n-fixed point p, of prime period n,*
    (a) *either none of the eigenvalues of $Dh^n(p)$ is equal to 1,*
    (b) *or, if one of the eigenvalues of $Dh^n(p)$ is 1, then the topological index of h^n at p is zero and actually, both eigenvalues are equal to 1 (see the proof of Lemma 2.10). Moreover, p is of Lojasiewicz type for h^n.*

Let $K \subset \mathbb{R}^2$ be an $\widetilde{h}$-invariant continuum and let O denote an $\widetilde{h}$-invariant connected component of K^c. Write $\rho_{PE}(\widetilde{h}, O)$ to denote the prime ends rotation number of $\widetilde{h}$ restricted to O. Then the following statements are true:

  i) *If $\rho_{PE}(\widetilde{h}, O)$ is rational and O is bounded, or $\rho_{PE}(\widetilde{h}, O)$ is zero, then ∂O contains only saddle-like $\widetilde{h}$-periodic points, and connections between these saddle-like periodic points.*

  ii) *If O is not equal to K^c and $\rho_{PE}(\widetilde{h}, O)$ is irrational, then there is no $\widetilde{h}$-periodic point in ∂O.*

Sketch of the proof. Let us show item (i). Assume $\rho_{PE}(\widetilde{h}, O) = \frac{p}{q}$ which is in reduced form. Then, as h is non-wandering, a prime chain corresponding to a q-periodic prime end $\widehat{z}$ has the property that each of its crosscuts must intersect its image under $\widetilde{h}^q$, otherwise the corresponding crossections would contain wandering domains, even in the torus (this argument goes back to Cartwright-Littlewood, see for instance proposition 2.1 of [?]). So, the principal set of $\widehat{z}$ is made of $\widetilde{h}^q$-fixed points, something that implies the first assertion of item (i), i.e., there exists some $\widetilde{h}^q$-fixed point $z \in \partial O$.

The main results of [22] (see also Theorem 1.2, Corollary 1.3 and Theorem 1.4 of the report [24] from ICM 2018), imply:

  (1) if $q = 1$, then all $\widetilde{h}$-periodic points $z \in \partial O$ are fixed and the eigenvalues of $D\widetilde{h}(z)$ are both real and positive;

  (2) if O is bounded, for any value of p/q, all $\widetilde{h}$-periodic points $z \in \partial O$ have prime period q and the eigenvalues of $D\widetilde{h}^q(z)$ are both real and positive;

So, as h is non-wandering, a periodic point $z \in \partial O$ is either an hyperbolic saddle or both eigenvalues of $Dh^q(z)$ are equal to 1 and z has topological index 0. In this way, Lemma 2.10 implies that z is a saddle-like periodic point, either a hyperbolic saddle or a point with zero index and local dynamics as is Figure 1.

With this local description, Theorem 5.1 in [28] implies that the boundary ∂O contains connections between saddle-like periodic points, as described above. We should remark that although in reference [24], most statements assume preservation of area and boundedness of O as a planar subset, the arguments therein only use the fact that the dynamics is non-wandering restricted to a small neighbourhood of the compact set K. This completes the proof of item (i).

Item (ii) is a direct consequence of Theorem C of [21]. $\qquad\square$

3. Perturbations for Homeomorphisms with Unbounded Deviation

In this section and in the next, for any $w \in \mathbb{R}^2$, we will write $[w]$ to denote an integral vector which is the closest to w. Also, a condition assumed in all the theorems proved here is unbounded deviation for a fixed direction, along which, we want our rotation set to grow.

Theorem 3.1. *Let* $\widetilde{f} \in \widetilde{Homeo}_{0,nw}(\mathbb{T}^2)$ *whose rotation set* $\rho(\widetilde{f})$ *is a line segment from* $(0,0)$ *to some* $(\alpha, \beta) \in \mathbb{R}^2$ *which is totally irrational. Assume* $\widetilde{f}$ *has unbounded deviation along the direction* (α, β). *Then* $\widetilde{f}$ *can be* C^0-*approximated by* $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$ *such that* $\rho(\widetilde{g})$ *has non-empty interior and*

$$(3.1) \qquad\qquad\qquad \rho(\widetilde{f}) \backslash \{(0,0)\} \subset Int(\rho(\widetilde{g})).$$

When the rotation set is as above, we can also study a similar situation around the other endpoint, $(0,0)$.

Theorem 3.2. *Let* $\widetilde{f}$ *and* $\rho(\widetilde{f})$ *be as in Theorem 3.1. Assume* $\widetilde{f}$ *admits unbounded deviation along* $-(\alpha, \beta)$. *Then* $\widetilde{f}$ *can be* C^0-*approximated by* $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$, *such that* $(0,0) \in Int(\rho(\widetilde{g}))$.

3.1. **Some Preparations.**

Given the totally irrational vector (α, β), define

$$(3.2) \qquad\qquad\qquad L_0 : y = \frac{\beta}{\alpha} x.$$

$$(3.3) \qquad\qquad\qquad L_1 : \alpha x + \beta y = \alpha^2 + \beta^2,$$

which are straight lines along the directions (α, β), $(-\beta, \alpha)$, respectively, and intersecting at the point (α, β). Also define the four connected components of the complement of $L_0 \bigcup L_1$ in $\mathbb{R}^2$. See Figure 2.

$$(3.4) \qquad \Delta_0 = \{\widetilde{z} \in \mathbb{R}^2 \big| \mathrm{pr}_{(-\beta,\alpha)}(\widetilde{z}) > 0 \text{ and } \mathrm{pr}_{(\alpha,\beta)}(\widetilde{z} - (\alpha,\beta)) > 0\}.$$

$$(3.5) \qquad \Delta_1 = \{\widetilde{z} \in \mathbb{R}^2 \big| \mathrm{pr}_{(-\beta,\alpha)}(\widetilde{z}) < 0 \text{ and } \mathrm{pr}_{(\alpha,\beta)}(\widetilde{z} - (\alpha,\beta)) < 0\}.$$

$$(3.6) \qquad \Omega_0 = \{\widetilde{z} \in \mathbb{R}^2 \big| \mathrm{pr}_{(-\beta,\alpha)}(\widetilde{z}) < 0 \text{ and } \mathrm{pr}_{(\alpha,\beta)}(\widetilde{z} - (\alpha,\beta)) > 0\}.$$

$$(3.7) \qquad \Omega_1 = \{\widetilde{z} \in \mathbb{R}^2 \big| \mathrm{pr}_{(-\beta,\alpha)}(\widetilde{z}) > 0 \text{ and } \mathrm{pr}_{(\alpha,\beta)}(\widetilde{z} - (\alpha,\beta)) < 0\}.$$
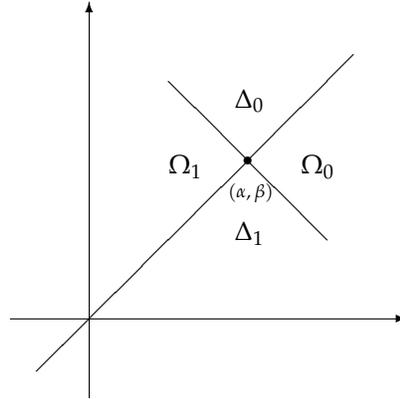
FIGURE 2. The Four Regions.

Lemma 3.3. *Let $\{x^{(0)}, \cdots, x^{(n-1)}\}$ be an ε-pseudo periodic orbit of some $f \in Homeo_0(\mathbb{T}^2)$. Then, f can be C^0-ε-perturbed to g, also in $Homeo_0(\mathbb{T}^2)$ such that $g^n(x^{(0)}) = x^{(0)}$. Moreover, if f preserves area, then so does g.*

Proof. See [13]. For the area-preserving case, just note that g is obtained from f by a series of perturbations supported in finitely many disjoint disks. And it is well-known that these perturbations, which are the identity in the boundary of each disk, can be chosen as area-preserving themselves. $\square$

Recall that Theorem 2.6 says that $\tilde{f}$ has bounded deviation along perpendicular directions. Based on this, the following lemma gives small displacements along the perpendicular directions, for some chosen iterates.

Lemma 3.4 (Small Displacement). *Let w be either $(-\beta, \alpha)$ or $(\beta, -\alpha)$. Then, for any $\delta > 0$ and any $\tilde{z}_0 \in \mathbb{R}^2$, there exists n_0 such that,*

$$(3.8) \qquad pr_w(\tilde{f}^n(\tilde{z}_0) - \tilde{f}^{n_0}(\tilde{z}_0)) < \delta, \text{ for any } n > n_0.$$

Proof. Observing Theorem 2.6, we can choose $n_0 \geqslant 1$, with

$$(3.9) \qquad \mathrm{pr}_w(\tilde{f}^{n_0}(\tilde{z}_0) - \tilde{z}_0) > \sup_{n \geqslant 1}\{\mathrm{pr}_w(\tilde{f}^n(\tilde{z}_0) - \tilde{z}_0)\} - \delta.$$

Then, for any $n > n_0$, (3.8) follows immediately. $\square$

3.2. The Irrational Endpoint.

Proof of Theorem 3.1. Fix $\varepsilon > 0$. By Theorem 1 of [1], for any ergodic measure with average rotation vector (α, β), around a typical point which is f-recurrent, there is some point y and a positive integer n, such that

$$(3.10) \qquad \mathrm{dist}_{\mathbb{T}^2}(f^n(y), y) < \varepsilon.$$

$$(3.11) \qquad \mathrm{pr}_{(\alpha, \beta)}([\tilde{f}^n(\tilde{y}) - \tilde{y}] - n(\alpha, \beta)) > 0.$$

Considering Figure 2, the last estimate implies that

$$(3.12) \qquad \frac{1}{n}[\tilde{f}^n(\tilde{y}) - \tilde{y}] \in \Omega_0 \bigcup \Delta_0.$$

From now on, we assume that $\tilde{y}_0$ and $n^* \geqslant 1$ satisfy that $f^{n^*}(y_0)$ is ε-close to y_0, and

$$(3.13) \qquad \frac{1}{n}[\tilde{f}^{n^*}(\tilde{y}_0) - \tilde{y}_0] \in \Delta_0.$$

The other possibility (the rotation vector obtained in the above expression belonging to Ω_0) is analogous. The goal is to show that, we can always find another 6ε-pseudo periodic orbit lifting to a 6ε-pseudo $\tilde{f}$-orbit segment starting at some $\tilde{z}'$ and ending at some $\tilde{z}''$, so that the rational vector

$$(3.14) \qquad \frac{1}{n}(\tilde{z}'' - \tilde{z}') \in \Omega_0.$$

We assume the totally irrational (α, β) has norm 1 for simplicity. For any $K > 0$, define $\mathcal{R}_K \subset \mathbb{T}^2$ to be the set of points x such that for at least K choices of positive integers n, we have $\mathrm{dist}_{\mathbb{T}^2}(f^n(x), x) \leqslant \frac{1}{K}$ and $\mathrm{pr}_{(\alpha, \beta)}(\tilde{f}^n(\tilde{x}) - \tilde{x} - n(\alpha, \beta)) \geqslant K$. We claim that $\mathcal{R}_K$ is non-empty for any positive constant K. In fact, we can cover $\mathbb{T}^2$ with N disks (for some integer N), all with diameter $\frac{1}{K}$. Then by the assumption

on unbounded deviation in the direction (α, β), it is not hard to find $\widetilde{x}$, and integers $0 = m_0 < m_1 < \cdots < m_{KN}$, such that, for any $k = 1, \cdots, KN$,

$$(3.15) \qquad \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^{m_k}(\widetilde{x}) - \widetilde{f}^{m_{k-1}}(\widetilde{x}) - (m_k - m_{k-1})(\alpha, \beta)\big) \geqslant K.$$

By the pigeonhole principle, for at least $K + 1$ choices of the indices among those m_j's, the corresponding iterates of x_0 lie in one single disk with diameter no more than $\frac{1}{K}$. Clearly, between any two of these chosen ones, say $m_i < m_j$, we see

$$(3.16) \qquad \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^{m_j}(\widetilde{x}) - \widetilde{f}^{m_i}(\widetilde{x}) - (m_j - m_i)(\alpha, \beta)\big) \geqslant K,$$

and in particular the claim follows by taking the first iterate among those.

Next we take an accumulation point x^* of the sets $\mathcal{R}_K$ as K tends to infinity. By Lemma 2.2, the point $x^* - 4\varepsilon(-\beta, \alpha)$ is approximated by an f-recurrent point. Then, with the help of Lemma 3.4, one can find a point z_1 which is some forward iterate of the recurrent point we just found, and a positive integer n_1, such that both z_1 and $f^{n_1}(z_1)$ are ε-close to $x^* - 4\varepsilon(-\beta, \alpha)$ and

$$(3.17) \qquad \mathrm{pr}_{(-\beta,\alpha)}\big(\widetilde{f}^{n_1}(\widetilde{z}_1) - \widetilde{z}_1\big) < \varepsilon.$$

Next, by choosing another f-recurrent point near the point $f^{n_1}(z_1) - 4\varepsilon(-\beta, \alpha)$ and then applying Lemma 3.4, we can find another orbit segment satisfying similar estimates. In fact, we can inductively find $z_1, z_2, \cdots, z_{K_0}$ with disjoint orbits, and integers $n_1, n_2, \cdots, n_{K_0}$, such that, for any $i = 2, \cdots, K_0$, both z_i and $f^{n_i}(z_i)$ are ε-close to $f^{n_{i-1}}(z_{i-1}) - 4\varepsilon(-\beta, \alpha)$ and

$$(3.18) \qquad \mathrm{pr}_{(-\beta,\alpha)}\big(\widetilde{f}^{n_i}(\widetilde{z}_i) - \widetilde{z}_i\big) < \varepsilon.$$

Moreover, K_0 is chosen as the smallest integer such that $f^{K_0}(z_{K_0}) - 4\varepsilon(-\beta, \alpha)$ is ε-close to x^* in $\mathbb{T}^2$. It is not difficult to ensure K_0 is found to be finite and the process stops. Then we sum all the deviations (in the direction (α, β)) of each of the above orbit segments.

$$(3.19) \qquad M := \sum_{i=1}^{K_0} \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^{n_i}(\widetilde{z}_i) - \widetilde{z}_i - n_i(\alpha, \beta)\big).$$

When we consider a point x in $\mathcal{R}_K$ with $K > |M| + \varepsilon(K_0 + 1)$, by definition, for at least K positive integers, its corresponding iterates all lie in a same disk of diameter at most $1/K$, and pairwise the deviation along the direction (α, β) is at least $|M| + \varepsilon(K_0 + 1)$. Moreover, the dynamics has bounded deviation along the perpendicular direction (see Theorem 2.6). So when K is sufficiently large, we can find two of these iterates so that along the direction $(-\beta, \alpha)$, the deviation is at most ε (with a similar argument as in Lemma 3.4).

Recall now that x^* is an accumulation point of $\mathcal{R}_K$. Therefore, by the above paragraph, we can find z_0 and an integer n_0, such that both z_0 and $f^{n_0}(z_0)$ are ε-close to x^*,

$$(3.20) \qquad \mathrm{pr}_{(-\beta,\alpha)}\big(\widetilde{f}^{n_0}(\widetilde{z}_0) - \widetilde{z}_0\big) < \varepsilon \text{ and}$$

$$(3.21) \qquad \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^{n_0}(\widetilde{z}_0) - \widetilde{z}_0 - n_0(\alpha, \beta)\big) > |M| + \varepsilon(K_0 + 1).$$

Moreover, we can require that the orbits of z_i, $i = 0, \cdots, K_0$ and y_0 are all pair-wise disjoint. Then we write down the following $K_0 + 1$ point-wise disjoint f-orbit segments, namely

$$(3.22) \qquad \{z_1, f(z_1), \cdots, f^{n_1}(z_1)\}, \cdots, \{z_{K_0}, \cdots, f^{n_{K_0}}(z_{K_0})\}, \{z_0, \cdots, f^{n_0}(z_0)\}.$$

They together form a 6ε-pseudo periodic orbit of period

$$(3.23) \qquad \ell = \sum_{j=0}^{K_0} n_j.$$

Note that, (3.21) implies the final deviation of the whole pseudo-orbit along (α, β) is positive. Among those $K + 1$ segments in (3.22), the way we jump between two consecutive orbit segment gives at least 2ε deviation along the direction $(\beta, -\alpha)$, and within each segment the deviation along $(-\beta, \alpha)$ is at most ε. So in the end we see positive deviation along $(\beta, -\alpha)$. It follows that this pseudo orbit sees a rotation vector in the region Ω_0.

Thus, we can apply Lemma 3.3 twice in order to close this and the ε-pseudo periodic orbits containing y_0 which we found at the beginning of this proof. So we obtain $\widetilde{g}$ which is 6ε-close to $\widetilde{f}$ in the C^0 topology, and admits two periodic points y_0 and z_1. By (3.13), $\rho(\widetilde{g}, y_0) \in \Delta_0$. And by the above construction, it follows that $\rho(\widetilde{g}, z_1) \in \Omega_0$.

Since $(0, 0)$ is an extremal point of $\rho(\widetilde{f})$, by [13] f admits a contractible fixed point p^*. Now as we look back at the whole perturbation process above, we can choose all the orbit segments far from p^*. This means that the perturbations can be made away from p^*. Thus, the rotation set of $\widetilde{g}$ satisfies $\rho(\widetilde{g}) \supset \{(0, 0), \rho(\widetilde{g}, z_1), \rho(\widetilde{g}, y_0)\}$. Since any rotation set is convex (Theorem 2.4), $\rho(\widetilde{g})$ is as required. As ε can be arbitrarily small, the proof of Theorem 3.1 is completed. $\quad\square$

3.3. **Origin as Endpoint.**

Proof of Theorem 3.2. This proof is similar to the above one. We fix ε now. First let us define two new regions as follows.

$$(3.24) \qquad \mathcal{D}_0 := \{\widetilde{z} \in \mathbb{R}^2 \,|\, \mathrm{pr}_{(\alpha, \beta)}(\widetilde{z}) < 0 \text{ and } \mathrm{pr}_{(-\beta, \alpha)}(\widetilde{z}) > 0\}.$$

$$(3.25) \qquad \mathcal{D}_1 := \{\widetilde{z} \in \mathbb{R}^2 \,|\, \mathrm{pr}_{(\alpha, \beta)}(\widetilde{z}) < 0 \text{ and } \mathrm{pr}_{(-\beta, \alpha)}(\widetilde{z}) < 0\}.$$

For any $K > 0$, define $\mathcal{R}'_K$ to be the set of points z such that for at least K choices of integers $n > 0$, the following holds.

$$(3.26) \qquad \mathrm{dist}_{\mathbb{T}^2}(f^n(z), z) \leqslant \frac{1}{K}.$$

$$(3.27) \qquad \mathrm{pr}_{(\alpha, \beta)}(\widetilde{f}^n(\widetilde{z}) - \widetilde{z}) \leqslant -K.$$

With the condition of unbounded deviations in the direction $-(\alpha, \beta)$, by a similar argument as in previous subsection, we can show $\mathcal{R}'_K$ is non-empty for any $K > 0$.

By taking $K \geqslant \frac{1}{\varepsilon}$, we can find an ε-pseudo periodic point y_0 of period n^*, which realizes a rotation vector in the region $\mathcal{D}_0 \cup \mathcal{D}_1$. Without loss of generality, assume it lies in $\mathcal{D}_0$ (the other possibility is analogous). More precisely, for a lift $\widetilde{y}_0$ of y_0 and some integer $n_* > 0$, $\mathrm{dist}_{\mathbb{T}^2}(f^{n*}(y_0) - y_0) < \varepsilon$, and

$$(3.28) \qquad v = \frac{1}{n_*}[\widetilde{f}^{n*}(\widetilde{y}_0) - \widetilde{y}_0] \in \mathcal{D}_0.$$

Then, very similarly to the previous subsection, let y^* be an accumulation point of $\mathcal{R}'_K$ as K tends to infinity. With the help of Lemmas 2.2 and Lemma 3.4, as well as the definition of $\mathcal{R}'_K$, we can choose finitely many orbit segments, which altogether form a 6ε-pseudo periodic orbit, lifting to a 6ε-pseudo orbit for $\widetilde{f}$, namely $\{\widetilde{z}_0, \widetilde{z}_1, \cdots, \widetilde{z}_{n'}\}$, such that

$$(3.29) \qquad v' = \frac{1}{n'}[\widetilde{z}_{n'} - \widetilde{z}_0] \in \mathcal{D}_1.$$

Thus, we obtain two pseudo periodic orbits seeing rotation vectors $v \in \mathcal{D}_0$ and $v' \in \mathcal{D}_1$. Since (α, β) is an extremal point of $\rho(\widetilde{f})$, by Theorem 2.4, there exists some ergodic f-invariant measure μ satisfying $\rho_\mu(\widetilde{f}) = (\alpha, \beta)$. Moreover, for a μ-typical point $x \in \mathbb{T}^2$, for some increasing integer sequence n_j, and any lift $\widetilde{x}$,

$$(3.30) \qquad \lim_{j \to \infty} f^{n_j}(x) = x.$$

$$(3.31) \qquad \lim_{j \to \infty} \frac{1}{n_j}\big(\widetilde{f}^{n_j}(\widetilde{x}) - \widetilde{x}\big) = (\alpha, \beta).$$

Then for sufficiently large n_j, $\{x, f(x), \cdots, f^{n_j}(x)\}$ forms an ε-pseudo periodic orbit, such that the vectors v, v' and

$$(3.32) \qquad w = \frac{1}{n_j}[\widetilde{f}^{n_j}(\widetilde{x}) - \widetilde{x}]$$

spans a triangle, which contains the origin $(0,0)$ in its interior. Note that we can choose the three pseudo orbits to be pair-wise disjoint. Then, applying Lemma 3.3 three times, we obtain a 6ε-perturbation $\widetilde{g}$ of $\widetilde{f}$. The rotation set $\rho(\widetilde{g})$ contains at least the three rational points v, v' and w. By the convexity of the rotation set, $(0,0)$ is contained in $\text{Int}\big(\rho(\widetilde{g})\big)$. We have completed the proof.   □

4. PERTURBATIONS FOR HOMEOMORPHISMS WITH BOUNDED DEVIATION

4.1. **The Totally Irrational Rigid Rotation.**

We start by showing a simple perturbation result for the rigid rotation $R_{(\alpha,\beta)}$ on $\mathbb{T}^2$, where (α, β) is totally irrational.

Proposition 4.1. *For any $\varepsilon > 0$, there exists $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$ with $dist_{C^0}(\widetilde{g}, R_{(\alpha,\beta)}) < \varepsilon$, such that, $\rho(\widetilde{g})$ has interior, and $(\alpha, \beta) \in Int(\rho(\widetilde{g}))$.*

Proof. Since (α, β) is totally irrational, the rigid rotation $R_{(\alpha,\beta)}$ is minimal. For any $x_0 \in \mathbb{T}^2$ and for any small $\varepsilon > 0$, consider a small disk $B = B(x_0, \varepsilon)$, which is divided into four regions as follows.

$$(4.1) \qquad \Delta_1(x_0, \varepsilon) := \{x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha,\beta)}(x - x_0) < 0, \text{pr}_{(-\beta,\alpha)}(x - x_0) < 0\}.$$

$$(4.2) \qquad \Delta_0(x_0, \varepsilon) := \{x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha,\beta)}(x - x_0) > 0, \text{pr}_{(-\beta,\alpha)}(x - x_0) > 0\}.$$

$$(4.3) \qquad \Omega_1(x_0, \varepsilon) := \{x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha,\beta)}(x - x_0) < 0, \text{pr}_{(-\beta,\alpha)}(x - x_0) > 0\}.$$

$$(4.4) \qquad \Omega_0(x_0, \varepsilon) := \{x \in B(x_0, \varepsilon) | \text{pr}_{(\alpha,\beta)}(x - x_0) > 0, \text{pr}_{(-\beta,\alpha)}(x - x_0) < 0\}.$$

By minimality of $R_{(\alpha,\beta)}$, we can choose some integer n, such that $R^n_{(\alpha,\beta)}(x_0) \in \Delta_1(x_0, \varepsilon)$. Then, for proper choices of lifts $\widetilde{R}$ and $\widetilde{x}_0$ of $R_{(\alpha,\beta)}$ and x_0, respectively,

$\widetilde{R}^n(\widetilde{x}_0)$ is ε-close to $\widetilde{x}_0 + (a, b)$ for some $(a, b) \in \mathbb{Z}^2$. Moreover, we can write $v_1 = (\frac{a}{n}, \frac{b}{n})$, and then clearly $v_1 \in \Delta_0((\alpha, \beta), \frac{\varepsilon}{n})$.

In other words, we can find an ε-pseudo periodic orbit for the rigid rotation $R_{(\alpha,\beta)}$, which sees a rational rotation vector in the region $\Delta_0((\alpha, \beta), \frac{\varepsilon}{n})$. We argue in a similar way with respect to the other three regions. Then, we obtain four ε-pseudo periodic orbits for $R_{(\alpha,\beta)}$, starting with x_0, y_0, z_0, w_0, respectively. We can also require these orbit segments to be point-wise disjoint. Then, applying Lemma 3.3 four times, these four pseudo orbits can be closed via an ε-perturbation, which produces four periodic orbits. These periodic orbits will have four rational rotation vectors v_1, v_2, v_3, v_4, respectively, whose convex hull contains (α, β) in the interior. Therefore $(\alpha, \beta) \in \text{Int}(\rho(\widetilde{g}))$. $\qquad\square$

Remark 4.1. *This proposition can be compared with Theorem 1 in* [19], *where with respect to sufficiently high regularity, due to the "KAM" phenomenon, the perturbed rotation set either misses (α, β), or it equals $\{(\alpha, \beta)\}$, provided (α, β) satisfies certain Diophantine conditions.*

4.2. **Bounded Deviations.**

In this subsection, we assume that $\widetilde{f}$ has bounded deviation along the direction (α, β). We consider this case for the sake of completeness, but it is possible that it might not happen at all (cf. Theorem 1.6 and Question 1).

Theorem 4.2. *Suppose $\widetilde{f} \in \widetilde{Homeo}_0(\mathbb{T}^2)$, whose rotation set $\rho(\widetilde{f})$ is the segment from $(0, 0)$ to the totally irrational point (α, β). Assume $\widetilde{f}$ has bounded deviation along the direction (α, β). Then $\widetilde{f}$ can be C^0-approximated by $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$, such that $\rho(\widetilde{g})$ has interior, and $\rho(\widetilde{f}) \backslash \{(0, 0)\} \subset \text{Int}(\rho(\widetilde{g}))$.*

Assume for some $M > 0$, for any $\widetilde{x} \in \mathbb{R}^2$ and any $n \geqslant 1$,

$$(4.5) \qquad \text{pr}_{(\alpha,\beta)}(\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n(\alpha, \beta)) \leqslant M.$$

Definition 7. *Let $\mathcal{M}_{(\alpha,\beta)}$ denote the set of ergodic f-invariant Borel probability measures, which have (α, β) as rotation vector. Then write*

$$(4.6) \qquad \mathcal{S}_{(\alpha,\beta)} := \overline{\bigcup_{\mu \in \mathcal{M}_{(\alpha,\beta)}} supp(\mu)},$$

where $supp(\mu)$ denotes the support of μ.

The following Lemma, whose proof depends on Atkinson's theorem on Cocycles (see [6]), appears as Lemma 6 of [2] or Proposition 65 of [27].

Lemma 4.3. *Suppose $\widetilde{f}$ satisfies condition (4.5). Then for any $x \in \mathcal{S}_{(\alpha,\beta)}$ with a lift $\widetilde{x}$, and for any $n \geqslant 1$,*

$$(4.7) \qquad pr_{(\alpha,\beta)}(\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n(\alpha, \beta)) \geqslant -M.$$

In particular, any invariant ergodic measure μ such that $supp(\mu) \subset \mathcal{S}_{(\alpha,\beta)}$ is contained in $\mathcal{M}_{(\alpha,\beta)}$.

Proof of Theorem 4.2. Fix $\varepsilon > 0$. Choose any minimal set $K \subset \mathcal{S}_{(a,b)}$. Then $\rho(\widetilde{f}, K) = \{(\alpha, \beta)\}$. By assumption (4.5), Theorem 2.6 and Lemma 4.3, $\widetilde{f}|_{\pi^{-1}(K)}$ has bounded

deviation along every direction. So we can apply Lemma 2.5 in order to find a semi-conjugacy ϕ between $(K, f|_K)$ and $(\mathbb{T}^2, R)$, where $R = R_{(\alpha,\beta)}$ denotes the rigid rotation on $\mathbb{T}^2$ by (α, β). There is a lift $\widetilde{\phi}$ of ϕ, which conjugates $\widetilde{f}|_{\pi^{-1}(K)}$ and $\widetilde{R}$. Note that the pre-image of every point under $\widetilde{\phi}$ has diameter uniformly bounded from above, say by a constant $C_\phi > 0$. Up to renormalizing the torus to a finite cover, we can assume that

$$(4.8) \qquad\qquad\qquad\qquad C_\phi < 1/6.$$

Then, there exists a positive integer n_0, such that for any $n \geqslant n_0$, the following can be ensured. For any $x \in \mathbb{T}^2$ and its pre-image $\gamma = \phi^{-1}(x)$, and for any points $p \in \gamma$ and $q \in f^n(\gamma)$, one can find an ε-pseudo orbit segment of length n, starting at p and ending at q, such that every jump happens within a leaf $\phi^{-1}(f^k(\gamma))$, for $k \geqslant 1$.

Recall the four regions defined from (4.1) to (4.4). Since $R = R_{(\alpha,\beta)}$ is minimal, for some sufficiently large positive integer $n > n_0$, and for a point $x_0 \in \mathbb{T}^2$ with its pre-image $\gamma_0 = \phi^{-1}(x_0)$, we have that

$$(4.9) \qquad\qquad\qquad\qquad R^n(x) \in \Omega_1(x, \varepsilon).$$

$$(4.10) \qquad\qquad\qquad\qquad \mathrm{dist}_{\mathbb{T}^2}(f^n(\gamma_0), \gamma_0) < \varepsilon.$$

Estimate (4.9) means that, by choosing a lift $\widetilde{x}_0$ of x_0, and writing $(a, b) = [\widetilde{R}^n(\widetilde{x}_0) - \widetilde{x}_0]$, we have

$$(4.11) \qquad\qquad v = \frac{[\widetilde{R}^n(\widetilde{x}_0) - \widetilde{x}_0]}{n} = (\frac{a}{n}, \frac{b}{n}) \in \Omega_0.$$

Equivalently (see Figure 2),

$$(4.12) \qquad\qquad\qquad \mathrm{pr}_{(\alpha,\beta)}\big(v - (\alpha, \beta)\big) > 0.$$

$$(4.13) \qquad\qquad\qquad \mathrm{pr}_{(-\beta,\alpha)}\big(v - (\alpha, \beta)\big) < 0.$$

On the other hand, by estimate (4.10) and noting $n > n_0$, we can find an ε-pseudo periodic orbit, containing a point $p \in \gamma_0$ so that for some $q \in f^n(\gamma_0)$,

$$(4.14) \qquad\qquad \mathrm{dist}_{\mathbb{T}^2}(p, q) = \mathrm{dist}_{\mathbb{T}^2}(\gamma_0, f^n(\gamma_0)) < \varepsilon.$$

This pseudo-orbit is obtained in the following way. For each $k = 1, \cdots, n - 1$, the orbit has a jump within the pre-image $\phi^{-1}(f^k(\gamma_0))$, so that at the $(n-1)$-th time, it arrives at $f^{-1}(q)$. Then the final jump happens from q to p. This pseudo-orbit lifts to an ε-pseudo orbit for $\widetilde{f}$ in $\mathbb{R}^2$. Recall from (4.8) that, the pre-image under the lifted semi-conjugacy $\widetilde{\phi}$ of any point has diameter bounded by $1/6$. It follows that the pseudo-orbit must see the same integer translate as the rigid rotation. More precisely, the ε-pseudo-orbit above must start at some point $\widetilde{p}$ and end at $\widetilde{p} + (a, b)$, so that it sees the rotation vector $v = (\frac{a}{n}, \frac{b}{n}) \in \Omega_0$.

In a similar way, we can find another ε-pseudo f-periodic orbit which sees a rational rotation vector in Δ_0. Moreover, as we explained before, there is also at least one contractible fixed point p_*. Note that we can choose the two pseudo orbits and the fixed point to be pairwise disjoint. So, if we apply Lemma 3.3 twice, we obtain an ε-perturbation $\widetilde{g} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$, with at least three periodic orbits, p_*, p_0, u_0, such that, $\rho(\widetilde{g}, p_*) = 0, \rho(\widetilde{g}, p_0) = v$ and $\rho(\widetilde{g}, u_0) = u$ span a triangle which contains $\rho(\widetilde{f}) \setminus \{(0, 0)\}$ in its interior. By Theorem 2.4, this triangle is clearly included in $\rho(\widetilde{g})$, and the proof of the Theorem is completed. $\qquad\square$

Proof of Theorem 1.1. Theorem 1.1 follows immediately from Theorem 3.1 and Theorem 4.2. □

5. GENERIC DIFFEOMORPHISMS

In this section, we prove the following theorem.

Theorem 5.1. *[Theorem 1.2 restated] Let $f \in \mathcal{G}^r$. Then for any lift $\widetilde{f}$, the rotation set $\rho(\widetilde{f})$ can not be a segment from $(0,0)$ to a totally irrational point (α, β).*

Proof. The proof of this theorem will go through the whole section. We assume the following conditions and arrive at a contradiction in the end of the proof.

$$(5.1) \qquad f \in \mathcal{G}^r.$$

$$(5.2) \qquad \rho(\widetilde{f}) \text{ is the segment from}(0,0) \text{ to the totally irrational point } (\alpha, \beta).$$

Since (α, β) is an extremal point of $\rho(\widetilde{f})$, by Theorem 2.4, we can choose an f-recurrent point z_*, such that

$$(5.3) \qquad \rho(\widetilde{f}, z_*) = \lim_{n \to \infty} \frac{1}{n}\left(\widetilde{f}^n(\widetilde{z}_*) - \widetilde{z}_*\right) = (\alpha, \beta).$$

Now, consider the family of all the f-invariant open topological disks. We can define a partial order among this family with respect to the usual inclusion relation. It is standard to check that with respect to this order, the family forms a partially ordered set, for which every chain has an upper bound. So by Zorn's lemma, we conclude the existence of maximal elements. As f is non-wandering, Lemma 2.3 implies that there exists a constant $M > 0$, such that, every connected component $\widetilde{D}$ of the lift of a maximal open f-invariant disk D is $\widetilde{f}$-invariant, and

$$(5.4) \qquad \qquad \text{diam}(\widetilde{D}) < M.$$

Also, if D is an f-invariant maximal open topological disk, then it contains fixed points. This follows from a classical argument: pick some point $p \in D$. If p is not fixed, then for a sufficiently small open ball B centered at p, contained in D, we have: B is disjoint from $f(B)$ and $f^n(B)$ intersects B for a sufficiently large $n > 0$ (there are no wandering points). And this implies the existence of a fixed point inside D, see Lemma 2.7. Since for $f \in \mathcal{G}^r$, there are finitely many fixed points, it follows that there are at most finitely many maximal open f-invariant disks.

Note that z_* is disjoint from the closure of the union of these finitely many maximal open f-invariant disks, because the orbit of any $\widetilde{z}_* \in \pi^{-1}(z_*)$ is unbounded in $\mathbb{R}^2$. Thus, we can choose some $\delta > 0$ such that, the open disk $B(z_*, \delta)$ is still disjoint from the closure of the union of these maximal open f-invariant disks. Define the set

$$(5.5) \qquad U := \text{ the connected component of } \bigcup_{n \in \mathbb{Z}} f^n\big(B(z_*, \delta)\big) \text{ which contains } z_*.$$

The following lemma allows us to find a good hyperbolic saddle fixed point.

Lemma 5.2. *There exists at least one fixed hyperbolic saddle point Q_* which is contained in $\overline{U}$.*

Proof. Clearly U is open. Since f is non-wandering, U is f^{n^*}-invariant for some $n^* > 0$. Recall the notions in subsection 2.1, and claim that U is essential. Suppose otherwise and let U_{filled} be the union of U with all the connected components of the complement of U which are contractible. This construction implies that U_{filled} is an open disk and, by Lemma 2.3 applied to f^{n^*}, all connected components of the lift of U_{filled} to the plane have bounded diameter and are $\widetilde{f}^{n^*}$-invariant (because $(0, 0)$ is the only rational point contained in the rotation set). In particular, any lift of z_* has bounded orbit, a contradiction with (5.3).

The next claim is that U is in fact fully essential. Suppose it is not, then all the homotopically non-trivial loops contained in U are homotopic to each other. Fix one homotopically non-trivial loop $\gamma \subset U$ and choose connected components of their lifts, $\widetilde{\gamma}$ and $\widetilde{U}$, such that,

$$(5.6) \qquad\qquad\qquad \widetilde{\gamma} \subset \widetilde{U}.$$

Clearly, there exists some integer vector $(a, b) \neq (0, 0)$, such that $\widetilde{\gamma} = \widetilde{\gamma} + (a, b)$. Moreover, since by assumption U is not fully essential, $\widetilde{U} \bigcap (\widetilde{U} + i(-b, a)) = \varnothing$ for any integer $i \neq 0$. Therefore,

$$(5.7) \qquad \widetilde{U} \text{ is contained in the strip bounded by } \widetilde{\gamma} - (-b, a) \text{ and } \widetilde{\gamma} + (-b, a).$$

This is a contradiction with the choice of z_* and the fact that (α, β) is totally irrational. So U is fully essential. Moreover, if $\overline{U}$ is not the whole torus, then any connected component of $(\overline{U})^c$ is a periodic open disk (the periodicity follows from the non-wandering hypothesis). In particular, by the choice of $\delta > 0$, each f-invariant maximal open disk (if any) is a connected component of $(\overline{U})^c$.

By assumption (5.1), every fixed point has non-zero topological index. In this case, the fixed point is also called a non-degenerate fixed point. In general there are two types of such points:

(1) $p \in \text{Fix}(f)$ has topological index 1.
(2) $p \in \text{Fix}(f)$ has topological index -1. In this case, p is a hyperbolic saddle, and both eigenvalues of $Df(p)$ are positive real numbers, one larger than 1 and the other smaller.

Now by Lemma 2.13, condition (2) of Definition 3 implies that each f-invariant open disk D has prime ends rotation number $\rho_{\text{PE}}(f, D) \notin \mathbb{Q}$. In particular, such a prime ends rotation number is not zero, so the sum of the indices of fixed points contained in the union of all the (finitely many) maximal f-invariant open disks is positive (or zero, in case $\overline{U}$ is the whole torus). By the Lefschetz fixed point formula (Lemma 2.8), the sum of the indices at all the fixed points is zero. And as $(0, 0)$ is an extremal point of the rotation set, f must have fixed points (see [13]). So it follows that there exists at least one negatively indexed fixed point, denoted Q_*, contained in the complement of the union of these maximal open f-invariant disks. Thus, Q_* is a fixed hyperbolic saddle point, which belongs to $\overline{U}$, and we have finished the proof. $\square$

For a fixed hyperbolic saddle point Q_* (or a fixed saddle-like point), let an *unstable branch* (respectively, *stable branch*) at Q_* be one of the connected components of $W^u(Q_*) \backslash \{Q_*\}$ (respectively, one of the connected components of $W^s(Q_*) \backslash \{Q_*\}$). Choose a lift $\widetilde{Q}_*$ of the hyperbolic saddle point Q_*, which is fixed by $\widetilde{f}$. We can

then lift the corresponding stable and unstable branches at Q_* to those branches at $\widetilde{Q}_*$.

Proposition 5.3. *It is not possible that some unstable branch $\widetilde{\lambda}_u$ and some stable branch $\widetilde{\lambda}_s$ at the hyperbolic saddle $\widetilde{Q}_*$ intersect.*

Proof. Suppose by contradiction that a stable branch $\widetilde{\lambda}_s$ at $\widetilde{Q}_*$ intersects an unstable branch $\widetilde{\lambda}_u$ at $\widetilde{Q}_*$. We can then choose an intersection point $\widetilde{w}$, such that, the arc along $\widetilde{\lambda}_u$ from $\widetilde{Q}_*$ to $\widetilde{w}$ and the arc along $\widetilde{\lambda}_s$ from $\widetilde{Q}_*$ to $\widetilde{w}$ are disjoint, except at their endpoints. It follows that, the union of these two arcs bounds a topological disk $\widetilde{D}$. Now we define

$$\tag{5.8} \widetilde{D}_{\mathrm{sat}} = \bigcup_{n \in \mathbb{Z}} \widetilde{f}^n(\widetilde{D}).$$

Note that $\widetilde{D}_{\mathrm{sat}}$ is an open and connected $\widetilde{f}$-invariant subset of the plane.

If there exists some integer vector $(a, b) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$, such that

$$\tag{5.9} \widetilde{D}_{\mathrm{sat}} \bigcap \left(\widetilde{D}_{\mathrm{sat}} + (a, b) \right) \neq \varnothing,$$

then either $\widetilde{\lambda}_u$ intersects $\widetilde{\lambda}_s + (a, b)$ topologically transversely, or $\widetilde{\lambda}_u$ intersects $\widetilde{\lambda}_s - (a, b)$ topologically transversely (see definition 4). In both cases, it follows from Lemma 2.9 that there exists a periodic orbit p_0 whose rotation vector $\rho(\widetilde{f}, p_0)$ is non-zero and rational, which is a contradiction with assumption (5.2).

Thus, we are left with the case when $\widetilde{D}_{\mathrm{sat}}$ does not intersect any of its non-trivial integer translations. As before, we consider the filled open set $\mathrm{Fill}(\widetilde{D}_{\mathrm{sat}})$, which is given by the union of $\widetilde{D}_{\mathrm{sat}}$ and all the bounded connected components of its complement. It is not hard to see that $\mathrm{Fill}(\widetilde{D}_{\mathrm{sat}})$ is an open topological disk, which does not intersect any of its non-zero integer translations. Thus, we can consider its projection $D_{\mathrm{Fill}} := \pi(\mathrm{Fill}(\widetilde{D}_{\mathrm{sat}}))$, which is an f-invariant open disk. By Lemma 2.3, $\mathrm{Fill}(\widetilde{D}_{\mathrm{sat}})$ has bounded diameter. Since $f \in \mathcal{G}^r$, in particular there are no saddle connections, by Lemma 2.13, the prime ends rotation number $\rho_{\mathrm{PE}}(\widetilde{f}, \mathrm{Fill}(\widetilde{D}_{\mathrm{sat}}))$ is irrational, and so the boundary $\partial\mathrm{Fill}(\widetilde{D}_{\mathrm{sat}})$ does not contain any periodic point. This implies that

$$\tag{5.10} \widetilde{Q}_* \in \mathrm{Fill}(\widetilde{D}_{\mathrm{sat}}).$$

This is a contradiction because Q_* does not to belong to a fixed open disk, see Lemma 5.2. So $\widetilde{\lambda}_u$ does not intersect any stable branch at $\widetilde{Q}_*$. $\square$

Remark 5.1. *The same conclusion is also true for the unstable and the stable branch at an index 0 saddle-like fixed point. The proof follows the same lines as above, with a difference that, when we apply Lemma 2.9 in the arguments, we actually need the statement for saddle-like fixed points, as was explained in Remark 2.5. Note also, the conditions stated in Lemma 2.13 are such that in both cases it can be applied.*

Lemma 5.4. *Each stable or unstable branch at $\widetilde{Q}_*$ is unbounded in $\mathbb{R}^2$.*

Proof. For definiteness, fix any unstable branch $\widetilde{\lambda}_u$ at $\widetilde{Q}_*$, and assume by contradiction that it is bounded.

The first claim is that, the closure $\mathrm{cl}(\widetilde{\lambda}_u)$ must intersect all the other branches at $\widetilde{Q}_*$. To see the claim, assume by contradiction that $\mathrm{cl}(\widetilde{\lambda}_u)$ does not intersect some

branch $\widetilde{\lambda}$. Then there exists some connected component $\widetilde{U}$ of the complement of $\mathrm{cl}(\widetilde{\lambda}_u)$, containing $\widetilde{\lambda}$. Since $\widetilde{\lambda}$ is $\widetilde{f}$-invariant, so is $\widetilde{U}$. Note that $\widetilde{Q}_* \in \partial\widetilde{U}$, and it is in fact accessible through the branch $\widetilde{\lambda}$, from the interior of $\widetilde{U}$. Thus, the prime ends rotation number $\rho_{\mathrm{PE}}(\widetilde{f}, \widetilde{U})$ must be equal to 0. And so, Lemma 2.13 implies the existence of saddle-connections, something that contradicts item (2) of Definition 3.

The second claim is that, if $\widetilde{\lambda}$ is any other branch at $\widetilde{Q}_*$, then $\mathrm{cl}(\widetilde{\lambda}_u) \supset \widetilde{\lambda}$. To prove this claim, note first that if $\widetilde{\lambda}$ is another unstable branch, then $\widetilde{\lambda}$ does not intersect $\widetilde{\lambda}_u$. And if $\widetilde{\lambda}$ is a stable branch, then by Proposition 5.3, we again obtain that $\widetilde{\lambda}$ does not intersect $\widetilde{\lambda}_u$. The following argument is a variation of one due to Fernando Oliveira in the area-preserving case (see Lemma 2 of [34]). We include it here for completeness.

Assume by contradiction that

$$\tag{5.11} \mathrm{cl}(\widetilde{\lambda}_u) \not\supset \widetilde{\lambda}.$$

Since $\mathrm{cl}(\widetilde{\lambda}_u)$ is a connected $\widetilde{f}$-invariant compact subset, there is a compact simple arc γ contained in $\widetilde{\lambda}$, such that $\mathrm{cl}(\widetilde{\lambda}_u) \bigcap \gamma$ consists of exactly the two endpoints of γ. Then there are two possibilities:

(1) for all non-zero integer vectors (m,n), $\gamma \bigcap \left(\mathrm{cl}(\widetilde{\lambda}_u) + (m,n)\right) = \varnothing$.
(2) for some $(m_0, n_0) \in \mathbb{Z}^2\backslash\{(0,0)\}$, $\gamma \bigcap \left(\mathrm{cl}(\widetilde{\lambda}_u) + (m_0, n_0)\right) \neq \varnothing$.

In case (1), we can find a bounded connected component of the complement of $\mathrm{cl}(\widetilde{\lambda}_u) \cup \gamma$, whose boundary contains γ. We denote it as $\widetilde{O}$. Then look at the set $O = \pi(\widetilde{O})$. It is not hard to see that, O contains a wandering domain for f, which is a contradiction.

In case (2), as γ is contained in a branch at $\widetilde{Q}_*$, $\mathrm{cl}(\widetilde{\lambda}_u)$ intersects $\mathrm{cl}(\widetilde{\lambda}_u) + (m_0, n_0)$. So we define

$$\tag{5.12} \widetilde{L} := \bigcup_{k\in\mathbb{Z}} \mathrm{cl}(\widetilde{\lambda}_u) + k(m_0, n_0).$$

Then $\widetilde{L}$ is closed, connected, $\widetilde{L} + (m_0, n_0) = \widetilde{L}$ and it is bounded in the direction perpendicular to (m_0, n_0). Moreover, $\widetilde{f}(\widetilde{L}) = \widetilde{L}$. But this shows that $\rho(\widetilde{f})$ must be contained in a line of rational slope (parallel to the vector (m_0, n_0)), which is a contradiction with assumption (5.2). This shows our second claim.

Note that we could have started with any stable branch $\widetilde{\lambda}_s$ as well. So these two claims above show that, under the assumptions (5.1) and (5.2), if one branch at the hyperbolic fixed point $\widetilde{Q}_*$ is bounded, then all four branches are bounded. Moreover, in this case, they all have the same closure. Then the final arguments follow exactly from Oliveira in [34], page 582, namely we get an intersection between a stable and an unstable branch. As this homoclinic intersection contradicts Proposition 5.3, all four branches at $\widetilde{Q}^*$ are unbounded.

To conclude, we present a sketch of Oliveira's argument used above. Let us denote the four branches at $\widetilde{Q}_*$ by $\widetilde{\lambda}_{1,u}, \widetilde{\lambda}_{2,u}, \widetilde{\lambda}_{1,s}$ and $\widetilde{\lambda}_{2,s}$. We are assuming that $\mathrm{cl}(\widetilde{\lambda}_{i,j})$ is bounded and equal to the same continuum for all $i \in \{1,2\}$ and $j \in \{u,s\}$. This implies that there exist two branches, one stable and one unstable, and a local quadrant $Quad$ at $\widetilde{Q}^*$ adjacent to at least one of them, such that both branches

accumulate on $\widetilde{Q}^*$ through *Quad*. It is not hard to show that a picture similar to one of the possibilities in Figure 3 must happen. More precisely, there always exists a Jordan curve separating the local $\widetilde{\lambda}_{1,s}$ from the hatched area of *Quad*.
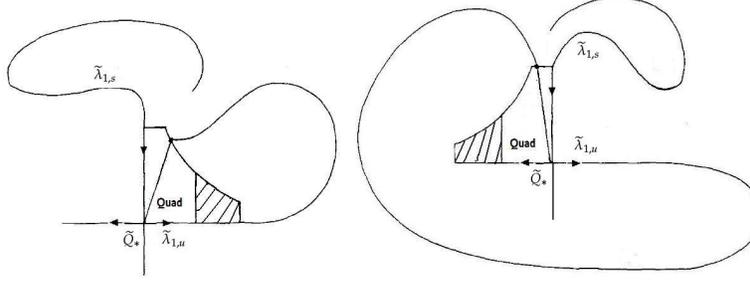
FIGURE 3. The contractible case

And it is easy to see that the stable branch can not accumulate on $\widetilde{Q}^*$ through *Quad* without intersecting the unstable branch: the only way it can enter *Quad* is through the hatched area. And it has to intersect the unstable branch in order to reach that area. □

Remark 5.2. *Similar to the previous proposition, the same conclusion holds for any index 0 saddle-like fixed point. The proof follows the same lines of the above proof.*

Proposition 5.5. *The projections of the four branches are two by two disjoint and they have the same closure, denoted as follows.*

$$(5.13) \qquad K = cl(\pi(\widetilde{\lambda}_{1,s})) = cl(\pi(\widetilde{\lambda}_{2,s})) = cl(\pi(\widetilde{\lambda}_{1,u})) = cl(\pi(\widetilde{\lambda}_{2,u})).$$

Moreover, each connected component of the complement of K is a periodic open disk.

Proof. Let us consider one branch, for instance, $\widetilde{\lambda}_{1,u}$, which is unbounded in $\mathbb{R}^2$ by Lemma 5.4. Then, the closure of its projection, $cl(\pi(\widetilde{\lambda}_{1,u}))$, is an essential subset of $\mathbb{T}^2$. If $cl(\pi(\widetilde{\lambda}_{1,u}))$ is not fully essential, then some connected component U of its complement is itself essential. Since f is non-wandering, it follows that U is periodic. But then, the rotation set $\rho(\widetilde{f})$ has to be contained in some affine line with rational slope, which is a contradiction with assumption (5.2). So, $cl(\pi(\widetilde{\lambda}_{1,u}))$ is fully essential. By item (1) of Definition 3 and assumption (5.1), the fixed point set of f^n is finite, for all $n \geqslant 1$. So, Theorem 2.3 implies that every connected component of the lift of the complement of $cl(\pi(\widetilde{\lambda}_{1,u}))$ is a bounded $\widetilde{f}$-periodic disk. Thus, if $\widetilde{\lambda}$ is any branch (possibly the same branch), and since $\widetilde{\lambda}$ is unbounded, then $\pi(\widetilde{\lambda})$ intersects $cl(\pi(\widetilde{\lambda}_{1,u}))$. If $\pi(\widetilde{\lambda})$ also intersects $cl(\pi(\widetilde{\lambda}_{1,u}))^c$, then by an argument very similar to that in the proof of Lemma 5.4 to treat possibility (1), we conclude that there exists a wandering domain inside an open periodic disk in the

torus, a contradiction. Thus $\pi(\widetilde{\lambda}) \subset \mathrm{cl}(\pi(\widetilde{\lambda}_{1,u}))$. Since we have chosen the two branches arbitrarily, the proof is over. $\qquad\square$

By Proposition 5.5, for the fixed saddle point Q_* in the torus, each of its four branches accumulates on all the other three branches, as well as on itself. Now we need to recall the final arguments of the proof of Theorem 2 in [34]. More precisely, from page 591 to page 594 of [34], the starting conditions are that all branches are unbounded in $\mathbb{R}^2$, and the closure of every branch in $\mathbb{T}^2$ accumulates on all the four branches. Under these conditions, following exactly the arguments in that paper, we get that

$$(5.14) \qquad \pi(\widetilde{\lambda}_{1,u} \bigcup \widetilde{\lambda}_{2,u}) \bigcap \pi(\widetilde{\lambda}_{1,s} \bigcup \widetilde{\lambda}_{2,s}) \neq \varnothing$$

with a topologically transverse intersection.

So, either

$$(5.15) \qquad (\widetilde{\lambda}_{1,u} \bigcup \widetilde{\lambda}_{2,u}) \bigcap (\widetilde{\lambda}_{1,s} \bigcup \widetilde{\lambda}_{2,s}) \neq \varnothing,$$

or

$$(5.16) \qquad (\widetilde{\lambda}_{1,u} \bigcup \widetilde{\lambda}_{2,u}) \bigcap (\widetilde{\lambda}_{1,s} \bigcup \widetilde{\lambda}_{2,s} + (m, n)), \text{ for some } (m, n) \in \mathbb{Z}^2 \backslash (0, 0),$$

in both cases, with topologically transverse intersections.

The first is a contradiction with Proposition 5.3, and for the second case, we can use a similar argument as in the proof of Proposition 5.3, to create a non-contractible periodic orbit. This is a contradiction with the assumption on the shape of $\rho(\widetilde{f})$, i.e., (5.2).

To conclude, as we did in the proof of Lemma 5.4, we present a sketch of the argument contained in the aforementioned pages of [34].

First, note that for any local quadrant, $Quad$ at Q_*, both adjacent branches accumulate on Q_* through it. Let $Quad$ be contained in the first quadrant and $\lambda_{1,u}$ and $\lambda_{1,s}$ be the branches adjacent to it. Follow $\lambda_{1,u}$ starting at Q_* until the first time it reaches $Quad$, at a point $z \in \partial Quad$. This could happen so that this sub arc from $\lambda_{1,u}$ whose endpoints are Q_* and z, united with a segment from z to Q_*, is either a contractible loop, or not. If the loop is contractible, then it bounds a disk D that separates the hatched area in the left case of Figure 3 from a local part of $\lambda_{1,s}$. As the only way for $\lambda_{1,s}$ to enter $Quad$ is through the hatched area, there must be an intersection between $\lambda_{1,s}$ and $\lambda_{1,u}$.

So we are left to consider the case when, for any of the four branches, whenever it returns to some adjacent local quadrant $Quad_i$ ($i = 1, 2, 3, 4$), it forms a non-contractible loop ($Quad_1$ is in the first quadrant and so on, rotating counterclockwise). In this case, the situation in the universal cover is as in Figure 4.

There, we consider $\widetilde{\lambda}_{1,s}$ and $\widetilde{\lambda}_{1,u}$ starting at $\widetilde{Q}_*$ until the first point each of them has in some connected component of $\pi^{-1}(\partial Quad_1)$ and then back to some translate of $\widetilde{Q}_*$ through a segment: we get a "web" on the plane, whose building blocks are all the integer translates of the curvilinear rectangle, denoted $Rect$ in Figure 4.

We know that $\widetilde{\lambda}_{2,s}$ and $\widetilde{\lambda}_{2,u}$ are both unbounded, so they have to leave $Rect$. If they do not intersect $\widetilde{\lambda}_{1,s}$ and $\widetilde{\lambda}_{1,u}$, the only possibilities are, for $\widetilde{\lambda}_{2,u}$ it leaves $Rect$ through the connected components of $\pi^{-1}(\partial Quad_1)$ that contain $\widetilde{Q}_* + r(1, u)$ or $\widetilde{Q}_* + r(1, u) + r(1, s)$. And $\widetilde{\lambda}_{2,s}$ leaves $Rect$ through the connected components of $\pi^{-1}(\partial Quad_1)$ that contain $\widetilde{Q}_* + r(1, s)$ or $\widetilde{Q}_* + r(1, u) + r(1, s)$. It is easy to see
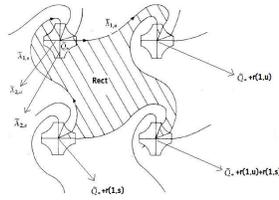
FIGURE 4. The curvilinear rectangle Rect

that, unless both $\widetilde{\lambda}_{2,s}$ and $\widetilde{\lambda}_{2,u}$ leave *Rect* through the connected component of $\pi^{-1}(\partial Quad_1)$ that contains $\widetilde{Q}_* + r(1,u) + r(1,s)$, the diagram in Figure 4 implies that there must be an intersection between $\widetilde{\lambda}_{2,s}$ and $\widetilde{\lambda}_{2,u}$ and we are done.

So, assume this is the case. Now we fall in the situation described in Figure 5.
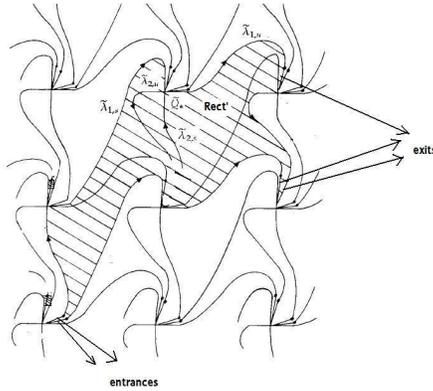
FIGURE 5. The curvilinear rectangle Rect′

Again, as $\widetilde{\lambda}_{2,s}$ and $\widetilde{\lambda}_{2,u}$ are both unbounded, they have to leave *Rect′*. If they do not intersect $\widetilde{\lambda}_{1,s}$ and $\widetilde{\lambda}_{1,u}$, from the position of the exits and entrances in *Rect′*, there must be an intersection between $\widetilde{\lambda}_{2,s}$ and $\widetilde{\lambda}_{2,u}$ (we are using the fact that unstable branches leave *Rect′* through one of the exits and stable branches leave *Rect′* through one of the entrances).

A final remark is that Figures 3,4 and 5 were taken from [34]: we just adapted them to our notation. □

6. A BROADER CLASS OF (NON GENERIC) DIFFEOMORPHISMS

In the proof of Theorem 3.1, we keep the fixed points away from the support of the perturbation. Thus, the rotation set after the perturbation still contains the point $(0,0)$. On the other hand, it seems possible that $(0,0)$ will be "mode locked" in the following sense. Possibly, for all sufficiently small perturbations, $(0,0)$ is not

contained in the interior of the perturbed rotation set. This intuition comes from the phenomenon called rational mode locking. The case when the rotation set has non-empty interior was treated in [4]. One of the theorems proved there states that rational mode locking happens under some conditions that are satisfied for generic one-parameter families.

This section has two objectives. First, we prove several results describing the dynamics of diffeomorphisms $f \in \mathcal{K}^r$, which together, imply Theorem 1.3. And then, using the previous results and some delicate topological arguments, we show the existence of lots of Brouwer lines in the universal cover, which are lifts of essential loops in the torus for all possible homotopy classes. In the end of the section we explain how Theorem 1.4 in the introduction can be deduced from the existence of Brouwer lines.

We start by describing the dynamics in $\mathcal{K}^r$. In this whole section, $f \in \mathcal{K}^r$ and $\tilde{f}$ is a lift of f whose rotation set is the segment from $(0,0)$ to a totally irrational point (α, β).

Proposition 6.1. *Every periodic point p is indeed a fixed point, with topological index 0, and the local dynamics around p can be described explicitly as in Lemma 2.10. In particular, p admits exactly one stable and one unstable branch.*

Proof. By the assumption on the shape of the rotation set $\rho(\tilde{f})$, all the periodic orbits of f must be contractible, that is, they lift to periodic orbits of $\tilde{f}$. Suppose by contradiction that there exists a periodic point which is not fixed, or there exists a fixed point which does not have topological index 0. In the first case, by Lemma 2.7, there exists some fixed point with positive topological index. On the other hand, it is easy to see from the definition of $\mathcal{K}^r$ that the indices at fixed points can only assume one of the following values: $-1, 0$ or 1. Observing Lemma 2.8, as in both possibilities above there exists fixed points with non-zero index, there must always exist a fixed point which has topological index -1. And from the definition of $\mathcal{K}^r$ this point is a hyperbolic saddle. By the same arguments used to prove Lemma 5.2, there exists a fixed hyperbolic saddle contained in $\overline{U}$, where U is defined in expression 5.5. Now the proof goes exactly as in Theorem 5.1, and as in that proof, we get a contraction with the shape of the rotation set $\rho(\tilde{f})$. So every fixed point has topological index 0 and there are no other periodic points. Since $f \in \mathcal{K}^r$, the local dynamics around a fixed point is given by Lemma 2.10. $\square$

The next lemma is Corollary 1.5. Note that it depends on Theorem 3.2.

Lemma 6.2 (Corollary 1.5 restated). *The lift $\tilde{f}$ has bounded deviation along the direction $-(\alpha, \beta)$. Equivalently, there exists $M > 0$, such that for any $\tilde{x}$ and $n \geq 1$,*

$$(6.1) \qquad pr_{-(\alpha,\beta)}(\tilde{f}^n(\tilde{x}) - \tilde{x}) \leq M.$$

Proof. By Proposition 2.12, for any $\tilde{g} \in \widetilde{\mathrm{Homeo}}_0(\mathbb{T}^2)$ which is a sufficiently small perturbation of $\tilde{f}$, it is not possible that $\rho(\tilde{g})$ contains $(0,0)$ in its interior. Then by Theorem 3.2, $\tilde{f}$ must have bounded deviation along $-(\alpha, \beta)$. $\square$

Lemma 6.3. *For any $\tilde{f}$-fixed point $\tilde{p}$, its stable and unstable branches are both unbounded. Their projections to the torus do not intersect. Moreover, the projection of each branch is dense in the torus.*

Proof. A first observation is that there is no periodic open disk. If such a disk existed, then from our hypotheses, its prime ends rotation number would be irrational. So f would have periodic points with positive index (see the end of the proof of Lemma 5.2), something that is not allowed by Proposition 6.2.

Fix some fixed point p in the torus. Consider any $\widetilde{p}$ in the plane that lifts p. The proofs of Proposition 5.3 and Lemma 5.4 imply that both the stable and the unstable branches at $\widetilde{p}$ are unbounded (see Remarks 5.1 and 5.2).

Now we show that $W^s(p)$ is dense in $\mathbb{T}^2$ (a similar argument works for $W^u(p)$). Since the lift $W^s(\widetilde{p})$ is unbounded, the closure $\overline{W^s(p)}$ must be an essential subset of $\mathbb{T}^2$. Assume its complement $\mathbb{T}^2 \backslash \overline{W^s(p)}$ is non-empty. If it contains an essential component, then as we have already done in many previous arguments, this implies that the rotation set is contained in a straight line with rational slope, a contradiction. Thus, $\mathbb{T}^2 \backslash \overline{W^s(p)}$ is inessential. So each connected component is a periodic open disk. As there are none, $W^s(p)$ is dense in $\mathbb{T}^2$.

Now, if the stable and unstable branches at p intersect, then for some $\widetilde{p}$ lift of p, either its stable and unstable branches intersect, or the unstable branch at $\widetilde{p}$ intersects (maybe in a tangency) the stable branch at $\widetilde{p} + (m, l)$ for some non-zero integer vector (m, l). In this second case, there exists a Jordan curve in the plane, which is given by the union of two arcs: one contained in the unstable branch at $\widetilde{p}$ and the other contained in the stable branch at $\widetilde{p} + (m, l)$. As this Jordan curve bounds a disk and $W^s(p)$ is dense in $\mathbb{T}^2$, we get that the stable branch at some integer translate $\widetilde{p} + (m', l')$ has a topologically transverse intersection with the unstable branch at $\widetilde{p}$. If (m', l') is non-zero, Lemma 2.10 and Remark 2.5 give a contradiction.

So we are left to consider the case when the stable and unstable branches at $\widetilde{p}$ intersect. As we did before, there exists an open disk $\widetilde{D}$ in the plane whose boundary is a Jordan curve containing $\widetilde{p}$, consisting of two compact arcs. One of the arcs is contained in the unstable branch at $\widetilde{p}$ and the other is contained in the stable. Now, either for some integer $n > 0$, $\widetilde{f}^n(\widetilde{D})$ intersects some non-zero integer translate of $\widetilde{D}$, something that is not allowed, again by Lemma 2.10 and Remark 2.5, or not. In this second possibility, if we consider $\widetilde{D}_{\text{sat}} = \bigcup_{n \in \mathbb{Z}} \widetilde{f}^n(\widetilde{D})$, then $Fill(\widetilde{D}_{\text{sat}})$ is open and disjoint from all its non-zero translates. Therefore, when projected to the torus, it is an f-invariant bounded open disk (see Lemma 2.3). As such disks do not exist, there are no homoclinic intersections in the torus. $\square$

The next lemma shows that f also does not admit heteroclinic intersections.

Lemma 6.4. *For any two fixed points p_1 and p_2, we have*

$$W^u(p_1) \bigcap W^s(p_2) = \varnothing. \tag{6.2}$$

Proof. Suppose f admits some fixed points p_1 and p_2, and

$$W^u(p_1) \bigcap W^s(p_2) \neq \varnothing. \tag{6.3}$$

We consider some lifts $W^u(\widetilde{p_1})$ and $W^s(\widetilde{p_2})$ of these branches, such that their intersection is non-empty. Then, since there are no saddle-like connections, we can find some Jordan curve, which is the union of one sub-arc of $W^u(\widetilde{p_1})$ and one sub-arc of $W^s(\widetilde{p_2})$, respectively. This Jordan curve bounds a topological disk, denoted $\widetilde{U}$. The projection $U = \pi(\widetilde{U})$ is a proper open subset of $\mathbb{T}^2$.

Since both $W^u(p_1)$ and $W^s(p_2)$ are dense in $\mathbb{T}^2$, each of them must intersect U. So, there are homoclinic intersections, which do not exist by the previous Lemma. This contradiction ends the proof. $\square$

The goal now is to show that both invariant branches at a fixed point tend to infinity.

Lemma 6.5. *Let $\widetilde{p}$ be a fixed point (for $\widetilde{f}$). Then both its stable and unstable branches intersect every compact set in a closed subset. More precisely, for the unstable branch $W^u(\widetilde{p})\backslash\{\widetilde{p}\}$ for example, if $\widetilde{\lambda} \subset W^u(\widetilde{p})\backslash\{\widetilde{p}\}$ denotes the closure of a fundamental domain, then for any compact set K, the set $\{n \geqslant 1 | \widetilde{f}^n(\widetilde{\lambda}) \bigcap K \neq \varnothing\}$ is finite.*

Proof. Consider the unstable branch $W^u(\widetilde{p})\backslash\{\widetilde{p}\}$ and choose a fundamental domain contained in it, whose closure we denote by $\widetilde{\lambda}$. Suppose by contradiction that there exists some compact set $K \subset \mathbb{R}^2$, an integer sequence $n_i \to +\infty$, and a sequence $\widetilde{q}_i \in \widetilde{\lambda}$, such that

$$(6.4) \qquad\qquad\qquad \widetilde{f}^{n_i}(\widetilde{q}_i) \in K.$$

By extracting a subsequence if necessary, we can assume $\widetilde{q} \in \lambda$ is the limit point of the sequence $\{\widetilde{q}_i\}_{i \geqslant 1}$. Considering the ω-limit set of $\widetilde{q}$, denoted $\omega(\widetilde{q})$, there are three possibilities:

Either $\omega(\widetilde{q}))$ is empty, a singleton, or it has more than one point. If it contains more than one point, then some point $\widetilde{w}$ in it is not fixed, because each fixed point is isolated. Then $\widetilde{w}$ is contained in some disk U, such that,

$$(6.5) \qquad\qquad\qquad \widetilde{f}(U) \bigcap U = \varnothing,$$

$$(6.6) \qquad\qquad\qquad \widetilde{f}^k(U) \bigcap U \neq \varnothing, \text{ for some } k \geqslant 2.$$

So Lemma 2.7 implies that $\widetilde{f}$ admits some fixed point with positive topological index, a contradiction with Proposition 6.1.

If $\omega(\widetilde{q})$ is a singleton, say, equal to $\{\widetilde{r}\}$, then $\widetilde{r}$ is necessarily a fixed point. So $\widetilde{q}$ belongs to the stable branch at the point $\widetilde{r}$, that is, there is an heteroclinic point, a contradiction with Lemma 6.5.

So, $\omega(\widetilde{q})$ is empty.

In particular, this means that the sequence $\{\widetilde{f}^n(\widetilde{q})\}_{n \geqslant 1}$ converges to infinity as n tends to $+\infty$. By Theorem 2.6 and Lemma 6.2, $\widetilde{f}$ has bounded deviation along the three directions $-(\alpha, \beta), (-\beta, \alpha)$ and $(\beta, -\alpha)$. So the sequence $\widetilde{f}^n(\widetilde{q})$ converges to infinity along the direction (α, β).

In particular, for some large n_0,

$$(6.7) \qquad\qquad\qquad \inf_{\widetilde{z} \in K} \left(\text{pr}_{(\alpha,\beta)}(\widetilde{f}^{n_0}(\widetilde{q}) - \widetilde{z}) \right) > 2M,$$

where $M > 0$ comes from estimate (6.1).

Then there exists some small disk B containing $\widetilde{q}$, such that, for every point $b \in B$, the above estimate also holds true. Now, we can choose a sufficiently large

i, with $\widetilde{q}_i \in B$, and $n_i > n_0$. Thus,

$$(6.8) \qquad \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^{n_0}(\widetilde{q}_i) - \widetilde{f}^{n_i}(\widetilde{q}_i)\big)$$

$$(6.9) \qquad \geqslant \inf_{\widetilde{z}\in K}\big(\mathrm{pr}_{(\alpha,\beta)}(\widetilde{f}^{n_0}(\widetilde{q}_i) - \widetilde{z})\big)$$

$$(6.10) \qquad \geqslant 2M.$$

As this is a contradiction with Lemma 6.2, the proof is completed. $\qquad\square$

Proof of Theorem 1.3. The proof now follows easily from Proposition 6.1, Lemmas 6.3, 6.4, 6.5 and Theorem 2.6.

Theorem 6.6. *Let $\widetilde{f}$ denote some lift of some $f \in \mathcal{K}^r$, and suppose $\rho(\widetilde{f})$ is the line segment from $(0,0)$ to (α,β). Then, for any coprime integer pair $(a,b) \neq (0,0)$, there exists a torus loop $\ell = \ell_{(a,b)}$, which can be lifted to an $\widetilde{f}$-Brouwer line $\widetilde{\ell}$, such that $\widetilde{\ell} + (a,b) = \widetilde{\ell}$.*

Proof. Up to a change of coordinates and/or considering f^{-1} if necessary, we reduce to the case when $(a,b) = (0,1)$ and $\alpha > 0$.

Proposition 6.7. *There exists an oriented properly embedded curve $\widetilde{\gamma} \subset \mathbb{R}^2$, with the following properties.*

(1) *$\widetilde{\gamma} + (0,1) = \widetilde{\gamma}$, and $\widetilde{\gamma}$ is oriented in the direction $(0,1)$.*
(2) *$\widetilde{\gamma}$ does not contain any $\widetilde{f}$-fixed points.*
(3) *Let $\mathcal{R}(\widetilde{\gamma})$ denote the unbounded complementary domain to the right of $\widetilde{\gamma}$. For any $\widetilde{f}$-fixed point contained in $\mathcal{R}(\widetilde{\gamma})$, its unstable branch does not intersect $\widetilde{\gamma}$.*
(4) *Analogously, let $\mathcal{L}(\widetilde{\gamma})$ denote the unbounded complementary domain to the left of $\widetilde{\gamma}$. For any $\widetilde{f}$-fixed point contained in $\mathcal{L}(\widetilde{\gamma})$, its stable branch does not intersect $\widetilde{\gamma}$.*

Proof. Start with a vertical line ℓ, oriented upwards, which does not contain any $\widetilde{f}$-fixed point. The complement, ℓ^c, consists of two unbounded connected components. We denote by $\mathcal{R}(\ell)$ (respectively, $\mathcal{L}(\ell)$) the right component (respectively, the left component). Let O_- (respectively, O_+) denote the union of the stable branches (respectively, unstable branches) of all the $\widetilde{f}$-fixed points belonging to $\mathcal{L}(\ell)$ (respectively, $\mathcal{R}(\ell)$). We claim that both O_- and O_+ are closed sets. The arguments are similar, so it suffices to prove the claim for O_-.

Let $\{z_i\}_{i\geqslant 1}$ be a sequence of points in O_- converging to some point z. Choose a small closed disk $B(z,1)$ containing z. By Theorem 2.6, Lemmas 6.2 and 6.5, there are only finitely many $\widetilde{f}$-fixed points in $\mathcal{L}(\ell)$, whose stable branches intersects $B(z,1)$. Moreover, the intersection of one such stable branch with $B(z,1)$ is a closed set. So $O_- \bigcap B(z,1)$ is closed. Therefore, z belongs to O_-, which implies that O_- is closed.

It is clear that $(O_-)^c$ has a connected component which is unbounded to the right. More precisely, this component contains some translated domain $\mathcal{R}(\ell) + (M',0)$, where M' is a positive constant obtained from the constant M in (6.1) by defining $M' = M\cos\theta$, where θ is the angle between the horizontal line and the vector (α,β).

Suppose by contradiction that $(O_-)^c$ is not connected. Then there exists a connected component B, which is contained in $\mathcal{L}(\ell) + (M',0)$. Observe B is open and its boundary ∂B is contained in O_-, and recall that the unstable branch of every

f-fixed point is dense in $\mathbb{T}^2$. It follows that, some unstable branch of some $\tilde{f}$-fixed point intersects B. Since the branch is unbounded to the right, it must intersect the boundary of B, which is a contradiction, because stable and unstable branches do not intersect. So, $(O_-)^c$ is connected. The same holds true for $(O_+)^c$.

Now consider the one point compactification of the plane, identified with $\mathbb{S}^2$. The two closed sets O_- and O_+ lift to $\widehat{O_-}$ and $\widehat{O_+}$, respectively, which are connected closed subsets, because every stable and unstable branch lifts to some closed set containing the point $\infty \in S^2$. Clearly, $\widehat{O_-} \bigcap \widehat{O_+} = \{\infty\}$. By Lemma 2.1, the complement of $O_- \bigcup O_+$ is an open connected subset of $\mathbb{R}^2$. Note that, if a point $\tilde{z} \in (O_- \bigcup O_+)^c$, then $\tilde{z} + (0,k) \in (O_- \bigcup O_+)^c$ for any $k \in \mathbb{Z}$, because of the relations $O_- = O_- + (0,1)$ and $O_+ = (O_+) + (0,1)$.

So, we can choose an arc δ connecting $\tilde{z}$ and $\tilde{z} + (0,1)$ such that

$$(6.11) \qquad \delta \bigcap (O_- \bigcup O_+) = \varnothing \text{ and}$$

$$(6.12) \qquad \delta \bigcap (\delta + (0,1)) \text{ contains exactly one point.}$$

Therefore, the union

$$(6.13) \qquad \tilde{\gamma} := \bigcup_{i \in \mathbb{Z}} (\delta + (0,i)).$$

is a properly embedded real line, which satisfies all four properties. $\square$

Lemma 6.8. *Let $\tilde{\gamma}$ be obtained from Proposition 6.7. For any fixed point $\tilde{q} \in \mathcal{R}(\tilde{\gamma})$, there exists a small closed neighbourhood $\tilde{K}$ containing $\tilde{q}$, such that $\{\tilde{f}^n(\tilde{K})\}_{n \geqslant 0} \subset \mathcal{R}(\tilde{\gamma})$. In particular, the forward iterates of $\tilde{K}$ do not intersect $\tilde{\gamma}$.*

Proof. Fix some fundamental domain $\tilde{\lambda}$ of $W^u(\tilde{q})$ very close to $\tilde{q}$, whose endpoints are $\tilde{y}$ and $\tilde{f}(\tilde{y})$. As $W^u(\tilde{q})$ is unbounded to the right, there exists $N > 0$ such that $\text{dist}(\tilde{f}^N(\tilde{\lambda}), \tilde{\gamma}) > 2M + C_{\tilde{f}}$, where M is the constant obtained in (6.1), and $C_{\tilde{f}}$ is given by:

$$(6.14) \qquad C_{\tilde{f}} = \max_{\tilde{z} \in \mathbb{R}^2} \|\tilde{f}(\tilde{z}) - \tilde{z}\|.$$
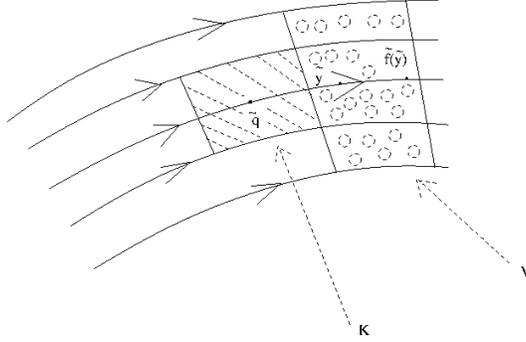
FIGURE 6. The Neighbourhoods K and V

Note that $\bigcup_{n=0}^{\infty} \widetilde{f}^n(\widetilde{\lambda})$ does not intersect $\widetilde{\gamma}$. We can choose a small open neighbourhood V of $\widetilde{\lambda}$, such that $\bigcup_{n=0}^{N} \widetilde{f}^n(V)$ is sufficiently close to $\bigcup_{n=0}^{N} \widetilde{f}^n(\widetilde{\lambda})$, so that it does not intersect $\widetilde{\gamma}$. Observing Lemma 6.2, we can also ensure that $\widetilde{f}^n(V)$ does not intersect $\widetilde{\gamma}$, for all $n > N$. Finally, choose a small neighbourhood K of $\widetilde{q}$, such that, for every point in K, either it belongs to the stable branch of $\widetilde{q}$, or it has a forward iterate belonging to V (See Figure 6 for the choices of these neighbourhoods). In fact, $\bigcup_{n=0}^{\infty} \widetilde{f}^n(K) \subset K \bigcup (\bigcup_{n=0}^{\infty} \widetilde{f}^n(V))$. Therefore, all non-negative iterates of K can not leave $\mathcal{R}(\widetilde{\gamma})$. $\qquad\square$

Lemma 6.9. *Let $\widetilde{\gamma}$ be the curve obtained in Proposition 6.7. Then there exists a positive integer N such that,*

(6.15) $$\widetilde{f}^N(\widetilde{\gamma}) \bigcap \widetilde{\gamma} = \varnothing.$$

Proof. Suppose by contradiction that there exists some sequence of points $\widetilde{z}_{(n)} \in \widetilde{\gamma}$ such that $\widetilde{f}^n(\widetilde{z}_{(n)}) \in \widetilde{\gamma}$. Noticing item (1) of Proposition 6.7, we can choose all the $\widetilde{z}_{(n's)}$ in a compact fundamental domain of $\widetilde{\gamma}$, denoted K. In particular, they have an accumulation point $\widetilde{z}_*$. Up to extracting a subsequence, simply assume $\widetilde{z}_{(n)} \to \widetilde{z}_*$.

Moreover, from Theorem 2.6, there exists a constant $M^* > 0$ such that, for all integers $n > 0$, $\widetilde{f}^n(K) \cap \widetilde{\gamma}$ is contained in the M^*-neighbourhood of K in $\widetilde{\gamma}$.

By Theorem 2.6 and Lemma 6.2, the forward orbit $\{\widetilde{f}^n(\widetilde{z}_*)\}_{n \geqslant 0}$ is bounded in three directions $(-\alpha, -\beta), (-\beta, \alpha), (\beta, -\alpha)$. Then, either $\{\widetilde{f}^n(\widetilde{z}_*)\}_{n \geqslant 0}$ is unbounded in the direction (α, β), or it is bounded. We seek contradictions in both cases.

If $\{\widetilde{f}^n(\widetilde{z}_*)\}_{n \geqslant 0}$ is bounded, then the omega limit set $\omega(\widetilde{z}_*)$ must be a single fixed point, otherwise, by the arguments used in the proof of Lemma 6.5, one finds a positive index fixed point, which does not exist by Proposition 6.1. So $\widetilde{z}_*$ belongs to some stable branch $W^s(\widetilde{q})$, for some fixed point $\widetilde{q} \in \mathcal{R}(\widetilde{\gamma})$. By Lemma 6.8, there exists a compact neighbourhood $\widetilde{K}$ of $\widetilde{q}$, whose forward iterates do not intersect $\widetilde{\gamma}$. And there exists a positive integer m_0, such that $\bigcup_{k=0}^{m_0} \widetilde{f}^{-k}(\widetilde{K})$ contains some neighbourhood N of $\widetilde{z}_*$. This is a contradiction, because for sufficiently large integers $m > m_0$, $\widetilde{z}_{(m)} \in N$ and $\widetilde{f}^m(\widetilde{z}_{(m)}) \in \widetilde{\gamma}$.

The other case is when the orbit of $\widetilde{z}_*$ is unbounded. Then, for sufficiently large k_0,

(6.16) $$\inf_{\widetilde{w} \in K} \mathrm{pr}_{(\alpha, \beta)}(\widetilde{f}^{k_0}(\widetilde{z}_*) - \widetilde{w}) > 10(M + M^* + 1).$$

Then, when $m \geqslant k_0$ is sufficiently large,

(6.17) $$\mathrm{pr}_{(\alpha, \beta)}(\widetilde{f}^{k_0}(\widetilde{z}_{(m)}) - \widetilde{z}_{(m)}) > 10(M + M^* + 1).$$

Noticing Lemma 6.2, this provides a contradiction, since

$$\widetilde{f}^m(\widetilde{z}_{(m)}) \in M^*\text{-neighbourhood of } K.$$

The proof is completed now. $\qquad\square$

For the oriented curve $\widetilde{\gamma}$ from Proposition 6.7 above, $\mathcal{R}(\widetilde{\gamma})$ denotes the unbounded connected component of $(\widetilde{\gamma})^c$ in the direction of (α, β). In Lemma 6.9, we have obtained the integer N such that $\widetilde{f}^N(\widetilde{\gamma}) \subset \mathcal{R}(\widetilde{\gamma})$.

The following is a standard argument. Consider the finite union of curves,

$$(6.18) \qquad Q := \bigcup_{j=0}^{N-1} \widetilde{f}^j(\widetilde{\gamma}).$$

Clearly, the complement of Q has a component which is unbounded in the direction of (α, β). If $\widetilde{\ell}$ is the boundary of this component, then $\widetilde{f}(\widetilde{\ell}) \cap \widetilde{\ell} = \varnothing$. This $\widetilde{\ell}$ is clearly the lift of a vertical loop in the torus. And so the proof of Theorem 6.6 is over. $\qquad \square$

We close this section by restating and proving the remaining part of Theorem 1.4.

Theorem 6.10 (Remains of Theorem 1.4). *Let $\widetilde{f}$ denote some lift of $f \in \mathcal{K}^r$, and $\rho(\widetilde{f})$ is the line segment from $(0,0)$ to (α, β). Let γ be any straight line passing through $(0,0)$, which does not contain $\rho(\widetilde{f})$. Then there exists $\varepsilon_0 > 0$ such that for any $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$, which is C^0-ε_0-close to $\widetilde{f}$, the rotation set $\rho(\widetilde{g})$ does not intersect the connected component of γ^c which does not intersect $\rho(\widetilde{f})$.*

Proof. Choose two reduced integer vectors, (a,b) and (a',b'), with the following properties.
  (1) the two rays from $(0,0)$ in the directions (a,b) and (a',b') define a closed cone C which contains the vector (α, β) in its interior.
  (2) the interior of C is contained in one of the connected component of γ^c.

By Theorem 6.6, there are two $\widetilde{f}$-Brouwer lines $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$, such that, $\widetilde{\ell}_1 + (a,b) = \widetilde{\ell}_1$, and $\widetilde{\ell}_2 + (a',b') = \widetilde{\ell}_2$. Since both $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are lifts of simple closed curves in $\mathbb{T}^2$, there exists ε_0, such that, for any $\widetilde{g} \in \widetilde{Homeo}_0(\mathbb{T}^2)$, with $\mathrm{dist}_{C^0}(\widetilde{g}, \widetilde{f}) < \varepsilon_0$, those two lines $\widetilde{\ell}_1$ and $\widetilde{\ell}_2$ are still Brouwer lines for $\widetilde{g}$. This implies that $\rho(\widetilde{g}) \subset C$. In particular, $\rho(\widetilde{g}) \backslash \{(0,0)\}$ is contained in the connected component of γ^c which contains $\rho(\widetilde{f}) \backslash \{(0,0)\}$. $\qquad \square$

7. UNBOUNDED DEVIATIONS

In this section, we show the following theorem.

Theorem 7.1. *[Theorem 1.6 restated] Suppose $\widetilde{f}$ is a lift of some $f \in \mathcal{K}^r$, and $\rho(\widetilde{f})$ is a segment from $(0,0)$ to a totally irrational point (α, β). Assume further that f preserves a foliation on $\mathbb{T}^2$. Then $\widetilde{f}$ has unbounded deviation along the direction (α, β).*

Proof. Let us assume by contradiction that there exists $M > 0$, such that

$$(7.1) \qquad \sup_{\widetilde{x} \in \mathbb{R}^2, n \geqslant 1} \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n(\alpha, \beta)\big) \leqslant M.$$

Recalling Definition 7, $\mathcal{S}_{(\alpha,\beta)}$ is the closure of the union of the support of all the f-invariant ergodic probability measures whose average rotation vector for $\widetilde{f}$ is (α, β). Then, Lemma 4.3 shows that for any lift $\widetilde{x} \in \mathbb{R}^2$ of some $x \in \mathcal{S}_{(\alpha,\beta)}$, and any $n \geqslant 1$,

$$(7.2) \qquad \mathrm{pr}_{(\alpha,\beta)}\big(\widetilde{f}^n(\widetilde{x}) - \widetilde{x} - n(\alpha, \beta)\big) \geqslant -M.$$

Clearly, for any fixed point p, there exists a small disk B containing p, such that for any point in B, its f-iterates will remain close to p for a long time, both in the future and in the past. Expression (7.2) implies immediately that, when B is sufficiently small, the whole orbit of an arbitrary point in $\mathcal{S}_{(\alpha,\beta)}$ does not intersect B.

Choose a fixed point p. Since f preserves the foliation $\mathcal{F}$, then the leaf $\mathcal{F}(p)$ containing p must be the union of $W^s(p)$ and $W^u(p)$. Choose a local leaf $L \subset \mathcal{F}(p)$, which connects some point $y \in W^s(p)$ and $y' \in W^u(p)$. Let V be an open neighbourhood of L such that for any local leaf in V, its forward and backward iterates under f also intersect V. Choose two small arcs γ and γ', both contained in V, transverse to the local foliation restricted to V, such that the arc γ connects y to a point x and γ' connects y' to a point x'. Moreover, $f(\gamma)$ and $f^{-1}(\gamma')$ are also both contained in V and x and x' bound a local leaf θ^+. It is also convenient to choose θ^+ so that it belongs either to $W^u(p)$ or $W^s(p)$. This is possible because both branches are dense in $\mathbb{T}^2$, see Lemma 6.3. Denote the closed region bounded by $\gamma, \theta^+, \gamma', L$ as K (See Figure 7). Note that K can be chosen arbitrarily close to L.
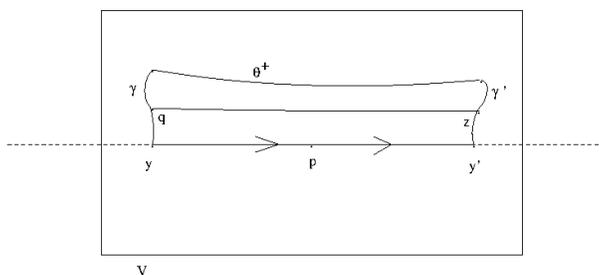
FIGURE 7. Local Foliation around p.

The claim is that, some local leaf in K, which is contained in $W^s(p)$ or $W^u(p)$, and different from L, must contain a fundamental domain of $W^s(p)$ or $W^u(p)$, that is, some sub-arc connecting a point and its image. Suppose by contradiction that this is not true.

Then one of the following cases must happen (see Figure 7).

(1) $f(\theta^+)$ intersects θ^+.
(2) $f(\theta^+)$ is above θ^+.
(3) $f(\theta^+)$ is below θ^+.

If case (1) happens, since θ^+ belongs to $W^s(p)$ or $W^u(p)$, then it contains a fundamental domain of $W^s(p)$ or $W^u(p)$ and we are done.

Up to considering the backward dynamics and exchanging the roles of stable and unstable branch, we can simply assume $f(\theta^+)$ is below θ^+. Then, $f(\gamma)$ is contained in K, provided the region is chosen sufficiently close to L.

Since $W^s(p)$ is dense, we can follow it until the first time it enters the region K. Denote by $z \in W^s(p)$ the first entering point ($z \in \gamma'$). The local leaf containing z, denoted T, intersects γ at a point q. If $f^{-1}(z) \in T$, then we find the fundamental domain in T. If $f^{-1}(z) \notin T$, then $f(q) \in K$, and this contradicts the fact that z is the first returning point to K along $W^s(p)$.

So, there is a fundamental domain of $W^s(p)$ contained in some local leaf in K, other than L. Now we pick a lift $\widetilde{p}$ of p, and consider the lifted leaf $\widetilde{\mathcal{F}}(\widetilde{p})$ containing $W^s(\widetilde{p})$ and $W^u(\widetilde{p})$. By previous paragraphs, there is some integer (a, b) such that, the curves $\widetilde{\mathcal{F}}(\widetilde{p})$ and $\widetilde{\mathcal{F}}(\widetilde{p}) + (a, b)$ bound an infinite strip $\widetilde{H}$, whose union with these two curves covers $\mathbb{T}^2$. Moreover, we can find a small fundamental domain for $\widetilde{f}$ restricted to $\widetilde{H}$, namely $\widetilde{K}' \subset \widetilde{K}$, such that for any point $\widetilde{z} \in \widetilde{H}$ whose orbit is positively and negatively unbounded, its orbit must intersect $\widetilde{K}'$ (See Figure 8).
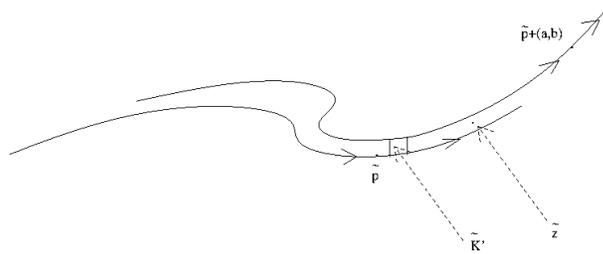
FIGURE 8. A Fundamental Domain that All Unbounded Trajectories cross.

On the other hand, we can choose $K \subset B$, where the disk B was obtained at the beginning of the proof. Therefore, $K' = \pi(\widetilde{K}') \subset K \subset B$ intersects the orbit of any chosen point in $\mathcal{S}_{(\alpha, \beta)}$ (one whose orbit is unbounded both in the future and past). And this is a contradiction with the fact that $\mathcal{S}_{(\alpha, \beta)}$ avoids B. □

REFERENCES

[1] Salvador Addas-Zanata. Instability for the rotation set of homeomorphisms of the torus homotopic to the identity. *Ergodic Theory and Dynamical Systems*, 24(2):319–328, 2004.

[2] Salvador Addas-Zanata. Uniform bounds for diffeomorphisms of the torus and a conjecture of boyland. *Journal of the London Mathematical Society*, 91(2):537–553, 2015.

[3] Salvador Addas-Zanata. A consequence of the growth of rotation sets for families of diffeomorphisms of the torus. *To appear in Ergodic Theory and Dynamical Systems*, 2018.

[4] Salvador Addas-Zanata and Patrice Le Calvez. Rational mode locking for homeomorphisms of the 2-torus. *Proceedings of the American Mathematical Society*, 146(4):1551–1570, 2018.

[5] Salvador Addas-Zanata and Fábio Armando Tal. On periodic points of area preserving torus homeomorphisms. *Far East Journal of Dynamical Systems*, 9:371–378, 2007.

[6] Giles Atkinson. Recurrence of co-cycles and random walks. *Journal of the London Mathematical Society*, 2(3):486–488, 1976.

[7] Morton Brown. Some applications of an approximation theorem for inverse limits. *Proceedings of the American Mathematical Society*, 11(3):478–483, 1960.

[8] Jack Carr. *Applications of centre manifold theory*, volume 35. Springer Science & Business Media, 2012.

[9] Mary Cartwright and John Littlewood. Some fixed point theorems. *Annals of Mathematics*, pages 1–37, 1951.

[10] Freddy Dumortier and Welington de Melo. A type of moduli for saddle connections of planar diffeomorphisms. *Journal of Differential Equations*, 75:88–102, 1988.

[11] Freddy Dumortier, Paulo R Rodrigues, and Robert Roussarie. *Germs of Diffeomorphisms in the Plane*. Lecture Notes in Mathematics 902, Springer, 1981.

[12] John Franks. Generalizations of the poincaré-birkhoff theorem. *Annals of Mathematics*, 128(1):139–151, 1988.

[13] John Franks. Recurrence and fixed points of surface homeomorphisms. *Ergodic theory and dynamical systems*, 8(8*):99–107, 1988.

[14] John Franks. The rotation set and periodic points for torus homeomorphisms. *arXiv preprint math/9605228*, 1996.

[15] Pierre-Antoine Guihéneuf. Instability for the rotation set of diffeomorphisms of the torus homotopic to the identity. *arXiv preprint arXiv:1603.00556*, 2016.

[16] Pierre-Antoine Guihéneuf and Andres Koropecki. Stability of the rotation set of area-preserving toral homeomorphisms. *Nonlinearity*, 30(3):1089, 2017.

[17] Michael Handel. Periodic point free homeomorphism of t^2. *Proceedings of the American Mathematical Society*, 107(2):511–515, 1989.

[18] Tobias Jäger. Linearization of conservative toral homeomorphisms. *Inventiones mathematicae*, 176(3):601–616, 2009.

[19] Nikolaos Karaliolios. Local rigidity of diophantine translations in higher-dimensional tori. *Regular and Chaotic Dynamics*, 23(1):12–25, 2018.

[20] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54. Cambridge university press, 1995.

[21] Andres Koropecki, Patrice Le Calvez, and Meysam Nassiri. Prime ends rotation numbers and periodic points. *Duke Mathematical Journal*, 164(3):403–472, 2015.

[22] Andres Koropecki, Patrice Le Calvez, and Meysam Nassiri. Prime ends and boundary dynamics. *Preprint*, 2017.

[23] Andres Koropecki, Patrice Le Calvez, and Fabio Tal. A triple boundary lemma for surface homeomorphisms. *Proceedings of the American Mathematical Society*, 147(2):681–686, 2019.

[24] Andres Koropecki and Meysem Nassiri. Boundary dynamics for surface homeomorphisms. In *Proceedings of ICM*, 2018.

[25] Andres Koropecki and Fabio Armando Tal. Strictly toral dynamics. *Inventiones mathematicae*, 196(2):339–381, 2014.

[26] Jaroslaw Kwapisz and Mark Mathison. Diophantine and minimal but not uniquely ergodic (almost). *Nonlinearity*, 25(7):2027, 2012.

[27] Patrice Le Calvez and Fabio Armando Tal. Forcing theory for transverse trajectories of surface homeomorphisms. *Inventiones mathematicae*, 212(2):619–729, 2018.

[28] John Mather. Invariant subsets for area-preserving homeomorphisms of surfaces. math. *Mathematical Analysis and Applications, Part B Advances in Mathematics Supplementary Studies, Vol 7B*, 7, 1981.

[29] John Mather. Topological proofs of some purely topological consequences of caratheodory's theory of prime ends. *Selected Studies: Physics-Astrophysics, Mathematics, History of Science*, pages 225–255, 1982.

[30] Kenneth Meyer. Generic bifurcation of periodic points. *Transactions of the American Mathematical Society*, 149(1):95–107, 1970.

[31] Michał Misiurewicz and Krystyna Ziemian. Rotation sets for maps of tori. *Journal of the London Mathematical Society*, 2(3):490–506, 1989.

[32] Michał Misiurewicz and Krystyna Ziemian. Rotation sets and ergodic measures for torus homeomorphisms. *Fundamenta Mathematicae*, 137(1):45–52, 1991.

[33] Maxwell Newman. *Elements of the topology of plane sets of points*. University Press Cambridge, 1951.

[34] Fernando Oliveira. On the generic existence of homoclinic points. *Ergodic Theory and Dynamical Systems*, 7(4):567–595, 1987.

[35] John Oxtoby. Diameters of arcs and the gerrymandering problem. *The American Mathematical Monthly*, 84(3):155–162, 1977.

[36] Alejandro Passeggi. Rational polygons as rotation sets of generic homeomorphisms of the two torus. *Journal of the London Mathematical Society*, 89(1):235–254, 2013.

[37] Clark Robinson. Generic properties of conservative systems. *American Journal of Mathematics*, 92(3):562–603, 1970.

[38] Floris Takens. Forced oscillations and bifurcations. In *Global analysis of dynamical systems*, pages 11–71. CRC Press, 2001.

[39] Fabio Armando Tal and Guilherme Silva Salomão. Bounded deviation in the perpendicular direction. *In Preparation*, 2019.

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