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On the existence of a new type of periodic and quasi-periodic orbits for twist maps of the torus

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Abstract

We prove that for a large and important class of C^1 twist maps of the torus periodic and quasi-periodic orbits of a new type exist, provided that there are no rotational invariant circles (RICs). These orbits have a non-zero 'vertical rotation number' (VRN), in contrast to what happens to Birkhoff periodic orbits and Aubry–Mather sets. The VRN is rational for a periodic orbit and irrational for a quasi-periodic. We also prove that the existence of an orbit with a VRN $= a > 0$, implies the existence of orbits with VRN $= b$, for all $0 < b < a$. In this way, related to a generalized definition of rotation number, we characterize all kinds of periodic and quasi-periodic orbits a twist map of the torus can have. As a consequence of the previous results we obtain that a twist map of the torus with no RICs has positive topological entropy, which is a very classical result. At the end of the paper we present some examples, like the standard map, such that our results apply.

Mathematics Subject Classification: 37E40, 37E45

1. Introduction and statements of the principal results

Twist maps are C^1 diffeomorphisms of the cylinder (or annulus, torus) onto itself that have the following property: the angular component of the image of a point increases as the radial component of the point increases (more precise definitions will be given below). Such maps were first studied in connection with the three-body problem by Poincaré and later it was found that first return maps for many problems in Hamiltonian dynamics are actually twist

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maps. Although they have been extensively studied, there are still many open questions about their dynamics. Great progress has been achieved in the nearly integrable case, by means of KAM theory (see [23]), and many important results have been proved in the general case, concerning the existence of periodic and quasi-periodic orbits (Aubry–Mather sets) (see [19, 4, 14]). In this work a result is proved associating the non-existence of rotational invariant circles (RICs) with the appearance of periodic and quasi-periodic orbits of a new type for an important class of twist maps of the torus.

Notation and definitions

(a) Let (ϕ, I) denote the coordinates for the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$, where ϕ is defined modulo 1. Let $(\tilde{\phi}, \tilde{I})$ denote the coordinates for the universal cover of the cylinder, $\mathbb{R}^2$. For all maps $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ we define

$$(\phi', I') = \hat{f}(\phi, I) \qquad \text{and} \qquad (\tilde{\phi}', \tilde{I}') = f(\tilde{\phi}, \tilde{I}),$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $\hat{f}$.

(b) $D^1_r(\mathbb{R}^2) = \{f : \mathbb{R}^2 \to \mathbb{R}^2 / f$ is a C^1 diffeomorphism of the plane, $\tilde{I}'(\tilde{\phi}, \tilde{I}) \overset{\tilde{I}\to\pm\infty}{\to} \pm\infty$, $\partial_{\tilde{I}}\tilde{\phi}' > 0$ (twist to the right), $\tilde{\phi}'(\tilde{\phi}, \tilde{I}) \overset{\tilde{I}\to\pm\infty}{\to} \pm\infty$ and f is the lift of a C^1 diffeomorphism $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}\}$.

(c) $\mathrm{Diff}^1_r(S^1 \times \mathbb{R}) = \{\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R} / \hat{f}$ is induced by an element of $D^1_r(\mathbb{R}^2)\}$.

(d) Let $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ be the standard projections, respectively, in the $\tilde{\phi}$ and $\tilde{I}$ coordinates ($p_1(\tilde{\phi}, \tilde{I}) = \tilde{\phi}$ and $p_2(\tilde{\phi}, \tilde{I}) = \tilde{I}$). We also use p_1 and p_2 for the standard projections of the cylinder.

(e) Given a measure μ on the cylinder that is positive on open sets, absolutely continuous with respect to the Lebesgue measure and a map $\hat{T} \in \mathrm{Diff}^1_r(S^1 \times \mathbb{R})$, we say that $\hat{T}$ is μ-exact if μ is invariant under $\hat{T}$ and for any open set A homeomorphic to the cylinder we have

$$\mu(\hat{T}(A)\backslash A) = \mu(A\backslash \hat{T}(A)). \tag{1}$$

For an area-preserving map $\hat{T} \in \mathrm{Diff}^1_r(S^1 \times \mathbb{R})$, there is a simple criterion to know if it is exact or not. $\hat{T}$ is exact if and only if its generating function $h(\tilde{\phi}, \tilde{\phi}')$, defined on $\mathbb{R}^2$, satisfies $h(\tilde{\phi} + 1, \tilde{\phi}' + 1) = h(\tilde{\phi}, \tilde{\phi}')$ (see [20]).

(f) Let $TQ \subset D^1_r(\mathbb{R}^2)$ be such that for all $T \in TQ$ we have

$$\begin{gathered} T : \begin{cases} \tilde{\phi}' = T_\phi(\tilde{\phi}, \tilde{I}), \\ \tilde{I}' = T_I(\tilde{\phi}, \tilde{I}), \end{cases} \qquad \text{with } \partial_{\tilde{I}}\tilde{\phi}' = \partial_{\tilde{I}}T_\phi(\tilde{\phi}, \tilde{I}) > 0, \\ T_I(\tilde{\phi} + 1, \tilde{I}) = T_I(\tilde{\phi}, \tilde{I}), \\ T_I(\tilde{\phi}, \tilde{I} + 1) = T_I(\tilde{\phi}, \tilde{I}) + 1, \\ T_\phi(\tilde{\phi} + 1, \tilde{I}) = T_\phi(\tilde{\phi}, \tilde{I}) + 1, \\ T_\phi(\tilde{\phi}, \tilde{I} + 1) = T_\phi(\tilde{\phi}, \tilde{I}) + 1. \end{gathered} \tag{2}$$

Every $T \in TQ$ induces a map $\hat{T} \in \mathrm{Diff}^1_r(S^1 \times \mathbb{R})$ and a map $\bar{T} : T^2 \to T^2$, where $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is the 2-torus. Let $p : \mathbb{R}^2 \to T^2$ be the associated covering map.

(g) Given $T \in TQ$, we say that $\beta \in]0, \pi/2[$ is a uniform angle of deviation for T if

$$DT|_x \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in C_{\mathrm{I}}(\beta) \qquad \text{and} \qquad DT^{-1}|_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in C_{\mathrm{II}}(\beta),$$

for all $x, z \in \mathbb{R}^2$, where $C_{\mathrm{I}}(\beta)$ and $C_{\mathrm{II}}(\beta)$ are the angular sectors:

$$\begin{aligned} C_{\mathrm{I}}(\beta) &= \{(\phi, I) \in \mathbb{R}^2 : \phi > 0 \text{ and } -\operatorname{cotan}(\beta)\phi \leqslant I \leqslant \operatorname{cotan}(\beta)\phi\}, \\ C_{\mathrm{II}}(\beta) &= \{(\phi, I) \in \mathbb{R}^2 : \phi < 0 \text{ and } \operatorname{cotan}(\beta)\phi \leqslant I \leqslant -\operatorname{cotan}(\beta)\phi\}. \end{aligned}$$

(h) Let $\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ be the following covering map:

$$\pi(\tilde{\phi}, \tilde{I}) = (\tilde{\phi}(\text{mod } 1), \tilde{I}). \tag{3}$$

(i) For maps of the torus we can generalize the notion of rotation number, originally defined for circle homeomorphisms, as follows.

Given a map $\bar{f} : T^2 \to T^2$ and $x \in T^2$, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a lift of $\bar{f}$ and $\tilde{x} \in p^{-1}(x)$. The rotation vector $\rho(x, f)$ is defined as (let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$)

$$\rho(x, f) = \lim_{n\to\infty} \frac{f^n(\tilde{x}) - \tilde{x}}{n} \in \bar{\mathbb{R}}^2, \qquad \text{if the limit exists.} \tag{4}$$

Of course, for a map $\bar{f} : T^2 \to T^2$ that is not homotopic to the identity, the above limit may depend on the choice of $\tilde{x} \in p^{-1}(x)$. So, for all $T \in TQ$ we have to modify a little the above definition for rotation vector. The following lemma shows what type of changes must be done.

Lemma 1. *Given $T \in TQ$ and $\tilde{z} \in \mathbb{R}^2$ such that $\rho(\tilde{z}, T) = \lim_{n\to\infty} (T^n(\tilde{z}) - \tilde{z})/n$ exists, we have two possibilities:*

(a) $\rho(\tilde{z}, T) = (\omega, 0)$, *with* $\omega \in \mathbb{R}$,
(b) $\rho(\tilde{z}, T) = (+\infty, \omega)$ *or* $(-\infty, \omega)$, *with* $\omega \in \mathbb{R}$.

Given $i, j \in \mathbb{Z}$, we have

- $\rho(\tilde{z}, T) = (\omega, 0) \Rightarrow \rho(\tilde{z} + (i, j), T) = (\omega + j, 0)$,
- $\rho(\tilde{z}, T) = (\pm\infty, \omega) \Rightarrow \rho(\tilde{z} + (i, j), T) = (\pm\infty, \omega)$.

So, for all $\tilde{x} \in \mathbb{R}^2$ such that $\rho(\tilde{x}, T)$ exists, there are two different cases.

- *Case 1.* $p_1 \circ \rho(\tilde{x}, T) \in \mathbb{R}$. In this case we shall define the rotation vector of $x = p(\tilde{x}) \in T^2$ as follows:

$$\rho(x, T) = \left(\lim_{n\to\infty} \frac{p_1 \circ T^n(\tilde{x}) - p_1(\tilde{x})}{n} (\text{mod } 1), 0 \right). \tag{5}$$

- *Case 2.* $|p_1 \circ \rho(\tilde{x}, T)| = \infty$. In this case $\rho(x, T)$ is defined as in (4):

$$\rho(x, T) = \lim_{n\to\infty} \frac{T^n(\tilde{x}) - \tilde{x}}{n}, \qquad \text{where } x = p(\tilde{x}). \tag{6}$$

We just remark that even in this case $p_2 \circ \rho(x, T)$ may be zero.

When $f : \mathbb{R}^2 \to \mathbb{R}^2$ induces a map $\bar{f} : T^2 \to T^2$ homotopic to the identity map (in this case the rotation vector is given by expression (4)), in many situations (see [10, 18, 21]), we can guarantee the existence of a convex open set $B \subset \mathbb{R}^2$, such that for all $v \in B$, $\exists x \in T^2$ such that $\rho(x, f) = v$ and if $v = (r/q, s/q)$, then $x \in T^2$ can be chosen such that $f^q(x) = x + (r, s)$. A major difference in the situation studied here is that given $T \in TQ$, as we have already said, the diffeomorphism $\bar{T} : T^2 \to T^2$ induced by T is not homotopic to the identity. In fact, it is homotopic to the following linear map (where ϕ and I are taken mod 1):

$$\begin{pmatrix} \phi' \\ I' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \phi \\ I \end{pmatrix}. \tag{7}$$

Before presenting the first theorem we still need more definitions.

Definitions. Given $T \in TQ$, let $\bar{T} : T^2 \to T^2$ be the torus diffeomorphism induced by T.

- We say that $x \in T^2$ belongs to a n-periodic orbit (or set), if for some $n \in \mathbb{N}^*$ we have $\bar{T}^n(x) = x$ and for all $m \in \mathbb{N}^*$, $0 < m < n$, $\bar{T}^m(x) \neq x$. So the periodic orbit to which x belongs is $O_x = \{x, \bar{T}(x), \ldots, \bar{T}^{n-1}(x)\}$. In this case we have the following implications (now we consider the standard projections $p_i : \bar{\mathbb{R}}^2 \to \bar{\mathbb{R}}$, $i = 1, 2$):

$$p_2 \circ \rho(x, T) = 0 \quad \Rightarrow \quad p_1 \circ \rho(x, T) \text{ is a rational number,}$$
$$p_1 \circ \rho(x, T) = \pm\infty \quad \Rightarrow \quad p_2 \circ \rho(x, T) \text{ is a non-zero rational number.}$$

- We say that $Q \subset T^2$ is a quasi-periodic set for $\bar{T}$ in the following cases:
  - ∗ Q is the projection of an Aubry–Mather set $\hat{Q} \subset S^1 \times \mathbb{R}$. In this case $p_1 \circ \rho(z, T) \in [0, 1[$ is an irrational number which does not depend on the choice of $z \in Q$.
  - ∗ Q is a compact $\bar{T}$-*invariant* set such that $p_2 \circ \rho(z, T)$ is an irrational number which does not depend on the choice of $z \in Q$.

As before, let $T \in TQ$ and $\bar{T} : T^2 \to T^2$ be the diffeomorphism induced by T. So as a simple consequence of lemma 1, we have the following classification theorem.

Theorem 1. *Let $x \in T^2$ belong to a periodic or a quasi-periodic set. Then there are two different situations:*

(1) $\exists C > 0$ such that $|p_2 \circ T^n(\tilde{x}) - p_2(\tilde{x})| < C$, for all $n > 0$ and $\tilde{x} \in p^{-1}(x) \Rightarrow \rho(x, T) = (\omega, 0)$ for some $\omega \in [0, 1[$

(2) $p_2 \circ T^n(\tilde{x}) \overset{n\to\infty}{\to} \pm\infty$, for all $\tilde{x} \in p^{-1}(x) \Rightarrow \rho(x, T) = (\pm\infty, \omega)$ for some $\omega \in \mathbb{R}^*$.

Remarks.

- If we call $\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ the map of the cylinder induced by T, it is easy to see that case 1 above corresponds to periodic and quasi-periodic orbits for $\hat{T}$. These are the standard periodic and quasi-periodic orbits, whose existence is assured by theorem 2 (see [19, 4, 14, 16, 17]).
- Case 2 corresponds to orbits for $\hat{T}$ that either go up or down on the cylinder, depending on the sign of $\omega \in \mathbb{R}^*$. If $\omega > 0$ (<0), then $\rho(x, T) = (+\infty(-\infty), \omega)$.
- As we said, there may be a point $x \in T^2$ such that $\rho(x, T) = (\pm\infty, 0)$. It is clear that x is not periodic because its $\bar{T}$-orbit cannot be finite and x does not belong to a quasi-periodic set, because any component of $\rho(x, T)$ is irrational.

A periodic or quasi-periodic orbit O for $\bar{T}$ that belongs to case 2 in theorem 1 can (as the orbits belonging to case 1) be characterized by a single number, the 'vertical rotation number' (VRN), which is defined in the following way:

$$\rho_V(O) = p_2 \circ \rho(x, T) = \lim_{n\to\infty} \frac{p_2 \circ T^n(x) - p_2(x)}{n}, \qquad \text{for any } x \in O. \tag{8}$$

As we want to characterize all kinds of periodic and quasi-periodic orbits a twist map can have, we recall a well-known result.

Theorem 2. *Given a map $T \in TQ$ such that $\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ is μ-exact for some measure μ we have: for every $\omega \in \mathbb{R}$ there is a $\hat{T}$-periodic or quasi-periodic orbit $O \subset S^1 \times \mathbb{R}$, respectively, for rational and irrational values of ω, such that $\rho(O) = \omega$.*

See [19, 4, 14, 16, 17] for different proofs. We have the following corollary.

Corollary 1. *Given a map $T \in TQ$ such that $\hat{T}$ is μ-exact for some measure μ we have: for every $\omega \in [0, 1[$ there is a $\bar{T}$-periodic or quasi-periodic orbit $O \subset T^2$, respectively, for rational and irrational values of ω, such that $\rho(x, T) = (\omega, 0)$, for all $x \in O$.*

So the first type of orbit that appears in theorem 1 always exists. Before presenting the next results we need another definition.

Definition. Given a map $T \in TQ$ such that $\hat{T}$ is μ-exact, we say that C is a RIC for T if C is a homotopically non-trivial simple closed curve on the cylinder and $\hat{T}(C) = C$.

By a theorem essentially due to Birkhoff, C is the graph of some Lipschitz function $\psi : S^1 \to \mathbb{R}$ (see [15, p 430]).

The following theorems are the main results of this paper.

Theorem 3. *Let $T \in TQ$ be such that $\hat{T}$ is μ-exact. Then given $k \in \mathbb{Z}^*$, $\exists N > 0$, such that $\bar{T}$ has a periodic orbit with ρ_V (*VRN*) $= k/N$, if and only if, T does not have RICs.*

The next theorem shows how these periodic orbits appear.

Theorem 4. *Again, for all $T \in TQ$ such that $\hat{T}$ is μ-exact, if $\bar{T}$ has a periodic orbit with $\rho_V = k/N$, then for every pair $(k', N') \in \mathbb{Z}^* \times \mathbb{N}^*$, such that $0 < |k'/N'| < |k/N|$ and $kk' > 0$; $\bar{T}$ has at least two periodic orbits with* VRN *$\rho'_V = k'/N'$.*

About the quasi-periodic orbits we have the following theorem.

Theorem 5. *For all $T \in TQ$ such that $\hat{T}$ is μ-exact we have: if $\bar{T}$ has an orbit with $\rho_V = \omega$, then for all $\omega' \in \mathbb{R}\backslash\mathbb{Q}$ such that $0 < |\omega'| < |\omega|$ and $\omega\omega' > 0$, $\bar{T}$ has a quasi-periodic set with* VRN *$\rho'_V = \omega'$.*

As a consequence of the proof of theorem 5, we prove the following classical result.

Theorem 6. *Every $T \in TQ$ without RICs such that $\hat{T}$ is μ-exact induces a map $\bar{T} : T^2 \to T^2$ such that $h(\bar{T}) > 0$, where $h(\bar{T})$ is the topological entropy of $\bar{T}$.*

Theorem 1 is an immediate consequence of lemma 1, which is proved using simple ideas and the structure of the set TQ. Theorems 3–5 are proved using topological ideas, essentially due to the twist condition and some results due to Le Calvez (see [16, 17] and the next section). In the proofs of theorems 5 and 6 we also use some results from the Nielsen–Thurston theory of classification of homeomorphisms of surfaces up to isotopy, to isotope the map to a pseudo-Anosov one, and then some results due to M Handel, to prove the existence of quasi-periodic orbits with irrational VRN.

2. Basic tools

First we recall some topological results for twist maps essentially due to Le Calvez (see [16, 17]). Let $\hat{f} \in \mathrm{Diff}^1_r(S^1 \times \mathrm{IR})$ and $f \in D^1_r(\mathbb{R}^2)$ be its lifting. For every pair (p, q), $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, we define the following sets:

$$\tilde{K}(p, q) = \{(\tilde{\phi}, \tilde{I}) \in \mathbb{R}^2 : p_1 \circ f^q(\tilde{\phi}, \tilde{I}) = \tilde{\phi} + p\}, \qquad K(p, q) = \pi \circ \tilde{K}(p, q). \tag{9}$$

Then we have the following lemmas.

Lemma 2. *For every $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(p, q) \supset C(p, q)$, a connected compact set that separates the cylinder.*

Lemma 3. *Let $\hat{f} \in \mathrm{Diff}^1_r(S^1 \times \mathbb{R})$ be a μ-exact map. Then the following intersection holds: $\hat{f}(C(p, q)) \cap C(p, q) \neq \emptyset$.*

Now we need a few definitions.

For every $q \geqslant 1$ and $\bar{\phi} \in \mathbb{R}$, let

$$\mu_q(t) = f^q(\bar{\phi}, t), \qquad \text{for } t \in \mathbb{R}. \tag{10}$$

We say that the first encounter between μ_q and the vertical line through some $\phi_0 \in \mathbb{R}$ is for

$$t_F \in \mathbb{R} \ \text{ such that } t_F = \min\{t \in \mathbb{R}: p_1 \circ \mu_q(t) = \phi_0\},$$

and the last encounter is defined in the same way:

$$t_L \in \mathbb{R} \ \text{ such that } t_L = \max\{t \in \mathbb{R}: p_1 \circ \mu_q(t) = \phi_0\}.$$

Of course we have $t_F \leqslant t_L$.

Lemma 4. *For all $\phi_0, \bar{\phi} \in \mathbb{R}$, let $\mu_q(t) = f^q(\bar{\phi}, t)$, as in (10). So we have the following inequalities: $p_2 \circ \mu_q(t_L) \leqslant p_2 \circ \mu_q(\bar{t}) \leqslant p_2 \circ \mu_q(t_F)$, for all $\bar{t} \in \mathbb{R}$ such that $p_1 \circ \mu_q(\bar{t}) = \phi_0$.*

For all $s \in \mathbb{Z}$ and $N \in \mathbb{N}^*$, we can define the following functions on S^1:

$$\mu^-(\phi) = \min\{p_2(Q): Q \in K(s, N) \text{ and } p_1(Q) = \phi\},$$

$$\mu^+(\phi) = \max\{p_2(Q): Q \in K(s, N) \text{ and } p_1(Q) = \phi\}.$$

And we can define similar functions for $\hat{f}^N(K(s, N))$:

$$\nu^-(\phi) = \min\{p_2(Q): Q \in \hat{f}^N \circ K(s, N) \ \text{ and } \ p_1(Q) = \phi\},$$

$$\nu^+(\phi) = \max\{p_2(Q): Q \in \hat{f}^N \circ K(s, N) \ \text{ and } \ p_1(Q) = \phi\}.$$

Lemma 5. *Defining* $\text{Graph}\{\mu^\pm\} = \{(\phi, \mu^\pm(\phi)): \phi \in S^1\}$*, we have*

$$\text{Graph}\{\mu^-\} \cup \text{Graph}\{\mu^+\} \subset C(s, N).$$

So for all $\phi \in S^1$, we have $(\phi, \mu^\pm(\phi)) \in C(s, N)$.

And we have the following simple corollary to lemma 4.

Corollary 2. *$\hat{f}^N(\phi, \mu^-(\phi)) = (\phi, \nu^+(\phi))$ and $\hat{f}^N(\phi, \mu^+(\phi)) = (\phi, \nu^-(\phi))$.*

Now we are going to present a lemma due to Casdagli (see [6]) that together with lemma 3 guarantees the existence of periodic orbits with all rational rotation numbers, for all μ-exact $\hat{f} \in \text{Diff}^1_r(S^1 \times \mathbb{R})$.

Lemma 6. *If $z \in C(s, N) \cap \hat{f}(C(s, N)) \Rightarrow z$ is (s, N) periodic for $\hat{f}$.*

We say that z is (s, N) periodic for $\hat{f}$ if

$$\hat{f}^N(z) = z \qquad \text{and} \qquad \frac{p_1 \circ f^N(\tilde{z}) - p_1(\tilde{z})}{N} = \frac{s}{N},$$

where $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a lift of $\hat{f}$ and $\tilde{z} \in \pi^{-1}(z)$.

For proofs of all the previous results, see Le Calvez [16, 17]. The following is another classical result (due to Birkhoff) with some small changes.

Theorem 7. *Given $T \in TQ$ without RICs such that $\hat{T}$ is μ-exact, we have for all $s, l \in \mathbb{Z}$, $s > 0$ and $l < 0$, $\exists P, Q \in S^1 \times [0, 1]$ and numbers $1 < n_P, n_Q \in \mathbb{N}$ such that*

$$p_2 \circ \hat{T}^{n_P}(P) > s, \qquad p_2 \circ \hat{T}^{n_Q}(Q) < l.$$

For a proof, see [15].

As we have already said, in the proof of theorems 5 and 6 we use some results from the Nielsen–Thurston theory of classification of homeomorphisms of surfaces up to isotopy and some results due to Handel.

The following is a brief summary of these results, taken from [18]. For more information and proofs, see [25, 9, 13].

Let M be a compact, connected oriented surface possibly with boundary, and $f : M \to M$ be a homeomorphism. Two homeomorphisms are said to be isotopic if they are homotopic via homeomorphisms. In fact, for closed orientable surfaces, all homotopic pairs of homeomorphisms are isotopic [7].

There are two basic types of homeomorphisms which appear in the Nielsen–Thurston classification: the finite order homeomorphisms and the pseudo-Anosov ones.

A homeomorphism f is said to be of finite order if $f^n = id$ for some $n \in \mathbb{N}$. The least such n is called the order of f. Finite order homeomorphisms have topological entropy zero.

A homeomorphism f is said to be pseudo-Anosov if there is a real number $\lambda > 1$ and a pair of transverse measured foliations F^S and F^U such that $f(F^S) = \lambda^{-1}F^S$ and $f(F^U) = \lambda F^U$. Pseudo-Anosov homeomorphisms are topologically transitive, have positive topological entropy and have Markov partitions [9].

A homeomorphism f is said to be reducible by a system

$$C = \bigcup_{i=1}^{n} C_i$$

of disjoint simple closed curves $C_1, \ldots, C_n$ (called reducing curves) if

(a) $\forall i$, C_i is neither homotopic to a point, nor to a component of ∂M,
(b) $\forall i \neq j$, C_i is not homotopic to C_j,
(c) C is invariant under f.

Theorem 8. *If the Euler characteristic $\chi(M) < 0$, then every homeomorphism $f : M \to M$ is isotopic to a homeomorphism $F : M \to M$ such that either*

(a) F is of finite order,
(b) F is pseudo-Anosov, or
(c) F is reducible by a system of curves C.

Homeomorphisms F as in theorem 8 are called Thurston canonical forms for f.

Theorem 9. *If f is pseudo-Anosov and g is isotopic to f, then $h(g) \geqslant h(f)$.*

Some results due to Handel can be trivially adapted to the situation studied here. To be more precise, we can change in [12, propositions 1.1 and 1.2], annulus homeomorphisms by torus homeomorphisms homotopic to the map $LM : T^2 \to T^2$, which is the torus map induced by the following linear map of the plane:

$$\begin{pmatrix} \tilde{\phi}' \\ \tilde{I}' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\phi} \\ \tilde{I} \end{pmatrix}. \tag{11}$$

In our case we also have to present appropriate definitions for rotation number and rotation set.

Given a homeomorphism $\bar{f} : T^2 \to T^2$ that is homotopic to LM and a lift of $\bar{f}$ to the cylinder, $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$, we define the vertical rotation set as

$$\rho_V(\hat{f}) = \cup \rho_V(\hat{f}, z),$$

where the union is taken over all $z \in T^2$ such that the VRN ($\hat{z} \in S^1 \times \mathbb{R}$ is any lift of $z \in T^2$),

$$\rho_V(\hat{f}, z) = \lim_{n\to\infty} \frac{p_2 \circ \hat{f}^n(\hat{z}) - p_2(\hat{z})}{n},$$

exists.

We say that $\bar{f} : T^2 \to T^2$ is pseudo-Anosov relative to a finite invariant set $Q \subset T^2$ if it satisfies all of the properties of a pseudo-Anosov homeomorphism except that the associated stable and unstable foliations may have one-prolonged singularities at points in Q. As a last definition, for every set $A \subset T^2$ let $\hat{A} \subset S^1 \times \mathbb{R}$ be the full (cylinder) pre-image of A. Now we are ready to state the modified versions of [12, propositions 1.1 and 1.2].

Proposition 1 (modified 1.1). *If $\bar{f} : T^2 \to T^2$ homotopic to LM, is pseudo-Anosov relative to some finite invariant set Q, then $\rho_V(\hat{f})$ is a closed interval. For each $\omega \in \rho_V(\hat{f})$, there is a compact invariant set $E_\omega \subset T^2$ such that $\rho_V(\hat{f}, z) = \omega$ for all $z \in E_\omega$. Moreover, if $\omega \in \operatorname{int}(\rho_V(\hat{f}))$, then we may choose $E_\omega \subset T^2 \backslash Q$.*

Proof. As in [12]. □

Proposition 2 (modified 1.2). *Suppose that $\bar{f} : T^2 \to T^2$ is pseudo-Anosov relative to a finite invariant set Q and that $\bar{T} : T^2 \to T^2$ (induced by some element of TQ) is homotopic to $\bar{f}$ relative to Q. If $\hat{f} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ and $\hat{T} : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ are lifts that are equivariantly homotopic relative to $\hat{Q}$, then $\rho_V(\hat{T}) \supset \operatorname{int}(\rho_V(\hat{f}))$. Moreover, for each $\omega \in \operatorname{int}(\rho_V(\hat{f}))$, there is a compact $\bar{T}$-invariant set $E_\omega \subset T^2$ such that $\rho_V(\hat{T}, z) = \omega$ for all $z \in E_\omega$.*

Proof. Also as in [12]. □

3. Proofs

From now, for simplicity, we will omit the 'tilde' in the coordinates $(\tilde{\phi}, \tilde{I})$ of the plane. First of all we prove lemma 1. This lemma is a trivial consequence of the following result.

Lemma 7. *Let $T \in TQ$ and $z, w_+, w_- \in \mathbb{R}^2$ be points such that*

(a) $|p_2 \circ T^n(z)| < C$, for all $n \geqslant 0$ and some constant $C > 0$,

(b) $p_2 \circ T^n(w_\pm) \overset{n\to\infty}{\to} \pm\infty$.

So we have

(a) $\exists K > 0$, such that for all $n > 0$, $|(p_1 \circ T^n(z) - p_1(z))/n| < K$,

(b) $(p_1 \circ T^n(w_\pm) - p_1(w_\pm))/n \overset{n\to\infty}{\to} \pm\infty$.

Proof. As the proofs for w_+ and w_- are equal, we only analyse w_+, which will be called just w. For all $n > 0$ we define $z = (\phi_z^0, I_z^0)$, $\phi_z^n = p_1 \circ T^n(z)$, $I_z^n = p_2 \circ T^n(z)$ and $w = (\phi_w^0, I_w^0)$, $\phi_w^n = p_1 \circ T^n(w)$, $I_w^n = p_2 \circ T^n(w)$. From the initial hypothesis, $|I_z^j| < C$ for all $j > 0$, so defining the following ϕ-periodic function $\tilde{T_\phi}(\phi, I) = T_\phi(\phi, I) - \phi$, there is a constant $K > 0$ such that

$$\left|\frac{p_1 \circ T^n(z) - p_1(z)}{n}\right| \leqslant \frac{\sum_{j=0}^{n-1} |\tilde{T_\phi}(\phi_z^j, I_z^j)|}{n} < \frac{nK}{n} = K.$$

Now we write $I_w^n = I_{w0}^n + k_n$, with $I_{w0}^n \in [0, 1)$ and $k_n \in \mathbb{Z}$. Of course, the hypothesis in the lemma implies that $k_n \overset{n\to\infty}{\to} \infty$, because $p_2 \circ T^n(w) \overset{n\to\infty}{\to} \infty$. As above $\exists \bar{K} > 0$ such that for all $j > 0$,

$$\left|\tilde{T_\phi}\left(\phi_w^j, I_{w0}^j\right)\right| \leqslant \max_{(\phi,I)\in[0,1]^2} |\tilde{T_\phi}(\phi, I)| < \bar{K}.$$

So for all $n > 0$,

$$\frac{p_1 \circ T^n(w) - p_1(w)}{n} = \frac{\sum_{j=0}^{n-1} \tilde{T}_\phi(\phi_w^j, I_{w0}^j) + \sum_{j=0}^{n-1} k_j}{n} > -\bar{K} + \frac{\sum_{j=0}^{n-1} k_j}{n}.$$

In order to finish the proof that

$$\lim_{n\to\infty} \frac{p_1 \circ T^n(w) - p_1(w)}{n} = \infty,$$

we remember the Cesaro theorem, which says that

$$\lim_{j\to\infty} k_j = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\sum_{j=0}^{n-1} k_j}{n} = \infty. \qquad \square$$

The following is a very important lemma.

Lemma 8. *Given $T \in TQ$ such that $\hat{T}$ is μ-exact and T does not have RICs, we have for all $k \in \mathbb{N}^*$, $\exists N > 0$ and $\tilde{P} = (\phi_{\tilde{P}}, I_{\tilde{P}}) \in [0,1]^2$, such that $T^N(\tilde{P}) = T^N(\phi_{\tilde{P}}, I_{\tilde{P}}) = (\phi_{\tilde{P}} + s, I_{\tilde{P}}^N)$ for some $s \in \mathbb{Z}$, with $I_{\tilde{P}}^N > I_{\tilde{P}} + k$.*

Proof. The proof will be done by contradiction. Suppose there exists $k_0 \geqslant 1$, such that $\forall N > 0$, there is no $\tilde{P} \in [0,1]^2$ such that $T^N(\tilde{P}) = T^N(\phi_{\tilde{P}}, I_{\tilde{P}}) = (\phi_{\tilde{P}} + s, I_{\tilde{P}}^N)$ for some $s \in \mathbb{Z}$, with $I_{\tilde{P}}^N > 1 + k_0 \geqslant I_{\tilde{P}} + k_0$.

First let us note that given a map $T \in TQ$, $\exists a > 0$, such that $\forall Q \in \mathbb{R}^2$, $p_2 \circ (T(Q) - Q) > -a$. In fact, from the definition of the set TQ, we just have to take $a > -\inf_{Q\in[0,1]^2} p_2 \circ (T(Q) - Q)$, because as $[0,1]^2$ is compact, $a < \infty$.

All $T \in TQ$ can be written in the following way:

$$T : \begin{cases} \phi' = T_\phi(\phi, I), \\ I' = T_I(\phi, I), \end{cases}$$

and for all $(\phi, I) \in \mathbb{R}^2$, we have the following estimates:

$$\exists b > 0, \quad \text{such that} \quad \left|\frac{\partial T_\phi}{\partial \phi}\right| < b, \tag{12}$$

$$\exists K > 0, \quad \text{such that} \quad \frac{\partial T_\phi}{\partial I} \geqslant K \quad \text{(twist condition)}. \tag{13}$$

As T does not have RICs, theorem 7 implies that

$$\exists P = (\phi_P, I_P) \in [0,1]^2 \quad \text{and} \quad N_1 > 1 \quad \text{such that } p_2 \circ T^{N_1}(P) > \left[(k_0 + 2) + \frac{(2+b)}{K}\right] + a.$$

A very natural thing is to look for a point $\tilde{P}$ as described above, in the line segment $r = \{(\phi, I) \in [0,1]^2 : \phi = \phi_P\}$.

First, let us define

$$\operatorname{Max} HL(T^n(r)) = \sup_{x,y\in[0,1]} |p_1 \circ T^n(\phi_P, x) - p_1 \circ T^n(\phi_P, y)|. \tag{14}$$

It is clear that

$$\operatorname{Max} HL(T^n(r)) \geqslant |p_1 \circ T^n(\phi_P, 0) - p_1 \circ T^n(\phi_P, 1)| = n \overset{n\to\infty}{\to} \infty. \tag{15}$$

So, for all $n > 1$, $\exists$ at least one $s \in \mathbb{Z}$, such that $\phi_P + s \in p_1(T^n(r))$.

The hypothesis we want to contradict implies that for all $n > 0$ and $Q \in r$, such that

$$p_1 \circ T^n(Q) = \phi_P (\text{mod } 1), \tag{16}$$

we have

$$p_2 \circ T^n(Q) \leqslant (k_0 + 1). \tag{17}$$

As $p_2 \circ T^{N_1}(P) > [(k_0 + 2) + (2 + b)/K] + a$, $\exists P_1 \in r$ such that

$$p_2 \circ T^{N_1}(P_1) = (k_0 + 2) + a$$

and

$$\forall Q \in \overline{PP}_1 \subset r,$$

$$p_2 \circ T^{N_1}(Q) \geqslant (k_0 + 2) + a.$$

The reason why such a point P_1 exists is the following: as $N_1 > 1$, $\exists$ at least one $s \in \mathbb{Z}$ such that $\phi_P + s \in p_1(T^{N_1}(r))$. Thus, from (16) and (17), $T^{N_1}(r)$ must cross the line l given by $l = \{(\phi, (k_0 + 2) + a)$, with $\phi \in \mathbb{R}\}$.

Also from (16) and (17) we have

$$\sup_{Q,R \in \overline{PP}_1} |p_1 \circ T^{N_1}(R) - p_1 \circ T^{N_1}(Q)| < 1.$$

Now let $\gamma_{N_1} : J \to \mathbb{R}^2$ be the following curve:

$$\gamma_{N_1}(t) = T^{N_1}(\phi_P, t), t \in J = \text{interval whose extremes are } I_P \text{ and } I_{P_1}. \tag{18}$$

It is clear that it satisfies the following inequalities:

$$p_2 \circ \gamma_{N_1}(I_P) - p_2 \circ \gamma_{N_1}(I_{P_1}) > \frac{2 + b}{K},$$

$$\sup_{t,s \in J} |p_1 \circ \gamma_{N_1}(t) - p_1 \circ \gamma_{N_1}(s)| < 1.$$

Claim 1. *Given a continuous curve $\gamma : J = [\alpha, \beta] \to \mathbb{R}^2$, with*

$$\sup_{t,s \in J} |p_1 \circ \gamma(t) - p_1 \circ \gamma(s)| < 1, \tag{19}$$

$$|p_2 \circ \gamma(\beta) - p_2 \circ \gamma(\alpha)| > \frac{(2 + b)}{K}. \tag{20}$$

Then $\exists s \in \mathbb{Z}$, such that $\phi_P + s \in p_1 (T \circ \gamma(J))$.

Proof.

$$\sup_{t,s \in J} |p_1 \circ T \circ \gamma(t) - p_1 \circ T \circ \gamma(s)| = \sup_{t,s \in J} |T_\phi \circ \gamma(t) - T_\phi \circ \gamma(s)|$$

$$\geqslant |T_\phi \circ \gamma(\beta) - T_\phi \circ \gamma(\alpha)| \geqslant -b + K\frac{2 + b}{K} = 2$$

So the claim is proved. □

$\gamma_{N_1}(t)$ (see (18)) satisfies the claim hypothesis, by construction. So $\exists s \in \mathbb{Z}$ such that $\phi_P + s \in p_1(T \circ \gamma_{N_1}(J)) = p_1(T^{N_1+1}(\overline{PP}_1))$.

As $\inf_{t \in J} p_2(\gamma_{N_1}(t)) = p_2(\gamma_{N_1}(I_{P_1})) = (k_0 + 2) + a$, from the choice of $a > 0$ we get that $\inf_{t \in J} p_2(T \circ \gamma_{N_1}(t)) > (k_0 + 2)$.

So there is $\bar{t} \in J$ and $\bar{P} = (\phi_P, \bar{t}) \in r$ such that

$$p_1 \circ T^{N_1+1}(\bar{P}) = \phi_P \ (\text{mod } 1),$$

$$p_2 \circ T^{N_1+1}(\bar{P}) > (k_0 + 2).$$

This contradicts (16) and (17). So for all $k \geqslant 1$, $\exists N > 0$ and $\tilde{P} \in r$, such that $T^N(\tilde{P}) = T^N(\phi_P, I_{\tilde{P}}) = (\phi_P + s, I_{\tilde{P}}^N)$ for some $s \in \mathbb{Z}$, with $I_{\tilde{P}}^N > I_{\tilde{P}} + k$. □

Remark. Of course for all $k \leqslant -1$, $k \in \mathbb{Z}$, there are also $N > 0$ and $\tilde{Q} = (\phi_{\tilde{Q}}, I_{\tilde{Q}}) \in [0, 1]^2$, such that $T^N(\tilde{Q}) = T^N(\phi_{\tilde{Q}}, I_{\tilde{Q}}) = (\phi_{\tilde{Q}} + s, I_{\tilde{Q}}^N)$ for some $s \in \mathbb{Z}$, with $I_{\tilde{Q}}^N < I_{\tilde{Q}} + (k - 1)$. The proof in this case is completely similar to that above, because as T does not have RICs, again by theorem 7 for all $l < 0$ there exist $Q = (\phi_Q, I_Q) \in [0, 1]^2$ and $n_Q > 1$ such that $p_2 \circ T^{n_Q}(Q) < l$.

Below we prove the main results of this paper.

Proof of theorem 3. As the two cases, $k > 0$ and $k < 0$, are completely similar, let us fix $k > 0$.

($\Rightarrow$)

If $\bar{T}$ has a periodic point P, with $\rho_V(P) = k/N$, for some $k > 0$ and $N > 0$, then $p_2 \circ \hat{T}^n(P) \overset{n\to\pm\infty}{\to} \pm\infty$, which implies that there can be no RIC.

($\Leftarrow$)

To prove the existence of a periodic orbit with $\rho_V = k/N$, for a given $k > 0$ and some $N > 0$ sufficiently large, it is enough to show that there exists a point $P \in S^1 \times \mathbb{R}$ such that

$$\hat{T}^N(P) = P + (0, k). \tag{21}$$

As $T \in TQ$, for each $(s, l) \in \mathbb{Z}^2$ and $N > 0$ the sets $C(s, N)$, defined in lemma 2, satisfy $C(s + lN, N) = C(s, N) + (0, l)$. So, for each fixed $N > 0$, there are only N distinct sets of this type: $C(0, N), C(1, N), \ldots, C(N - 1, N)$.

The others are just integer vertical translations of them. Another trivial remark about the sets $C(s, N)$ is: $C(s, N) \cap C(r, N) = \emptyset$, if $s \neq r$. For all $s \in \mathbb{Z}$, we get from lemma 3 that $\hat{T}(C(s, N)) \cap C(s, N) \neq \emptyset$. So we can apply lemma 6 and conclude that $\exists \bar{P}_s \in C(s, N)$ such that $\hat{T}^N(\bar{P}_s) = \bar{P}_s$.

From lemma 8, for the given $k > 0$, $\exists N > 0$ and $\tilde{P} = (\phi_{\tilde{P}}, I_{\tilde{P}}) \in S^1 \times \mathbb{R}$, such that $\hat{T}^N(\tilde{P}) = \hat{T}^N(\phi_{\tilde{P}}, I_{\tilde{P}}) = (\phi_{\tilde{P}}, I_{\tilde{P}}^N)$, with $I_{\tilde{P}}^N > I_{\tilde{P}} + k$. So $\tilde{P} \in K(\tilde{s}, N)$ (see expression (9)), for a certain $\tilde{s} \in \mathbb{Z}$ and $p_2 \circ \hat{T}^N(\tilde{P}) - p_2(\tilde{P}) > k$. As $\tilde{P} \in K(\tilde{s}, N)$, we get that $\mu^-(\phi_{\tilde{P}}) \leqslant p_2(\tilde{P})$ and $\nu^+(\phi_{\tilde{P}}) \geqslant p_2 \circ \hat{T}^N(\tilde{P})$, which implies that $\nu^+(\phi_{\tilde{P}}) - \mu^-(\phi_{\tilde{P}}) \geqslant p_2 \circ \hat{T}^N(\tilde{P}) - p_2(\tilde{P}) > k$.

From corollary 2 we get that $\hat{T}^N(\phi_{\tilde{P}}, \mu^-(\phi_{\tilde{P}})) = (\phi_{\tilde{P}}, \nu^+(\phi_{\tilde{P}}))$, so defining $\tilde{\tilde{P}} = (\phi_{\tilde{P}}, \mu^-(\phi_{\tilde{P}})) \in C(\tilde{s}, N)$ (see lemma 5) we have $p_2 \circ \hat{T}^N(\tilde{\tilde{P}}) - p_2(\tilde{\tilde{P}}) > k$. As we proved above, $\exists \bar{P}_{\tilde{s}} \in C(\tilde{s}, N)$ such that $p_2 \circ \hat{T}^N(\bar{P}_{\tilde{s}}) - p_2(\bar{P}_{\tilde{s}}) = 0$. So as $C(\tilde{s}, N)$ is connected, $\exists P \in C(\tilde{s}, N)$ such that

$$p_2 \circ \hat{T}^N(P) = p_2(P) + k.$$

And the theorem is proved.

Now we present an alternative proof, suggested by a referee, which is much shorter. We decided to mantain the original proof because it is based on lemma 8, which will be used in future works, so we wanted to keep it in this paper.

For a given $k > 0$, we are going to prove the existence of a point $P \in C(0, \bar{N})$ such that $\hat{T}^{\bar{N}}(P) = P + (0, k)$, for a sufficiently large $\bar{N}$. As $\hat{T}$ is μ-exact, we get that there is a point $P_0 = (\phi_0, I_0) \in C(0, 1)$ such that $\hat{T}(P_0) = P_0$. For any given $N > 0$, let $\mu^-, \mu^+, \nu^-, \nu^+$ be the maps associated with $C(0, N)$. From the choice of P_0 we get $\mu^-(\phi_0) \leqslant I_0 \leqslant \nu^+(\phi_0)$. In the proof of lemma 4 (see [16]), the following property for the lift of the map μ^- to $\mathbb{R}$ is obtained (we are denoting the lift also by μ^-): for any $n \in \{1, 2, \ldots, N\}$, the point $T^n(\phi, \mu^-(\phi))$ is the first point where the image of $\phi \times \mathbb{R}$ by T^n meets the vertical passing through $T^n(\phi, \mu^-(\phi))$, and for the same reasons, we get $p_1 \circ T^n(\phi', \mu^-(\phi')) < p_1 \circ T^n(\phi, \mu^-(\phi))$ if $\phi' < \phi$. So the image by T^n of the graph of μ^- is also a graph and the order given by p_1 is preserved. Moreover, as T is a twist map, we can prove that (see [15, lemma 13.1.1, p 424]) if $\phi > \phi'$, then $\mu^-(\phi) - \mu^-(\phi') \geqslant -\text{cotan}(\beta)(\phi - \phi')$, where β is a uniform angle of deviation for T. By periodicity of μ^- we get $\max \mu^- - \min \mu^- \leqslant \text{cotan}(\beta)$ and analogous inequalities for the other maps.

As in the above proof, we know that $p_2 \circ \hat{T}^N - p_2$ vanishes on $C(0, N)$. Suppose that this map does not take the value k on $C(0, N)$. Then as $C(0, N)$ is compact, it is strictly smaller and we have $\nu^+(\phi) - \mu^-(\phi) < k$, for all $\phi \in S^1$. So for any $\phi \in S^1$, we get the following estimates:

$$\mu^-(\phi) = \mu^-(\phi) - \mu^-(\phi_0) + \mu^-(\phi_0) - \nu^+(\phi_0) + \nu^+(\phi_0) > -\text{cotan}(\beta) - k + I_0,$$

$$\nu^+(\phi) = \nu^+(\phi) - \nu^+(\phi_0) + \nu^+(\phi_0) - \mu^-(\phi_0) + \mu^-(\phi_0) < \text{cotan}(\beta) + k + I_0,$$

and the above inequalities imply that

$$\hat{T}^N(S^1 \times]-\infty, -\text{cotan}(\beta) - k + I_0]) \subset S^1 \times]-\infty, \text{cotan}(\beta) + k + I_0],$$

which cannot hold for all $N > 0$ by theorem 7. □

Proof of theorem 4. Again we fix $k > 0 \Rightarrow k' > 0$. The case $k < 0$ is completely similar. By contradiction, suppose that for some $0 < k'/N' < k/N$ and any fixed $s \in \mathbb{Z}$:

$$p_2 \circ \hat{T}^{N'}(Q) - p_2(Q) - k' \leqslant 0, \qquad \forall Q \in C(s, N').$$

So, in particular, we have $\nu^+(\phi) - \mu^-(\phi) - k' \leqslant 0$ for all $\phi \in S^1$. This means that the unbounded connected component of $C(s, N')^c$, which is below $C(s, N')$ and we denote by U, satisfies the following equation: $\hat{T}^{N'}(U) - (0, k') \subset U$, so $\hat{T}^{iN'}(U) - (0, ik') \subset U$, for all $i > 0$. Now let us choose a point $P \in U$, such that

$$\lim_{n\to\infty} \frac{p_2 \circ \hat{T}^n(P) - p_2(P)}{n} = \frac{k}{N}. \tag{22}$$

So we get that for all $i > 0$, $[p_2 \circ \hat{T}^{iN'}(P) - p_2(P) - ik'] \leqslant C - p_2(P)$, where $C = \sup\{p_2(x) : x \in C(s, N')\}$. This implies that

$$\lim_{i\to\infty} \frac{p_2 \circ \hat{T}^{iN'}(P) - p_2(P)}{iN'} \leqslant \frac{k'}{N'} \quad \Rightarrow \quad \lim_{n\to\infty} \frac{p_2 \circ \hat{T}^n(P) - p_2(P)}{n} \leqslant \frac{k'}{N'},$$

which contradicts (22). So there is a point $P_1 = (\phi_1, I_1) \in C(s, N')$ such that $p_2 \circ \hat{T}^{N'}(P_1) - p_2(P_1) - k' > 0$ and from the μ-exactness of $\hat{T}$, $\exists P_0 = (\phi_0, I_0) \in C(s, N')$ such that $p_2 \circ \hat{T}^{N'}(P_0) - p_2(P_0) < 0$. Now let Δ_0 and Δ_1 be the proper simple arcs given by

$$\Delta_0 = \{\phi_0\} \times [\mu^+(\phi_0), +\infty[\cup \hat{T}^{-N'}(\{\phi_0\} \times]-\infty, \nu^-(\phi_0)],$$

$$\Delta_1 = \{\phi_1\} \times]-\infty, \mu^-(\phi_1)] \cup \hat{T}^{-N'}(\{\phi_1\} \times [\nu^+(\phi_1), +\infty[.$$

It is easy to see that $\Delta_0 \cap C(s, N') = (\phi_0, \mu^+(\phi_0))$, $\Delta_1 \cap C(s, N') = (\phi_1, \mu^-(\phi_1))$ and that $(\Delta_0 \cup \Delta_1)^c$ is an open set that divides $C(s, N')$ into two connected components, C_1 and C_2

$(C(s, N') = C_1 \cup C_2)$, such that $C_1 \cap C_2 = (\phi_0, \mu^+(\phi_0)) \cup (\phi_1, \mu^-(\phi_1))$. Therefore, the function $p_2 \circ \hat{T}^{N'} - p_2 - k'$ has at least one zero in each C_i. □

Proof of theorem 5. The proof will be divided into two cases (as before we fix $\omega > 0 \Rightarrow \omega' > 0$)

Case I. ($\omega \in \mathbb{Q}$). As $\omega' \in \mathbb{R}\backslash\mathbb{Q}$, there is a sequence

$$\frac{p_i}{q_i} \overset{i\to\infty}{\to} \omega', \qquad \text{with } 0 < \frac{p_i}{q_i} < \omega, \quad \forall i > 0$$

and (from theorem 4) a family of periodic orbits

$$E_i = \{P_1^i, P_2^i, \ldots, P_{q_i}^i\} \subset T^2, \qquad \text{with } \rho_V(E_i) = \frac{p_i}{q_i}.$$

So in the Hausdorff topology there is a subsequence $E_{i_n} \overset{n\to\infty}{\to} E \subset T^2$ that for simplicity we will call E_n. The convergence in the Hausdorff topology means that: given $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$, such that for all $n \geqslant n_0$, $E_n \subset B_\epsilon(E)$ and $E \subset B_\epsilon(E_n)$, where $B_\epsilon(\bullet)$ is the ϵ neighbourhood of the given set.

In this way, for all $z \in E$ there is a sequence $z_n \overset{n\to\infty}{\to} z$, such that $z_n \in E_n$. But there is still a problem to obtain that for all $z \in E$, $\rho_V(z) = \omega'$, because we do not have any control under the uniformity of the VRNs of the family of orbits E_n. In the Aubry–Mather case, the periodic orbits, whose limit in the Hausdorff topology is a quasi-periodic set, have a very strong uniformity condition; they are of Birkhoff type (see, e.g., [14]). Indeed, if we knew that given $\epsilon > 0$, $\exists \hat{i}(\epsilon) > 0$ ($\hat{i}$ independent of n), such that $\forall n > 0$ and $\forall z_n \in E_n$,

$$\left| \frac{p_2 \circ T^i(z_n) - p_2(z_n)}{i} - \frac{p_n}{q_n} \right| < \epsilon, \qquad \text{for all } i > \hat{i},$$

then the problem would be solved. In order to overcome this problem, we use the following important lemma that is a consequence of propositions 1 and 2, some ideas from [18] and some results from the Nielsen–Thurston theory.

Lemma 9. *Under the hypothesis of theorem 5, for all $\omega' \in (0, \omega)\backslash\mathbb{Q}$, there is a quasi-periodic set $\bar{E}$, such that $\rho_V(\bar{E}) = \omega'$.*

Proof. See the end of section 3. □

Case II. ($\omega \notin \mathbb{Q}$). From case I, we just have to prove that for all $0 < p/q < \omega$ there is a q-periodic orbit with $\rho_V = p/q$. The proof of this fact is identical to the proof of theorem 4; so we omit it. □

We still have to prove lemma 9 and theorem 6. The following are auxiliary results that are important in these proofs.

Lemma 10. *Let $\bar{T} : T^2 \to T^2$ ($T^2 = \mathbb{R}^2/\mathbb{Z}^2$) be a homeomorphism homotopic to LM (see (11)), and let $C \subset T^2$ be a homotopically non-trivial simple closed curve, $\bar{T}^s$-invariant, for some $s > 0$. Then C is a rotational simple closed curve on the cylinder $S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. Moreover, $[C]$ (homotopy class of C) is the only homotopy class of simple closed curves on the torus that is preserved by iterates of $\bar{T}$.*

Proof. The action of $\bar{T}$ on $\pi_1(T^2)$ is given by

$$\bar{T}_*([C]) = \bar{T}_*(c_\phi, c_{\text{I}}) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_\phi \\ c_{\text{I}} \end{pmatrix},$$

and the eigenvector corresponding to the eigenvalue 1 is $(1, 0)$. □

Lemma 11. *Let $\bar{f} : T^2 \to T^2$ be a homeomorphism isotopic to LM. If $\exists l > 0$ such that $\bar{f}^l$ has a rotational invariant curve γ with $[\gamma] = x_\phi = (1, 0)$, then $\bar{f}$ cannot have a periodic orbit with* VRN $\rho_V \neq 0$ *and another with* $\rho_V = 0$.

Proof. Let $\hat{F}_0$ be a lift of $\bar{f}^l$ to the cylinder which fixes $\hat{\gamma}$, a lift of $\gamma \subset T^2$. This implies that the VRN for $\hat{F}_0$ of every point is zero. So, given any lift $\hat{F}$ of $\bar{f}^l$, the VRN of every point are equal. In particular, given a lift $\hat{f}$ of $\bar{f}$, the VRN of every point is the same, which is what we wanted to prove. □

We have already seen that all $T \in TQ$ such that $\hat{T}$ is μ-exact and T does not have RICs induces a map $\bar{T}$ defined on the torus that has periodic orbits with non-zero VRNs. Suppose that $\bar{T}$ has a periodic orbit with $\rho_V = \omega > 0$. Given $\omega' \in \mathbb{R}\backslash\mathbb{Q}$, with $0 < \omega' < \omega$, let us choose irreducible fractions a_1/b_1 and a_2/b_2, such that

$$0 < \frac{a_1}{b_1} < \omega' < \frac{a_2}{b_2} \leqslant \omega,$$

and periodic orbits Q_1 and Q_2 with $\rho_V(Q_i) = a_i/b_i$ and $\#\{Q_i\} = b_i$, for $i = 1, 2$ (this is possible by theorem 4).

As $T \in TQ$ and $\hat{T}$ is μ-exact, from lemmas 3 and 6 it is clear that $\exists R \in T^2$ such that $\bar{T}(R) = R$ and

$$\begin{aligned} p_2 \circ T(\tilde{R}) &= p_2(\tilde{R}), \\ p_1 \circ T(\tilde{R}) &= p_1(\tilde{R}) \pmod 1, \end{aligned} \qquad \text{for any } \tilde{R} \in p^{-1}(R).$$

Let $Q = Q_1 \cup Q_2 \cup R$. Now we blow up each $x \in Q$ to a circle S_x. Let T_Q^2 be the compact manifold (with boundary) thereby obtained; T_Q^2 is the compactification of $T^2\backslash Q$, where S_x is a boundary component where x was deleted. Now we extend $\bar{T} : T^2\backslash Q \to T^2\backslash Q$ to $\bar{T}_Q : T_Q^2 \to T_Q^2$ by defining $\bar{T}_Q : S_x \to S_x$ via the derivative; we just have to think of S_x as the unit circle in T_xT^2 and define

$$\bar{T}_Q(v) = \frac{D\bar{T}_x(v)}{\|D\bar{T}_x(v)\|}, \qquad \text{for } v \in S_x.$$

$\bar{T}_Q$ is continuous on T_Q^2 because $\bar{T}$ is C^1 on T^2. Let $b : T_Q^2 \to T^2$ be the map that collapses each S_x onto x. Then $\bar{T} \circ b = b \circ \bar{T}_Q$. This gives $h(\bar{T}_Q) \geqslant h(\bar{T})$ (see [15, p 111]). Actually $h(\bar{T}_Q) = h(\bar{T})$, because each fibre $b^{-1}(y)$ is a simple point or an S_x and the entropy of $\bar{T}$ on any of these fibres is 0 (the map on the circle induced from any linear map has entropy 0). This construction is due to Bowen (see [5]).

Now we have the following theorem.

Theorem 10. *The map $\bar{T}_Q : T_Q^2 \to T_Q^2$ is isotopic to a pseudo-Anosov homeomorphism of T_Q^2.*

Proof. By theorem 8, $\bar{T}_Q$ is isotopic to a homeomorphism $F_Q : T_Q^2 \to T_Q^2$ (Thurston canonical form) such that either

(a) F_Q has finite order,
(b) F_Q is pseudo-Anosov,
(c) F_Q is reducible by a system of curves C.

We must think of T_Q^2 as a torus with round discs removed, all of the same size, each one centred at a point $x \in Q$. Let $F : T^2 \to T^2$ be the completion of F_Q, i.e. the homeomorphism obtained by radially extending F_Q into all the holes (see [8]).

It is easy to see that F_Q does not have finite order, because there are points with different rotation numbers (by construction of Q).

We say that a simple closed curve γ on a torus with holes is rotational if, after filling in the holes, γ is homotopically non-trivial. Suppose that F_Q has a rotational reducing curve γ and let $[\gamma] \in \pi_1(T^2)$ be its homotopy class in the torus without holes. Then, for some $n > 0$, we have

$$F_Q^n(\gamma) = \gamma \quad \Rightarrow \quad F^n(\gamma) = \gamma.$$

And, as F_Q is isotopic to $\bar{T}_Q$, F is isotopic to LM. In this way, from lemma 10 the homotopy class of γ in the torus $T^2 = S^1 \times S^1$ is $[\gamma] = x_\phi = (1, 0)$ (γ is a rotational simple closed curve in the cylinder $S^1 \times \mathbb{R}$). So, from the existence of the periodic orbits Q_1 (or Q_2) and R, applying lemma 11 we conclude that F and thus F_Q do not have any rotational reducing curves.

And if γ is a non-rotational reducing curve, then γ must surround at least two holes (because γ is not homotopic to a component of ∂T_Q^2). These holes must have the same rotation number and this is impossible, because $\rho_V(Q_1) \neq \rho_V(Q_2) \neq \rho_V(R) = 0$ and two points from the same orbit cannot be surrounded by the same curve (by construction of Q_1 and Q_2). So $F_Q : T_Q^2 \to T_Q^2$ is a pseudo-Anosov homeomorphism. □

Now we prove theorem 6.

Proof of theorem 6. We just have to see that after all the previous work, the map $\bar{T}_Q : T_Q^2 \to T_Q^2$ is isotopic to a pseudo-Anosov homeomorphism of T_Q^2. Then $h(\bar{T}) = h(\bar{T}_Q)$ and $h(\bar{T}_Q) > 0$, by theorem 9. □

Finally, we prove lemma 9.

Proof of lemma 9. By theorem 10, $\bar{T}_Q : T_Q^2 \to T_Q^2$ is isotopic to a pseudo-Anosov homeomorphism, $F_Q : T_Q^2 \to T_Q^2$. So, as Q is an invariant and finite set we just have to apply propositions 1 and 2. □

4. Examples and applications

We conclude by giving some examples.

(a) It is obvious that the well-known standard map $S_M : T^2 \to T^2$ given by

$$S_M : \begin{cases} \phi' = \phi + I' \pmod 1 \\ I' = I - \dfrac{k}{2\pi} \sin 2\pi\phi \pmod 1 \end{cases}$$

is induced by an element of TQ. Also, it is easy to see that its generating function is

$$h_{S_M}(\phi, \phi') = \frac{(\phi' - \phi)^2}{2} + \frac{k}{4\pi^2} \cos 2\pi\phi \quad \Rightarrow \quad h_{S_M}(\phi + 1, \phi' + 1) = h_{S_M}(\phi, \phi'),$$

so S_M is an exact map. In this way, as we know that for sufficiently large $k > 0$, S_M does not have RICs, we can apply our previous results to this family of maps. In fact, theorems 3 and 4 can be used to produce a new criterion to obtain estimates for the parameter value k_{cr}, which is defined in the following way: if $k > k_{\mathrm{cr}}$, then S_M does not have RIC and for $k \leqslant k_{\mathrm{cr}}$ there is at least one RIC. This happens because for each $1/n$, $n \in \mathbb{N}^*$, there is a number k_n, such that for $k \geqslant k_n$, S_M has a n-periodic orbit with $\rho_V = 1/n$ and for $k < k_n$ it does not have such an orbit. From the theorems cited above, if $n > m$ then $k_n \leqslant k_m$ and $\lim_{n\to\infty} k_n = k_{\mathrm{cr}}$. In a future work, we will try to obtain estimates for k_{cr} using this method.

(*b*) In [1] the dynamics near a homoclinic loop to a saddle-centre equilibrium of a two degrees of freedom, Hamiltonian system was studied by means of an approximation of a certain Poincaré map. In an appropriate coordinate system this map is given by

$$\hat{F} : S^1 \times \,]0, +\infty[\to S^1 \times \,]0, +\infty[, \qquad \text{where } \hat{F} : \begin{cases} \phi' = \mu(\phi) + \gamma \log(I') \pmod{\pi}, \\ I' = J(\phi) I, \end{cases}$$

and

$$J(\phi) = \alpha^2 \cos^2 \phi + \alpha^{-2} \sin^2 \phi,$$

$$\mu(\phi) = \arctan\left(\frac{\tan\phi}{\alpha^2}\right), \qquad \mu(0) = 0.$$

So $J(\phi)$ is π-periodic and $\mu(\phi + \pi) = \mu(\phi) + \pi$. In this case S^1 will be identified with $\mathbb{R}/\pi\mathbb{Z}$.

A direct calculation shows that

$$h_{\hat{F}}(\phi, \phi') = \gamma \exp\left(\frac{\phi' - \mu\ (\phi)}{\gamma}\right).$$

So

$$h_{\hat{F}}(\phi + \pi, \phi' + \pi) = h_{\hat{F}}(\phi, \phi'), \qquad \text{because } \mu(\phi + \pi) = \mu(\phi) + \pi.$$

Thus $\hat{F}$ is also exact. Applying the following coordinate change

$$\tilde{\phi} = \phi,$$

$$\tilde{I} = \gamma \log I,$$

we get (omitting the 'tilde')

$$\hat{F} : S^1 \times \mathbb{R} \hookleftarrow : \begin{cases} \phi' = F_\phi(I, \phi) = \mu(\phi) + I' \pmod{\pi}, \\ I' = F_I(I, \phi) = \gamma \log(J(\phi)) + I. \end{cases}$$

It is obvious that in these coordinates, $\hat{F}$ is μ-exact and μ is given by

$$\mu(A) = \int_A \mathrm{e}^{I/\gamma}\, \mathrm{d}\phi\, \mathrm{d}I.$$

It is also easy to see that $\hat{F}$ induces a map $\bar{F} : T^2 \to T^2$ ($T^2 = \mathbb{R}^2/(\pi\mathbb{Z})^2$) given by

$$\bar{F} : \begin{cases} \phi' = \bar{F}_\phi(I, \phi) = \mu(\phi) + I' \pmod{\pi}, \\ I' = \bar{F}_I(I, \phi) = \gamma \log(J(\phi)) + I \pmod{\pi}, \end{cases}$$

that is also induced by an element of TQ.

And from [11], $\exists \alpha_{\mathrm{cr}}(\gamma)$ such that for $\alpha > \alpha_{\mathrm{cr}}(\gamma)$, $\bar{F}$ does not have RICs. In this case, we can apply the same criteria explained for the standard map. But as there are two parameters, we do not obtain a critical value, we obtain a critical set in the (γ, α) plane. Another important application of this theory is to obtain properties about the structure of the unstable set of the above mentioned homoclinic loop (to the saddle-centre equilibrium), when the former is unstable. The periodic orbits given by theorem 3 were analysed in [2] and it was proved that for every VRN $m/n > 0$, there is an open set in the parameter space with an (m/n)-periodic orbit which is topologically a sink. In particular, it can be proved that, for a fixed value of $\gamma > 0$, given an $\epsilon > \alpha_{\mathrm{cr}}(\gamma) > 1$, where $\alpha_{\mathrm{cr}}(\gamma)$ is analogous to the constant k_{cr} defined for the standard map, there is a number $m/n > 0$ and an open interval $I_{m/n} \subset (\alpha_{\mathrm{cr}}(\gamma), \epsilon)$, such that for $\alpha \in I_{m/n}$, $\bar{F}$ has a vertical periodic orbit with $\rho_V = m/n$ which is also a topological sink. So we can say that one of the mechanisms that cause the lost of stability of the homoclinic loop is the creation of periodic sinks for $\bar{F}$. In [3] it was proved that the existence of a topological sink for $\bar{F}$ implies many interesting properties on the topology of the set of orbits that have the saddle-centre loop as their α-limit set (a set analogous to the unstable manifold of a hyperbolic periodic orbit). More precisely, in this case, given an arbitrary neighbourhood of the original homoclinic loop, a set of positive measure contained in this neighbourhood escapes from it, following (or clustering around) a finite set of orbits that, in a certain sense, correspond to the topological sinks for $\bar{F}$. In a forthcoming paper, we will analyse the following function:

$$\rho_V^{\max}(\gamma,\alpha) = \sup_{P\in T^2} \rho_V(P) = \sup_{P\in T^2}\left[\lim_{n\to\infty} \frac{p_2\circ F^n(P) - p_2(P)}{n}\right],$$

where the supremum is taken over all $P \in T^2$ such that $\rho_V(P)$ exists. Using a method developed in [2] and results from [22], we plan to prove the density of periodic sinks in the subset of the parameter space (γ,α) where $\bar{F}$ does not have RICs.

(c) Given a C^2 circle diffeomorphism $f : S^1 \to S^1$ ($f(\phi+1) = f(\phi)+1$), we can define the following generating function:

$$h_f(\phi,\phi') = \exp(\phi' - f(\phi)).$$

As $h_f(\phi+1,\phi'+1) = h_f(\phi,\phi')$ the associated twist map $\hat{T}_f : S^1\times\,]0,+\infty[\hookleftarrow$ is exact

$$\hat{T}_f : \begin{cases} \phi' = f(\phi) + \log(I') \pmod 1, \\ I' = \dfrac{1}{f'(\phi)} I. \end{cases}$$

By the same coordinate change applied to $\hat{F}$,

$$\tilde{\phi} = \phi,$$
$$\tilde{I} = \log(I).$$

we can write $\hat{T}_f$ in the following way:

$$\phi' = f\ (\phi) + I' \pmod 1,$$
$$I' = \log\left(\frac{1}{f'(\phi)}\right) + I.$$

As above, in these coordinates $\hat{T}_f$ is μ-exact for the following measure:

$$\mu(A) = \int_A \mathrm{e}^I\, \mathrm{d}\phi\, \mathrm{d}I.$$

$\hat{T}_f$ induces a torus map

$$\bar{T}_f : \begin{cases} \phi' = f\ (\phi) + I' \pmod 1, \\ I' = \log\left(\dfrac{1}{f'(\phi)}\right) + I \pmod 1, \end{cases}$$

such that our results apply.

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