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Maximizing measures for endomorphisms of the circle

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Abstract
We study a given fixed continuous function $\phi : S^1 \to \mathbb{R}$ and an endomorphism $f : S^1 \to S^1$, whose f-invariant probability measures maximize $\int \phi \, d\mu$. We prove that the set of endomorphisms having a ϕ maximizing invariant measure supported on a periodic orbit is $\mathcal{C}_0$ dense.

Mathematics Subject Classification: 37E10, 37A99

1. Introduction

While ergodic theory and optimization are two of the most renowned and studied theories of the last century, it was only in the last decade or so that works relating these two fields were systematically produced, consolidating into the so-called ergodic optimization, see [4–7].

It is well known, from the Krylov–Bogolyubov theorem in ergodic theory, that given a compact metric space X and a continuous transformation $f : X \to X$, the set of f-invariant Borel probability measures, $M_{\rm inv}(f)$, is non-empty. It is also widely known that $M_{\rm inv}(f)$ is convex and compact in the weak* topology.

Optimization theory is concerned with finding the maxima (or minima) of a given continuous functional $P : K \to \mathbb{R}$, where K is a topological space. Its applications are wide ranging, from Lagrangian mechanics to economics and engineering.

One way to unify these fields is the following. Given a continuous function $\phi : X \to \mathbb{R}$, we can define the functional $P_\phi : M_{\rm inv}(f) \to \mathbb{R}$, $P(\mu) = \int \phi \, d\mu$. Ergodic optimization is the study of the maxima (or minima) of P_ϕ. Since $M_{\rm inv}(f)$ is compact and convex, and since P_ϕ is affine, there is always a maximum at an extreme point of $M_{\rm inv}(f)$, denoted by $\mu_{\max}$. But these extremes are precisely the ergodic measures.

Several relevant questions may be considered involving the relationship between f, ϕ and the maxima of P_ϕ. Recent works [1–3] have concentrated on two main lines:

1. If we fix f and let ϕ belong to a 'large' space, is it true that 'generically' P_ϕ has as a maximum an ergodic measure supported on a periodic orbit?

2. Given an f invariant ergodic measure μ, is there a continuous ϕ such that μ is the unique maximum of P_ϕ? If so, can ϕ be chosen in a smaller space? For instance can we pick a differentiable ϕ?

There have been several successful partial answers for both these questions, see for instance [1] for the first and [3] for the second. Question 1 seems to have been motivated by an important conjecture of Mañé, which says that minimizing measures for Lagrangian flows are generically supported on periodic trajectories. Of course, while in Mañé's conjecture the dynamics and the minimizing functional are directly connected, in question 1 we are allowed to change the target function without modifying the dynamics.

The problem we were concerned with in this paper is a variation of question 1. Instead of allowing ϕ to vary, we will keep ϕ fixed and allow f to vary. Is it still true that for a dense set of dynamics there is a ϕ-maximizing measure $\mu_{\max}$ supported in a periodic orbit? In [8] we show this to be true when f is a homeomorphism on a compact manifold, but the technique made use of the invertibility of f. On the other hand, in [9] we show that the set of homeomorphisms for which the maximizing measure is supported in a periodic orbit is meager in the usual topology.

In this paper we consider the case where f is an endomorphism of the circle, i.e. a continuous and surjective map. Our main result here is the following.

Theorem 1. *Given an endomorphism $f : S^1 \to S^1$, a continuous function $\phi : S^1 \to \mathbb{R}$ and $\varepsilon > 0$, there is an endomorphism $\tilde{f} : S^1 \to S^1$ such that $|f - \tilde{f}| = \max_{x \in S^1} |f(x) - \tilde{f}(x)| < \varepsilon$ and $\tilde{f}$ has a ϕ-maximizing invariant measure supported in a periodic trajectory.*

It should be noted that the actual 'topological size' of the set of circle endomorphisms with a ϕ-maximizing measure supported in a periodic orbit depends on the regularity of ϕ. This can be seen from the following theorem, also proved in [9].

Theorem 2. *Fix a continuous function $\phi : S^1 \to \mathbb{R}$. We have two possibilities.*

(a) ϕ is monotone on an interval of S^1. Then the set of endomorphisms of S^1 such that a ϕ-maximizing measure has support in a single fixed point has a nonempty interior.
(b) ϕ has a dense subset of strict local maxima (and, so, of strict local minima). Then the set of endomorphisms possessing a ϕ-maximizing periodic orbit is meager.

These results still leave open some interesting problems: for instance, can we require more regularity from f and $\tilde{f}$? And can we prove a similar result when X is a more general space?

The paper is organized as follows: in the next section we derive some general preliminary results and give an outline of the proof of our theorem, in section 3 we find an interval where we will perturb f, and in the last section we build $\tilde{f}$ and prove the theorem.

2. Preliminaries and outline of the proof

2.1. Initial assumptions and an important lemma

We can take f to be piecewise linear, since these functions form a dense subset in the set of endomorphisms of the circle. By piecewise linear we do not mean the usual notion, because we do not allow zero derivative. To be precise, f is supposed to be a continuous mapping of the circle, such that for all $x \in S^1 \backslash \{d_0, d_1, \ldots, d_K\}$ (for a certain finite set $\{d_0, d_1, \ldots, d_K\}$), $f'(x)$ exists and is locally constant in a neighbourhood of x and is different from zero at all points.

We assume that there is a maximizing measure $\mu_{\max}$, whose support does not contain any periodic orbit. As we said before, ergodic measures are the extreme points of the convex set $M_{\mathrm{inv}}(f) = \{f\text{-invariant Borel probability measures on } S^1\}$, so we can suppose that $\mu_{\max}$ is ergodic and, of course, without loss of generality, we can assume that $\int \phi \, \mathrm{d}\mu_{\max} = 0$.

We will need the next result (note that it is valid in a much more general situation).

Lemma 1. *Let $A \subset supp(\mu_{\max})$ be an f-forward invariant set ($f(A) \subset A$) of non-periodic points such that $\mu_{\max}(A) = 1$, and suppose that there exists $\varepsilon^* > 0$ such that for all $x \in A$ and $n > 0$ with $|f^n(x) - x| < \varepsilon^*$, the Birkhoff sum $\sum_{i=0}^{n-1} \phi(f^i(x)) \leqslant 0$. Then for all $x \in A$ and $n > 0$ with $|f^n(x) - x| < \varepsilon^*/2$, the Birkhoff sum must be equal to zero.*

Proof. Suppose that the lemma hypothesis is satisfied, but for some $x \in A$ and $n > 0$, $|f^n(x) - x| < \varepsilon^*/2$ and $\sum_{i=0}^{n-1} \phi(f^i(x)) = -\delta < 0$.

Let $M > 8$ be an integer such that $\min_{1 \leqslant i \leqslant n} |f^i(x) - x| > 2\frac{\varepsilon^*}{M}$. As $\mu_{\max}(A) = 1$ and $A \subset \mathrm{supp}(\mu_{\max})$, there exists a set $\tilde{A} \subset A \subset S^1$ of positive $\mu_{\max}$ measure, with $x \in \tilde{A} \subset [x - \varepsilon^*/M, x + \varepsilon^*/M] \cap A$ such that for all $y \in \tilde{A}$ we have

$$f^n(y) \in [x - \varepsilon^*/2, x + \varepsilon^*/2] \qquad \text{and} \qquad \sum_{i=0}^{n-1} \phi(f^i(y)) < -\delta/2 < 0.$$

As $\mu_{\max}(\tilde{A}) > c > 0$ for some real number c, $\mu_{\max}$-almost every point in $\tilde{A}$ is recurrent and returns to $\tilde{A}$ infinitely many times. If the diameter of the set $\tilde{A}$ is sufficient small, then for each recurrent $y \in \tilde{A}$, there exists $N(y) > n$ such that $f^{N(y)}(y) \in \tilde{A}$ and this is the first time it returns to $\tilde{A}$. Suppose that for all recurrent $y \in \tilde{A}$,

$$\sum_{i=0}^{N(y)-1} \phi(f^i(y)) < -\delta/4 < 0. \tag{1}$$

Birkhoff's ergodic theorem implies that for $\mu_{\max}$-a.e. recurrent $y \in \tilde{A}$,

$$\#\{i \in \{1, 2, \ldots, N\} : f^i(y) \in \tilde{A}\} > cN, \qquad \text{if } N \text{ is large enough.}$$

So, for such a recurrent $y \in \tilde{A}$, if $N_k \overset{k\to\infty}{\to} \infty$ is such that $f^{N_k}(y) \in \tilde{A}$, then we have $\sum_{i=0}^{N_k-1} \phi(f^i(y)) < (-\delta/4)(cN_k - 1)$, for all $k > 0$ sufficiently large. Thus,

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{i=0}^{N_k-1} \phi(f^i(y)) < -\delta c/4 < 0,$$

which is a contradiction, because the limit of the average above exists and is equal to zero for $\mu_{\max}$-almost every point. So (1) does not hold and there exists $y \in \tilde{A}$ such that $f^{N(y)}(y) \in \tilde{A}$ and $\sum_{i=0}^{N(y)-1} \phi(f^i(y)) \geqslant -\delta/4$. But as $f^n(y) \in [x - \varepsilon^*/2, x + \varepsilon^*/2]$ and $\sum_{i=0}^{n-1} \phi(f^i(y)) < -\delta/2$, we get that there exists $z = f^n(y) \in A \cap [x - \varepsilon^*/2, x + \varepsilon^*/2]$ such that $f^{N(y)-n}(z) \in \tilde{A} \subset [x - \varepsilon^*/M, x + \varepsilon^*/M] \Rightarrow |f^{N(y)-n}(z) - z| < \varepsilon^*$ and $\sum_{i=0}^{N(y)-n-1} \phi(f^i(z)) > \delta/4 > 0$. But this contradicts the lemma hypothesis, so for all $x \in A$ and $n > 0$, such that $|f^n(x) - x| < \varepsilon^*/2$, we get that $\sum_{i=0}^{n-1} \phi(f^i(x)) = 0$. $\square$

We will also make use of proposition 1.

Proposition 1. *For every constant $a > 0$, there exists a positive integer $m_0 = m_0(a)$ such that, for all $m \geqslant m_0$ and $x \in S^1$,*

$$\frac{1}{m} \sum_{i=0}^{m-1} \phi(f^i(x)) \leqslant \frac{a}{2}.$$

Proof. This follows from

$$\limsup_{n\to\infty} \max_{x\in S^1} \frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(x)) = \sup_{\mu\in M_{\mathrm{inv}}(f)} \int \phi \, \mathrm{d}\mu = 0,$$

which can be derived by the weak* compactness of probability measures in S^1, see proposition 2.1 of [6]. □

2.2. *Outline of the proof of theorem 1*

Here we present the main ideas behind the proof of theorem 1. Our argument is perturbative. We perform a small change to f, with support in a small interval in order to obtain $\tilde{f}$. The idea is to close a 'finitely'-recurrent orbit into a periodic one.

We use lemma 1 in order to find a recurrent point $\bar{x}$ in a special set C, defined in section 3, and $n > 0$ such that $|f^n(\bar{x}) - \bar{x}| < \varepsilon$ and the partial average

$$\frac{1}{n}\sum_{i=0}^{n-1}\phi(f^i(\bar{x})) \overset{\mathrm{def}}{=} a_0 \geqslant 0. \tag{2}$$

There are two cases.

(1) For all possible choices of $\bar{x}$ and $n > 0$, $a_0 = 0$.
(2) $a_0 > 0$.

In case 1 we have to consider other pre-images of $f^n(\bar{x})$ contained in the small interval $(\bar{x}, f^n(\bar{x})]$. Also, for each of these pre-images (if any), we look at the ϕ partial average over the finite piece of the orbit starting at the pre-image and ending at $f^n(\bar{x})$. There are 2 subcases.

(1.1) These partial averages are all less than or equal to zero; in this case we consider the point in the small interval $[\bar{x}, f^n(\bar{x})]$ which returns to this interval and has the largest possible partial average. This may be $\bar{x}$ itself. The rest of the proof, in this case, consists of performing a perturbation which turns this maximizing partial orbit into a periodic one.
(1.2) There is a partial average which is strictly positive; this subcase is handled together with case 2.

The proof of the main theorem in cases 1.2 and 2 has several details. The main difficulty is that, if a perturbation is performed in order to create a periodic orbit O_P with positive ϕ-average, other non-periodic invariant measures may appear. These new measures may have ϕ-average larger then the ϕ-average on O_P. So, care must be taken when we perturb f in order to create a periodic maximizing measure; we must guarantee that all non-periodic invariant measures created in the process have a 'small' average.

Therefore, in section 3.2 we derive some technical properties of the interval where the perturbation takes place. The perturbation itself is done in section 4.

3. Finding the return interval

We begin by considering the sets

$$\begin{aligned} A_0 = \{d_0, d_1, \ldots, d_K\} &= \{x \in S^1 : f' \text{ is not continuous at } x\} \\ &\supset \{x \in S^1 : x \text{ is a turning point of } f\} \end{aligned}$$

and

$$A = \bigcup_{i=0}^{\infty} f^{i}(A_0), \qquad B = \bigcup_{i=0}^{\infty} f^{-i}(A).$$

From the above definitions we get that A and B are denumerable, since every point in S^1 has at most a finite number of pre-images, and $\mu_{\max}(B) = 0$ because $\mu_{\max}$ is non-atomic. Also note that $f(A) \subset A$, $f^{-1}(A) \supset A$, $f(B) = B$ and $f^{-1}(B) = B$.

Now let us define two important sets.

- Let C be the set of points in $\operatorname{supp}(\mu_{\max}) \setminus B$ that are recurrent by both sides, that is, $x \in C$ if and only if there are sequences $n_i^l, n_i^r \to \infty$ as $i \to \infty$ and $f^{n_i^l}(x) \overset{i\to\infty}{\to} x$ through the left and $f^{n_i^r}(x) \overset{i\to\infty}{\to} x$ through the right.
- Let $C' \subset \operatorname{supp}(\mu_{\max}) \setminus B$ be the set of points whose orbits are dense in $\operatorname{supp}(\mu_{\max})$.

Lemma 2. *C has total $\mu_{\max}$ measure.*

Proof. As $\mu_{\max}$ is ergodic, Birkhoff's ergodic theorem implies that $\mu_{\max}(C') = 1$. Every $x \in C'$ is recurrent and if it is not recurrent by both sides, as the orbit of x is dense in the closed set $\operatorname{supp}(\mu_{\max})$, x belongs to the boundary of an open interval in $(\operatorname{supp}(\mu_{\max}))^c$. But this implies that the set of all $x \in C'$ which are not recurrent by both sides is denumerable, as it is contained in the boundary of the open set $(\operatorname{supp}(\mu_{\max}))^c$. As $\mu_{\max}$ is non-atomic, this set has zero $\mu_{\max}$ measure. So, $\mu_{\max}(C) = 1$. □

Now we prove lemma 3.

Lemma 3. *C satisfies the hypothesis of lemma 1.*

Proof. Clearly, if $x \in C \subset C' \subset \operatorname{supp}(\mu_{\max})\setminus B$, f is locally a linear homeomorphism (with constant slope) at x. Thus $f(x) \in C \Rightarrow f(C) \subset C$. Also, it is easy to see that if we denote by $\operatorname{Per}(f) = \{x \in S^1 : f^n(x) = x$, for some integer $n > 0\}$ the set of f-periodic points and

$$\operatorname{Pr}\operatorname{Per}(f) = \bigcup_{i=0}^{\infty} f^{-i}(\operatorname{Per}(f)),$$

we get that $C \cap \operatorname{Pr}\operatorname{Per}(f) = \emptyset$, so C satisfies the hypothesis of lemma 1. □

In the following we will start the proof of the main theorem, which will be divided into 2 cases and several sub-cases.

We proceed by choosing a point $\bar{x}$ in C with an iterate $y \overset{\text{def}}{=} f^n(\bar{x})$ which satisfies ($\varepsilon > 0$ comes from the statement of theorem 1) that

(i) if there exists $\varepsilon^* > 0$ as in lemma 1, then $|\bar{x} - y| < \min\{\varepsilon, \varepsilon^*/100\}$,
(ii) otherwise $|\bar{x} - y| < \varepsilon$.

In both cases the following holds:

$$\frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(\bar{x})) \overset{\text{def}}{=} a_0 \geqslant 0. \tag{3}$$

By lemma 1 this can always be achieved. Furthermore, for all $1 \leqslant i \leqslant n-1$, we can suppose that $f^i(\bar{x}) \notin [\bar{x}, y]$, just by taking appropriate points in $\{\bar{x}, f(\bar{x}), f^2(\bar{x}), \ldots, f^n(\bar{x})\} \subset C$ and renaming them as $\bar{x}$ and y, if necessary. Recalling that as $\bar{x} \notin B$ and $f(C) \subset C$, we get that $f^i(\bar{x}) \in C$ for all integer $i > 0$, so $y \in C$.

So, as y is recurrent by both sides, there is a positive integer k such that $f^k(y) \in (\bar{x}, y)$ and $f^i(y) \notin (\bar{x}, y)$, $0 < i < k$. Also, by lemma 1, if a strictly positive a_0 cannot be chosen (in other words, if $\varepsilon^* > 0$ exists), then $\sum_{i=0}^{k-1} \phi(f^i(y)) = 0$. By the notation $(\bar{x}, y)$, $[\bar{x}, y)$, $(\bar{x}, y]$, $[\bar{x}, y]$ we denote the interval (open, semi-closed and closed) of the circle with length less then ε whose extreme points are $\bar{x}$ and y.

As stated in section 2.2, there are two cases to consider, precisely.

(1) For all possible choices of $\bar{x}$ and $n > 0$, $a_0 = 0$,

(2) $a_0 > 0$.

In case 1, we look at the subset

$$\mathrm{PRE}_y = \{z \in [\bar{x}, y] \mid \exists n_z > 0, f^{n_z}(z) = y\}.$$

We have two subcases, namely.

(1.1) for all $z \in \mathrm{PRE}_y$,

$$\frac{1}{n_z} \sum_{i=0}^{n_z-1} \phi(f^i(z)) \leqslant 0,$$

(1.2) there is a z_0 in PRE_y such that

$$\frac{1}{n_{z_0}} \sum_{i=0}^{n_{z_0}-1} \phi(f^i(z_0)) > 0.$$

3.1. *Case 1.1*

Let us first consider the situation where a_0 (see (3)) cannot be chosen larger than 0 and such that for all $x \in (\bar{x}, y)$ and all positive n_x satisfying $f^{n_x}(x) = y$, we have $\sum_{i=0}^{n_x-1} \phi(f^i(x)) \leqslant 0$. The above hypotheses imply that if for some $x \in (\bar{x}, y)$ and $n'_x < n_x$, $f^{n'_x}(x) = \bar{x}$, then $\sum_{i=0}^{n'_x-1} \phi(f^i(x)) \leqslant 0$.

We further divide this case into two situations.

(1.1.1) For all $x \in (\bar{x}, y)$, if $f^j(x)$ also belongs to $(\bar{x}, y)$, then $\sum_{i=0}^{j-1} \phi(f^i(x)) \leqslant 0$.

(1.1.2) There exists a point $p \in (\bar{x}, y)$ such that its first return to $(\bar{x}, y)$ happens at a time $n_p > 0$ and such that $\frac{1}{n_p} \sum_{i=0}^{n_p-1} \phi(f^i(p)) = a_1 > 0$.

In the first case we can consider the transformation $\tilde{f} = T \circ f$, where T is a continuous transformation which is the identity outside $(\bar{x}, y)$, and $T(x) = \bar{x}$ for all points in $(\bar{x}, f^k(y))$ (see figure 1). Now $\bar{x}$ is a periodic point for $\tilde{f}$ with 0 average, because from lemma 1 $\sum_{i=0}^{n+k-1} \phi(f^i(\bar{x})) = 0$, and we did not create new orbits with ϕ average greater then 0.

In case 1.1.2, let $m_0(a_1)$ be given by proposition 1. For every integer $l \in \{1, 2, \ldots, m_0(a_1)\}$, we consider the compact set $K_l = [\bar{x}, y] \cup f^{-l}[\bar{x}, y]$, and let q_l be a point of K_l that maximizes $g_l(x) = \frac{1}{l} \sum_{i=0}^{l-1} \phi(f^i(x))$. Now let $q \in \{q_1, q_2, \ldots, q_{m_0(a_1)}\}$ be such that n_q is the first return of q to $[\bar{x}, y]$, and such that for all $x \in (\bar{x}, y)$, if $f^j(x)$ also belongs to $(\bar{x}, y)$, then

$$\frac{1}{j} \sum_{i=0}^{j-1} \phi(f^i(x)) \leqslant \frac{1}{n_q} \sum_{i=0}^{n_q-1} \phi(f^i(q)).$$

Since p must belong to a K_i for some i, we have

$$\frac{1}{n_q} \sum_{i=0}^{n_q-1} \phi(f^i(q)) \geqslant a_1 > 0.$$

This implies that $f^{n_q}(q) \notin \{\bar{x}, y\}$.

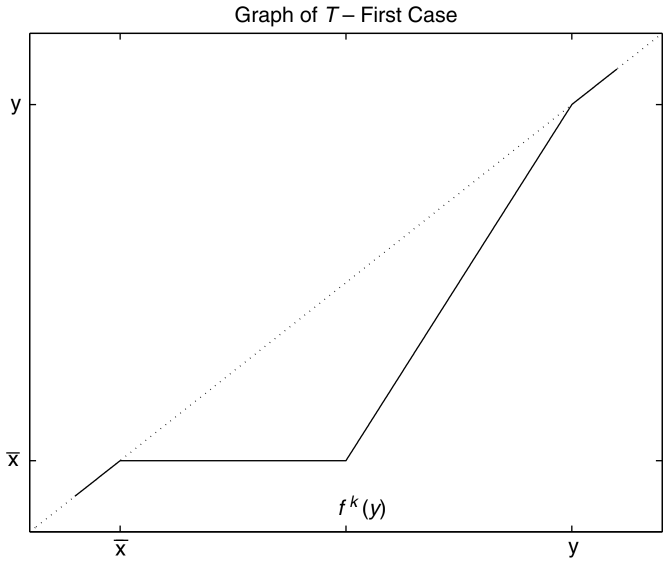

Figure 1. The graph of T in case 1.1.1. $\bar{x}$ is a periodic point of $\tilde{f} = T \circ f$.

We again consider an endomorphism of the form $\tilde{f} = T \circ f$, but now T is any continuous transformation such that $T(f^{n_q}(q)) = q$ and such that T is the identity outside $(\bar{x}, y)$. Now q is a periodic point for $\tilde{f}$, and its orbit is clearly the one maximizing the ϕ average.

In both cases, it is still possible that another trajectory has a ϕ average equal either to 0 in case 1.1.1 or to the partial average of q in case 1.1.2. But since every point whose positive orbit does not return to $(\bar{x}, y)$ has, at most, a 0 average, and for every point in $(\bar{x}, y)$, the ϕ average of its first return for $\tilde{f}$ is the same as for f, the maximum possible ϕ-average must be those of the periodic orbits.

3.2. The remaining case

Now we are left to consider the case where there exists $\tilde{x} \in [\bar{x}, y)$ (note that $\tilde{x}$ may equal $\bar{x}$ or not) and an integer $n_{\tilde{x}} > 0$ satisfying $f^{n_{\tilde{x}}}(\tilde{x}) = y$, with

$$\frac{1}{n_{\tilde{x}}} \sum_{i=0}^{n_{\tilde{x}}-1} \phi(f^i(\tilde{x})) \stackrel{\text{def}}{=} a > 0. \tag{4}$$

Note that this case includes 1.2 and 2 of page 4.

Proposition 1 implies that for some

$$n_{\max} < m_0 = m_0(a), \tag{5}$$

there exists a point $x_{\max} \in [\bar{x}, y]$ with $f^{n_{\max}}(x_{\max}) \in [\bar{x}, y]$, $f^i(x_{\max}) \notin [\bar{x}, y]$, for all $1 \leqslant i \leqslant n_{\max} - 1$, such that for every $x \in [\bar{x}, y]$ and $m > 0$ with $f^m(x) \in [\bar{x}, y]$, we have

$$\frac{1}{m} \sum_{i=0}^{m-1} \phi(f^i(x)) \leqslant \frac{1}{n_{\max}} \sum_{i=0}^{n_{\max}-1} \phi(f^i(x_{\max})) \geqslant a > 0.$$

It may be the case that more than one pair $\{x_{\max}, f^{n_{\max}}(x_{\max})\} \subset [\bar{x}, y]$ as above exists. We choose one which minimizes the distance $|x_{\max} - f^{n_{\max}}(x_{\max})|$.

Now we have two possibilities.

(I) $\{x_{\max}, f^{n_{\max}}(x_{\max})\} \subset (\bar{x}, y)$,
(II) possibility 1 does not hold.

In case I, as in 1.1.2 of section 3.1, consider the endomorphism $\tilde{f} = T \circ f$, where T is any continuous transformation such that $T(f^{n_{\max}}(x_{\max})) = x_{\max}$ and T is the identity outside $(\bar{x}, y)$. This turns $x_{\max}$ into a periodic point for $\tilde{f}$, and its orbit maximizes the ϕ average.

In case II, as $f(C) \subset C$ and $\bar{x}, y \in C$, we get that $f^{n_{\max}}(x_{\max}) \in C$. From now on, for notation's sake, we denote $x_{\max}$ by $\bar{x}$, $n_{\max}$ by n and $f^{n_{\max}}(x_{\max})$ by $y \in C$. In this way, we get

(i) $f^i(\bar{x}) \notin [\bar{x}, y]$, for all $1 \leqslant i \leqslant n-1$,
(ii) $\frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(\bar{x})) \geqslant a > 0$,
(iii) for every $x \in (\bar{x}, y]$ and $m > 0$ with $f^m(x) \in [\bar{x}, y]$, we have

$$\frac{1}{m}\sum_{i=0}^{m-1} \phi(f^i(x)) < \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(\bar{x})). \tag{6}$$

The rest of the paper deals with this situation.

As we said $y \in C$, so there must be a first positive integer $N_{\text{ret}}(y)$ such that $f^{N_{\text{ret}}(y)}(y) \in (\bar{x}, y)$.

The next step is to choose a $\delta_1 > 0$ sufficiently small such that the following conditions are satisfied (recall that m_0 comes from expression (5)).

- $\cup_{i=-m_0-10}^{m_0+10} f^i(A_0) \cap [y, y+\delta_1] = \emptyset$ (f^j is linear with constant slope in every connected component of $f^{-j}([y, y+\delta_1])$, $1 \leqslant j \leqslant m_0+1$). This is possible because $y \notin B \Rightarrow y \notin \cup_{i=-m_0-10}^{m_0+10} f^i(A_0)$, which is a finite set. So, if $\delta_1 > 0$ is sufficiently small, this condition is satisfied.
- For every $0 \leqslant j \leqslant m_0$, if x is in a connected component of $f^{-j}([y, y+\delta_1]) \cap [\bar{x}, y+\delta_1]$ and $\bar{x}$ is not, then $\frac{1}{j}\sum_{i=0}^{j-1} \phi(f^i(x)) < \frac{1}{n}\sum_{i=0}^{n-1} \phi(f^i(\bar{x}))$ (the average of ϕ over the trajectory of x until its first return is less than the average of ϕ on the orbit of $\bar{x}$).
- For all $x \in [y, y+\delta_1]$ and $0 < j < n_{\text{ret}(y)}$, $f^j(x) \notin [\bar{x}, y+\delta_1)$. Furthermore, $f^{n_{\text{ret}(y)}}([y, y+\delta_1]) \subset (\bar{x}, y)$ (that is, the first return, by f, of the interval $[y, y+\delta_1]$ is achieved at the iterate $n_{\text{ret}(y)}$ and lies inside the open segment $(\bar{x}, y)$).
- $f^n(y) \notin [y, y+\delta_1]$.

4. Building $\tilde{f}$

We will construct an endomorphism $\tilde{f} : S^1 \to S^1$ that satisfies

- $|f - \tilde{f}| = \max_{x \in S^1} |f(x) - \tilde{f}(x)| < \varepsilon$, where $\tilde{f}(x) = T(f(x))$ and $T(x) = x$, if x is not in $[\bar{x}, y+\delta_1]$ and of course $T([\bar{x}, y+\delta_1]) = [\bar{x}, y+\delta_1]$

Also, T will be non-decreasing, and $T(y) = \bar{x}$. Clearly, the dynamics of this new endomorphism differs significantly only at those points returning infinitely many times to the interval $I = [\bar{x}, y+\delta_1]$.

So let us consider the set

$$D = I \cap \left(\bigcup_{i=0}^{\infty} f^{-i}(I)\right) \tag{7}$$

of all those points in I returning to I, and let us define, for each point in D, the following functions,

$$\begin{aligned} N_{\mathrm{ret}}(x) &= \inf\{j \in \mathbb{N}^* \mid f^j(x) \in I\}, \\ f_2(x) &= f^{N_{\mathrm{ret}}(x)}(x) \text{ (first return to } I), \\ \psi(x) &= \frac{1}{N_{\mathrm{ret}}(x)} \sum_{i=0}^{N_{\mathrm{ret}}(x)-1} \phi(f^i(x)). \end{aligned} \tag{8}$$

Clearly, ψ gives the average of ϕ in the portion of a trajectory before returning to I. It should be clear that $\psi(\bar{x}) \geqslant a > 0$, see (6).

The following lemma will be useful.

Lemma 4. *: Let $x \in D \cap f_2^{-1}([y, y+\delta_1])$ be a point not in the connected component of $f^{-n}([y, y+\delta_1])$ that contains $\bar{x}$ (recall that $n > 0$ is such that $f^n(\bar{x}) = y$). Then $\psi(x) < \psi(\bar{x})$.*

Proof. The statement holds by the choice of $\bar{x}$ and $\delta_1 > 0$ for every x in the hypotheses such that $N_{\mathrm{ret}}(x) \leqslant m_0$. It must also hold for any x such that $N_{\mathrm{ret}}(x) > m_0$ by proposition 1. □

We recall that, by the requirements in the choice of δ_1, in each connected component of $f^{-n}([y, y+\delta_1])$, f^n is a linear isomorphism ($n \leqslant m_0 + 10$). There are three different cases, and we will choose T accordingly.

(a) The connected component of $f^{-n}([y, y+\delta_1]) \cap I$ that contains $\bar{x}$ is $\{\bar{x}\}$.
(b) There is a point w in I such that f^n is a linear isomorphism between $[\bar{x}, w]$ and $[y, y+\delta_1]$, and ψ is strictly increasing in $[\bar{x}, w]$.
(c) There is a point w in I such that f^n is a linear isomorphism between $[\bar{x}, w]$ and $[y, y+\delta_1]$, and ψ has a local maximum in $[\bar{x}, w)$.

In cases (b) and (c), since f^n is a linear isomorphism between $[\bar{x}, w]$ and $[y, y+\delta_1]$, it follows from the choice of δ_1 that $y \notin [\bar{x}, w]$.

In case (a), we will choose any T such that $T(x) = \bar{x}$ for all x in $[\bar{x}, y]$, $T(y+\delta_1) = y+\delta_1$ and T is non-decreasing, see figure 2.

In case (c), let α be a local maximum and $\alpha \in [c, d]$ a subset of $[\bar{x}, w)$ such that $\psi(x) \leqslant \psi(\alpha)$, for all x in $[c, d]$. We will choose T as

$$T(x) = \begin{cases} \bar{x}, & \text{if } x < f_2(c), \\ \alpha - \dfrac{(f_2(\alpha) - x)(\alpha - \bar{x})}{f_2(\alpha) - f_2(c)}, & \text{if } f_2(c) \leqslant x \leqslant f_2(\alpha), \\ \alpha - \dfrac{(f_2(\alpha) - x)(\alpha - (y+\delta_1))}{f_2(\alpha) - f_2(d)}, & \text{if } f_2(\alpha) \leqslant x \leqslant f_2(d), \\ y + \delta_1, & \text{if } f_2(d) < x. \end{cases} \tag{9}$$

Just like in case 1.1.1 of section 3.1, in case (a) a maximizing measure is supported in the periodic trajectory of $\bar{x}$.

In case (c), by lemma 4, if $x \in D \setminus [\bar{x}, w]$, $\psi(x) < \psi(\bar{x})$. If $x \in [c, d]$, $\psi(x) \leqslant \psi(\alpha)$. Finally, if $x \in [\bar{x}, w] \setminus [c, d]$, there is a positive iterate of x under $\tilde{f}$ which coincides with $\bar{x}$ (see figure 3). Thus, a maximizing measure is supported either in the periodic orbit of α or in the periodic orbit of $\bar{x}$.

Finally, in case (b), note that for all x in $[\bar{x}, w]$, as $N_{\mathrm{ret}}(x) = n$, ψ is a strictly increasing continuous function, so there must be a point α in the open interval $(\bar{x}, w)$ such that ψ is differentiable at α, and $\psi'(\alpha) = K$, for some real number K.

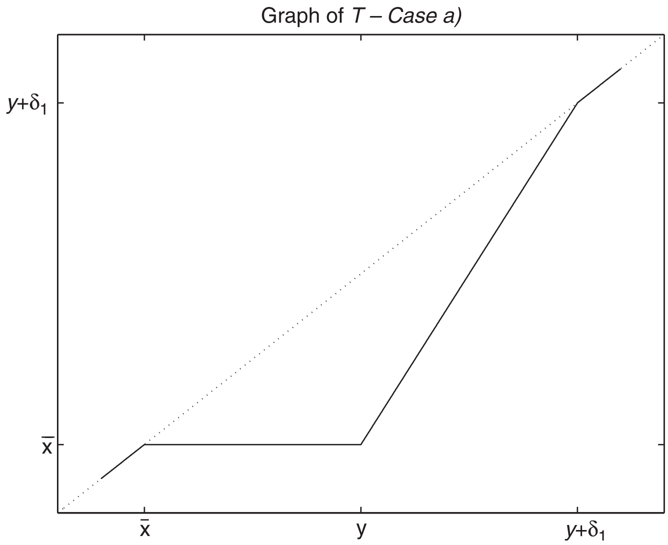

Figure 2. The graph of T in case (a). $\bar{x}$ is a periodic point of $\tilde{f} = T \circ f$.

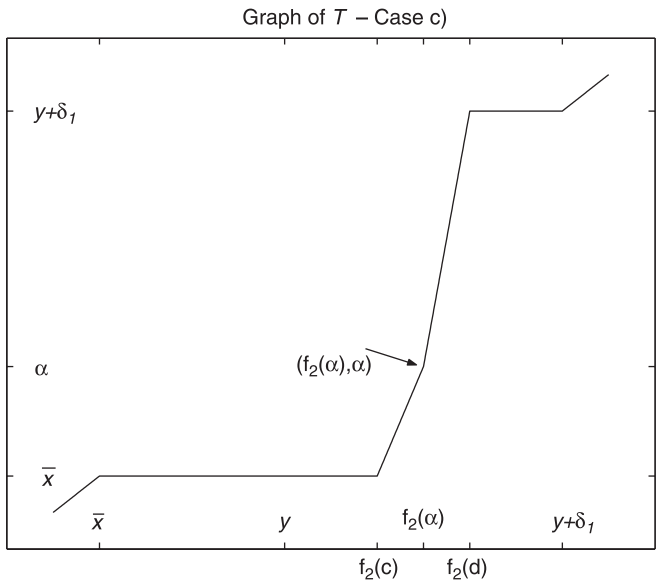

Figure 3. The graph of T in case (c). α is a periodic point of $\tilde{f} = T \circ f$.

We can choose $\delta_2 > 0$ sufficiently small such that all of the following happens.

- $[\alpha - \delta_2, \alpha + \delta_2] \subset (\bar{x}, w)$.
- For all x in $[\alpha - \delta_2, \alpha + \delta_2]$, we have $\mid \psi(x) - \psi(\alpha) \mid \leqslant (K+1) \mid x - \alpha \mid$.
- For all x in $[\alpha - \delta_2, \alpha + \delta_2]$, we have

$$\mid \psi(x) - \psi(\alpha) \mid \leqslant \frac{\psi(\alpha) - \psi(\bar{x})}{4m_0}. \tag{10}$$

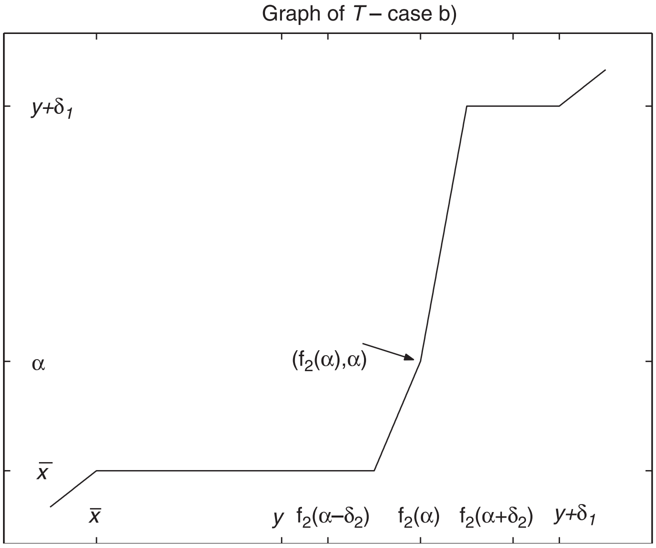

Figure 4. The graph of T in case (b). α is a periodic point of $\tilde{f} = T \circ f$.

In this context, we can choose T (see figure 4) as to satisfy the following conditions (recall that f_2 is defined in (8)).

1. T is non-decreasing and is the identity outside $[\bar{x}, y + \delta_1]$.
2. $T(x) = \bar{x}$ if $\bar{x} \leqslant x \leqslant f_2(\alpha - \delta_2)$, and $T(x) = y + \delta_1$ if $f_2(\alpha + \delta_2) \leqslant x \leqslant y + \delta_1$.
3. T is strictly increasing in some subinterval J of $[f_2(\alpha-\delta_2), f_2(\alpha+\delta_2)]$ such that $f_2(\alpha) \in J$. Also, $T(f_2(\alpha)) = \alpha$.
4. If x belongs to $[\alpha - \delta_2, \alpha + \delta_2]$, and
$$\tfrac{1}{2}(\psi(\alpha - \delta_2) - \psi(\alpha)) \leqslant \psi(x) - \psi(\alpha) \leqslant \tfrac{1}{2}(\psi(\alpha + \delta_2) - \psi(\alpha)),$$
then
$$\psi(T(f_2(x))) - \psi(\alpha) = 2(\psi(x) - \psi(\alpha)). \tag{11}$$
5. For every x in $[\alpha - \delta_2, \alpha)$, we have $(T \circ f(x))^{i.n} = \bar{x}$ for all sufficiently large $i > 0$.

Item 4 may be satisfied since both ψ and f_2 are strictly increasing in $[\alpha - \delta_2, \alpha + \delta_2]$. One useful fact is that (11) ensures that α is a repelling source for $T \circ f_2$.

Item 5 follows from the previous items but is included for simplicity. Figure 5 presents a sketch of the graph of $T \circ f_2$ in $[\bar{x}, w]$, the first return for $\tilde{f}$.

We claim that, in case (b), the invariant measure supported in the periodic orbit of α maximizes $\int \phi \, \mathrm{d}\mu$. To see this, first note that for any point in the circle, if its positive orbit by $\tilde{f} = T \circ f$ falls inside $[\bar{x}, w]$, but outside $[\alpha, \alpha + \delta_2]$, it must eventually end at $\bar{x}$ (since $T \circ f_2(\alpha - \delta_2) = \bar{x}$ and by the choice of δ_1, $f_2(y + \delta_1) \in (\bar{x}, y)$, so $T \circ f_2(y + \delta_1) = \bar{x}$, see figure 5).

Also, if a point $x \in D$ is such that $f_2(x) \notin [y, y + \delta_1]$, then $T(f_2(x)) = \bar{x}$, and since $\bar{x}$ is a periodic point for $\tilde{f}$ with ϕ average smaller than $\psi(\alpha)$ we need not be concerned.

If we consider those points x in $D \cap (w, y + \delta_1]$ such that $f_2(x) \in [y, y + \delta_1]$, then the choice of δ_1 gives $\psi(x) < \psi(\bar{x})$.

Therefore, if a trajectory is such that $\limsup_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \phi(\tilde{f}^i(x)) > \psi(\alpha)$, then x must return to $[\alpha, \alpha + \delta_2]$ infinitely many times.

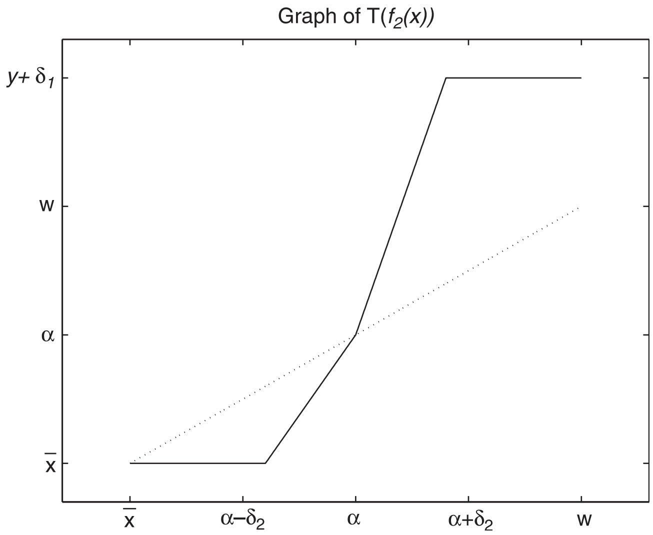

Figure 5. Sketch of the graph of $\tilde{f}^n$ in the domain $[\bar{x}, w]$ for case (b). α is a repelling source for this function.

Now, since α is repelling (see condition (11)), a point x in $(\alpha, \alpha+\delta_2]$ cannot have all its returns inside this interval without visiting $(w, y+\delta_1]$.

Therefore, we fix a point x in $(\alpha, \alpha+\delta_2]$ returning infinitely many times to this interval and consider the following increasing sequences of positive integers.

- n_j, where $n_0 = 0$ and $n_{i+1} = n_i + N_{\text{ret}}(\tilde{f}^{n_i}(x))$. This is the sequence of iterates for the return of x to $I = [\bar{x}, y+\delta_1]$.
- a_j, where $a_1 = 0$ and a_{i+1} is the smallest integer larger than a_i such that $\tilde{f}^{n_{a_{i+1}}}(x)$ is in $[\alpha, \alpha+\delta_2]$, but $\tilde{f}^{n_{(a_{i+1}-1)}}(x)$ is not.
- b_j, where b_i is the smallest integer larger than a_i such that $\tilde{f}^{n_{(b_i-1)}}(x)$ is in $[\alpha, \alpha+\delta_2]$, but $\tilde{f}^{n_{b_i}}(x)$ is not.

We have that

$$\frac{1}{n_{a_k}} \sum_{i=0}^{n_{a_k}-1} \phi(\tilde{f}^i(x)) = \frac{1}{n_{a_k}} \sum_{l=0}^{a_k-1} N_{\text{ret}}(\tilde{f}^{n_l})(x)\psi(\tilde{f}^{n_l}(x))$$
$$= \frac{1}{n_{a_k}} \sum_{j=1}^{k-1} \left(\sum_{l=a_j}^{b_j-1} N_{\text{ret}}(\tilde{f}^{n_l})(x)\psi(\tilde{f}^{n_l}(x)) + \sum_{l=b_j}^{a_{j+1}-1} N_{\text{ret}}(\tilde{f}^{n_l})(x)\psi(\tilde{f}^{n_l}(x)) \right). \tag{12}$$

Since, for all integers l in $[b_k, a_{k+1}-1)$, we have that $\psi(\tilde{f}^{n_l}(x)) \leqslant \psi(\bar{x})$ (because if the orbit of x does not belong to $[\alpha, \alpha+\delta_2]$, it must be outside $[\bar{x}, w]$, otherwise it would be attracted to $\bar{x}$), then

$$\sum_{l=b_j}^{a_{j+1}-1} N_{\text{ret}}(\tilde{f}^{n_l})(x)\psi(\tilde{f}^{n_l}(x)) \leqslant (n_{a_{j+1}} - n_{b_j})\psi(\bar{x}). \tag{13}$$

We recall that for any x in $[\bar{x}, w]$, $N_{\rm ret}(x) = N_{\rm ret}(\bar{x}) = n$ and so, by the construction of T and the sequences a_j, b_j,

$$\sum_{l=a_j}^{b_j-1} N_{\rm ret}(\tilde{f}^{n_l})(x)(\psi(\tilde{f}^{n_l}(x)) - \psi(\alpha)) = \sum_{i=0}^{b_j-a_j} 2^{-i} N_{\rm ret}(\bar{x})(\psi(\tilde{f}^{n_{(b_j)-1}}(x)) - \psi(\alpha))$$

$$< 2N_{\rm ret}(\bar{x})(\psi(\alpha+\delta_2) - \psi(\alpha)) < \frac{\psi(\alpha) - \psi(\bar{x})}{2}, \qquad (14)$$

where in the first equality we used (11) in the definition of T, and the last inequality made use of the choice of δ_2 and (10).

Substituting the two previous equations in (12) yields that

$$\frac{1}{n_{a_k}} \sum_{i=0}^{n_{a_k}-1} \phi(\tilde{f}^i(x)) < \psi(\alpha),$$

and since this is valid for a strictly increasing sequence n_{a_k}, there cannot be an ergodic measure ν, invariant by $\tilde{f}$, such that $\int \phi \, d\nu > \psi(\alpha)$, and we are done.

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References

[1] Bousch T 2000 Le poisson n'a pas d'arêtes *Ann. Inst. H Poincaré Probab. Stat.* **36** 489–508
[2] Jenkinson O 2000 Frequency locking on the boundary of the barycentre set *Exp. Math.* **9** 309–17
[3] Jenkinson O 2006 Every ergodic measure is uniquely maximizing *Discrete Contin. Dyn. Syst.* **16** 383–92
[4] Jenkinson O, Mauldin R D and Urbański M 2007 Ergodic optimization for noncompact dynamical systems *Dyn. Syst.* **22** 379–88
[5] Jenkinson O, Mauldin R D and Urbański M 2006 Ergodic optimization for countable alphabet subshifts of finite type *Ergod. Theory Dyn. Syst.* **26** 1791–803
[6] Jenkinson O 2006 Ergodic optimization *Discrete Contin. Dyn. Syst.* **15** 197–224
[7] Morris I D 2007 A sufficient condition for the subordination principle in ergodic optimization *Bull. London Math. Soc.* **39** 214–20
[8] Tal F and Addas-Zanata S 2008 On maximizing measures of homeomorphisms on compact manifolds *Fundam. Math.* **200** 145–59
[9] Tal F and Addas-Zanata S 2008 Support of maximizing measures for typical $\mathcal{C}^0$ dynamics on compact manifolds *Preprint*