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Instability for the rotation set of homeomorphisms of the torus homotopic to the identity

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Abstract. In this paper we consider homeomorphisms $f : T^2 \to T^2$ homotopic to the identity and their rotation sets $\rho(\tilde{f})$, which are compact convex subsets of the plane. We show that if $\rho(\tilde{f})$ has an extremal point (t, ω) which is not a rational vector, then arbitrarily C^0 close to f we can find a homeomorphism g such that $\rho(\tilde{g}) \cap \rho(\tilde{f})^c \neq \emptyset$. So in this case, we have instability for the rotation set.

1. *Introduction and statement of the principal result*

The dynamics of a homeomorphism of the circle has a well known and very important invariant, the so called rotation number. Roughly speaking, it measures the average speed the orbit of a point in the circle rotates around it, and its rationality or not has strong implications on the dynamics of the homeomorphism (see [**3**] for a nice didactic exposition). In particular, when the rotation number is irrational, by arbitrarily small perturbations we can increase its value.

Homeomorphisms of the 2-torus homotopic to the identity are the most natural two-dimensional generalization of circle homeomorphisms, but some care must be taken in this generalization. For example, in the torus we do not have rotation numbers, but rotation vectors, and there may be different rotation vectors for different points of the torus. So a rotation set must be defined, which in this case will be a subset of the plane. There are some different definitions for 'rotation set' in the literature, however the sets corresponding to these different definitions can differ only on their boundaries. In particular, they all have the same convex hull, which is equal to the Misiurewicz–Ziemian rotation set, the definition we use. Our main result is inspired in some way by the result we mentioned for circle homeomorphisms. More precisely, we prove that if the Misiurewicz–Ziemian rotation set has an extremal point $(t, \omega) \notin \mathbb{Q}^2$, then by appropriate arbitrarily small perturbations we can change the rotation set. Some consequences of this fact can be understood if we remember a series of results on the subject (due to Franks, Kwapisz, Misiurewicz and Ziemian) and this will be done in the last section.

This paper is organized as follows. In the first section, we present some basic definitions and the precise statement of our main result. In the second section, we present a brief summary of the results used in the paper and proofs for some simple facts. In the third section, we prove our main theorem. In the last section, we show some consequences and present some questions.

Notation and definitions.

(1) Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 'flat' torus and $p : \mathbb{R}^2 \to T^2$ be the associated covering map. We also fix the Euclidean metric on the plane and the corresponding metric on the torus. The distance between $x, y \in T^2$ ($\tilde{x}, \tilde{y} \in \mathbb{R}^2$) is denoted by $\|x - y\|$ ($\|\tilde{x} - \tilde{y}\|$).

(2) $D^0(T^2)$ is the set of homeomorphisms of the torus homotopic to the identity and $D^0(\mathbb{R}^2)$ is the set of lifts to plane of elements from $D^0(T^2)$.

(3) Let $p_1 : \mathbb{R}^2 \to \mathbb{R}$ and $p_2 : \mathbb{R}^2 \to \mathbb{R}$ be the standard projections $p_1(\tilde{x}) = \tilde{x}_1$ and $p_2(\tilde{x}) = \tilde{x}_2$, where $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$.

(4) Given a mapping $f \in D^0(T^2)$ and a lift $\tilde{f} \in D^0(\mathbb{R}^2)$, in **[5]** Misiurewicz and Ziemian define the rotation set as follows:

$$\rho(\tilde{f}) = \bigcap_{i=1}^{\infty} \overline{\bigcup_{n \geq i} \left\{ \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} : \tilde{x} \in \mathbb{R}^2 \right\}}; \tag{1}$$

that is, $(a, b) \in \rho(\tilde{f})$ if and only if there are sequences $\tilde{x}_i \in \mathbb{R}^2$ and $n_i \to \infty$, such that

$$\lim_{i \to \infty} \frac{\tilde{f}^{n_i}(\tilde{x}_i) - \tilde{x}_i}{n_i} = (a, b).$$

Among other things, they prove that $\rho(\tilde{f})$ is a compact convex subset of $\mathbb{R}^2$, which are important properties for this work.

Now we are ready to state our main result.

THEOREM 1. *Let $\tilde{f} \in D^0(\mathbb{R}^2)$ be such that $\rho(\tilde{f})$ has an extremal point $(t, \omega) \notin \mathbb{Q}^2$. Then there exists a $\tilde{g} \in D^0(\mathbb{R}^2)$ arbitrarily C^0 close to $\tilde{f}$ such that $\rho(\tilde{g}) \neq \rho(\tilde{f})$. Moreover, $\rho(\tilde{g}) \cap \rho(\tilde{f})^c \neq \emptyset$.*

2. *Basic tools*

First we recall some results about the rotation set due to Misiurewicz and Ziemian, Franks and Kwapisz.

Given $\tilde{f} \in D^0(\mathbb{R}^2)$, we can define a continuous function $\phi : T^2 \to \mathbb{R}^2$ by

$$\phi(x) = \tilde{f}(\tilde{x}) - \tilde{x}, \quad \text{where } \tilde{x} \in p^{-1}(x). \tag{2}$$

The rotation vector of an f-invariant measure μ is defined in the following way

$$\rho(\mu) = \int_{T^2} \phi(x)\, d\mu.$$

As we already said, in [**5**] it is proved that $\rho(\tilde{f})$, given by expression (1), is a compact convex subset of the plane. The first result we recall is Theorem 2.4 of [**5**].

THEOREM 2. *Given $\tilde{f} \in D^0(\mathbb{R}^2)$, for every extremal point (a, b) of $\rho(\tilde{f})$ there exists an f-invariant, ergodic measure μ with rotation vector $\rho(\mu) = (a, b)$.*

Now we present a theorem due to Franks (periodic case) [**2**] and Misiurewicz and Ziemian (general case) [**6**] that gives information about points in the interior of $\rho(\tilde{f})$.

THEOREM 3. *Let $\tilde{f} \in D^0(\mathbb{R}^2)$ and $(a, b) \in \operatorname{int}(\rho(\tilde{f}))$. Then there is an f-invariant compact set $X \subset T^2$ such that for all $x \in X$,*

$$\rho(x, \tilde{f}) = \lim_{n\to\infty} \frac{\tilde{f}^n(\tilde{x}) - \tilde{x}}{n} = (a, b), \quad \textit{for any } \tilde{x} \in p^{-1}(x).$$

If $(a, b) \in \mathbb{Q}^2$, then X can be chosen as a periodic orbit.

In [**6**] the following theorem is also proved.

THEOREM 4. *The function $\rho : D^0(\mathbb{R}^2) \to \{$space of all compact subsets of $\mathbb{R}^2\}$ with the Hausdorff metric is continuous at all $\tilde{f}$ with $\operatorname{int}(\rho(\tilde{f})) \neq \emptyset$.*

The next theorem about the existence of mappings f with certain types of rotation sets was proven by Kwapisz in [**4**].

THEOREM 5. *Given a convex polygon $Q \subset \mathbb{R}^2$ with vertices having rational coordinates, there is a mapping $\tilde{f} \in D^0(\mathbb{R}^2)$ such that $\rho(\tilde{f}) = Q$.*

So Theorems 4 and 5 imply that there are convex sets with non-rational extremal points that can be rotation sets for mappings from $D^0(\mathbb{R}^2)$.

In the following, we present a proof of a simple result, known as the C^0-closing lemma.

LEMMA 1. *Given $f \in D^0(T^2)$ and a lift $\tilde{f} \in D^0(\mathbb{R}^2)$, $\delta > 0$, $n \in \mathbb{N}^*$ and a non-periodic point $x \in T^2$ ($\tilde{x} \in p^{-1}(x)$), such that $\|\tilde{f}^n(\tilde{x}) - \tilde{x} - (l, m)\| < \delta$, for some $(l, m) \in \mathbb{Z}^2$, then there is a mapping $\tilde{g} \in D^0(\mathbb{R}^2)$ such that $\|\tilde{g} - \tilde{f}\|_0 < 2\delta$, $g^n(x) = x$ and $\tilde{g}^n(\tilde{x}) = \tilde{x} + (l, m)$, for any $\tilde{x} \in p^{-1}(x)$.*

Remark. As usual we denote:

$$\|g - f\|_0 \overset{\text{def.}}{=} \sup_{x\in T^2} \|g(x) - f(x)\| \quad \text{and} \quad \|\tilde{g} - \tilde{f}\|_0 \overset{\text{def.}}{=} \sup_{\tilde{x}\in\mathbb{R}^2} \|\tilde{g}(\tilde{x}) - \tilde{f}(\tilde{x})\|.$$

Proof. The geodesic segment $\overline{xf^n(x)}$ connecting x and $f^n(x)$ of length less than δ does not intersect the set $O_{1,n-1}(x) = \bigcup_{i=1}^{n-1} f^i(x)$ at its extremal points, because the orbit of x is not periodic. However, some element from $O_{1,n-1}(x)$ may still belong to $\overline{xf^n(x)}$. So we must make a finite number of changes in $\overline{xf^n(x)}$ in order to obtain a polygonal arc γ connecting x and $f^n(x)$ disjoint from $O_{1,n-1}(x)$ and with length less than $\frac{3}{2}\delta$. Let V be an open neighborhood of γ which also does not intersect $O_{1,n-1}(x)$. Let $h : T^2 \to T^2$ be a homeomorphism which is the identity outside V, $h(f^n(x)) = x$ and $\|\tilde{h}(\tilde{y}) - \tilde{y}\| < 2\delta$, for all $\tilde{y} \in p^{-1}(V)$. Finally, let $\tilde{g} \in D^0(\mathbb{R}^2)$ be defined by $\tilde{g}(\bullet) = \tilde{h} \circ \tilde{f}(\bullet)$. It is easy to see that $\|\tilde{g} - \tilde{f}\|_0 = \|\tilde{h} \circ \tilde{f} - \tilde{f}\|_0 < 2\delta$ and $\tilde{g}^n(\tilde{x}) = \tilde{x} + (l, m) \Rightarrow g^n(x) = x$. $\square$

Now we present some results and ideas from the theory of cocycles of ergodic transformation groups. A fundamental reference in this subject is the book of Schmidt [**7**]. In particular, our presentation will be directed towards the kind of application we need, so the definitions and results will be stated with no generality. Another fundamental result for us is due to Atkinson [**1**].

Given a mapping $f \in D^0(T^2)$, an f-invariant ergodic probability measure μ and a continuous function $\varphi : T^2 \to \mathbb{R}$, we define the cocycle for f given by φ to be the function $a : \mathbb{Z} \times T^2 \to \mathbb{R}$ given by

$$a(n,x) = \begin{cases} \displaystyle\sum_{i=0}^{n-1} \varphi \circ f^i(x), & \text{for } n > 0, \\ 0, & \text{for } n = 0, \\ -a(-n, f^n(x)), & \text{for } n < 0. \end{cases} \tag{3}$$

The skew-product extension of f, determined by φ, is given by the following mapping $V : T^2 \times \mathbb{R} \to T^2 \times \mathbb{R}$:

$$V(x,\alpha) = (f(x), \alpha + \varphi(x)). \tag{4}$$

So, the powers of V can be expressed as

$$V^n(x,\alpha) = (f^n(x), \alpha + a(n,x)).$$

We say that the cocycle a is recurrent, if and only if, for every $B \in \sigma_B(T^2) = \{$*Borel* σ-algebra of $T^2\}$ with $\mu(B) > 0$ and every $\epsilon > 0$, there is an $n \neq 0$ such that

$$\mu(B \cap f^{-n}(B) \cap \{x : |a(n,x)| < \epsilon\}) > 0.$$

Now we present a result from [**1**].

THEOREM 6. *Suppose that $(T^2, \sigma_B(T^2), \mu)$ is a non-atomic probability space, $f \in D^0(T^2)$ is ergodic with respect to μ and $\varphi : T^2 \to \mathbb{R}$ is a continuous function such that $\int_{T^2} \varphi(x)\, d\mu = 0$. Then the cocycle $a(n,x)$ (see (3)) is recurrent.*

It is easy to see that the skew-product V (see (4)) leaves the product measure $\mu \times \lambda$ invariant, where λ is the Lebesgue measure in $\mathbb{R}$. The problem here is that the space $T^2 \times \mathbb{R}$ is not compact, so we need to work a little more in order to get some kind of recurrence for V. An important definition for this purpose is the following (see Schmidt [**7**, ch. 1]).

We say that the skew-product V is conservative if for every $A \in \sigma_B(T^2 \times \mathbb{R})$ with $\mu \times \lambda(A) > 0$ and for $\mu \times \lambda$-a.e. $(x,\alpha) \in A$, the set

$$\left[\bigcup_{n \in \mathbb{Z}} V^n(x,\alpha) \right] \cap A$$

is infinite. Finally, we present a theorem relating the concepts of recurrence and conservativeness (see [**7**, ch. 5]).

THEOREM 7. *Suppose that $(T^2, \sigma_B(T^2), \mu)$ is a non-atomic probability space, the homeomorphism $f \in D^0(T^2)$ is ergodic with respect to μ and the cocycle $a(n,x)$ (see (3)) is recurrent. Then the skew-product $V(x,\alpha)$ given by (4) is conservative.*

So, Theorems 6 and 7 imply that for any continuous function $\varphi : T^2 \to \mathbb{R}$ such that $\int_{T^2} \varphi(x)\, d\mu = 0$, the cocycle $a(n, x)$ is recurrent and the skew-product $V(x, \alpha)$ is conservative. Another consequence of the conservativeness of a skew-product V as in (4) is the following.

LEMMA 2. *If a skew-product V as in (4) is conservative, then given any $B \in \sigma_B(T^2)$, with $\mu(B) > 0$ and any $\delta > 0$, for μ-a.e. $x \in B$ we have*

$$f^n(x) \in B \quad \textit{and} \quad |a(n, x)| < \delta, \quad \textit{for infinitely many } n \in \mathbb{Z}.$$

The proof is immediate from the definitions.

3. *Proof of the main result*

In this section we prove our main result, Theorem 1. First we present a sketch of the main idea, as an attempt to help the understanding of the reader.

Sketch of the proof. From the theorem hypothesis and the results in §2, we know that $\rho(\tilde{f})$ is a compact convex subset of the plane, which has an extremal point $(t, \omega) \notin \mathbb{Q}^2$. As (t, ω) is an extremal point of $\rho(\tilde{f})$, there is a line (called supporting line) passing through (t, ω) with the following property:

• $\rho(\tilde{f})$ does not intersect both components of the complement of this line; it is contained in the union of the line and one connected component of this complement.

Our main idea is to perturb $\tilde{f}$ in order to create a periodic point whose rotation vector belongs to the other side of the supporting line through (t, ω). As $\rho(\tilde{f})$ does not intersect this side of the line, we get that the rotation set for the perturbed mapping intersects $\rho(\tilde{f})^c$.

The reason why such a perturbation is possible is roughly the following. From Theorem 2, there is an ergodic measure with rotation vector equal to (t, ω). So, in particular, there are recurrent points for f with rotation vector equal to (t, ω). Consider a point $x_0 \in T^2$ of this type and an arbitrarily small neighborhood V_0 of it. As x_0 is recurrent for f, arbitrarily large iterates of x_0 fall inside V_0. As $\rho(x_0, \tilde{f}) = (t, \omega)$, there exists a sequence $n_i \overset{i\to\infty}{\to} \infty$ such that for any $\tilde{x}_0 \in p^{-1}(x_0)$, $\tilde{f}^{n_i}(\tilde{x}_0) - (l_i, m_i) \in \tilde{V}_0$, for some $(l_i, m_i) \in \mathbb{Z}^2$ with $(l_i/n_i, m_i/n_i) \overset{i\to\infty}{\to} (t, \omega)$. Finally, we have two possibilities:

(1) for some $i > 0$, $(l_i/n_i, m_i/n_i)$ belongs to the other side (the side that does not contain any point of $\rho(\tilde{f})$) of the supporting line through (t, ω);

(2) the above is not true.

Possibility (1) above is very easy to work with: we just have to perturb f in order to close the orbit of x_0. Clearly its rotation vector will be equal $(l_i/n_i, m_i/n_i)$. If we suppose that (1) does not happen, then using all the machinery developed in the previous section, we can prove that there exist $i_0, i_1 \in \mathbb{N}^*$ and a point x_1 arbitrarily close to x_0 such that

$$\|\tilde{f}^{n_{i_1}}(\tilde{x}_1) - (l_{i_1} - l_{i_0}, m_{i_1} - m_{i_0}) - \tilde{f}^{n_{i_0}}(\tilde{x}_1)\|$$

is less than the diameter of $V_0 \ni x_0$ and $((l_{i_1} - l_{i_0})/(n_{i_1} - n_{i_0}))$, $(m_{i_1} - m_{i_0})/(n_{i_1} - n_{i_0})$ belongs to the other side of the supporting line through (t, ω). And so, everything continues as in the above case. □

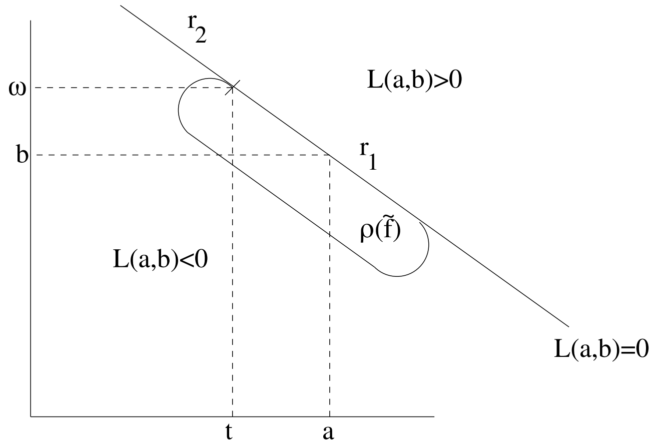

FIGURE 1. Diagram showing $\rho(\tilde{f})$, r_1 and r_2.

In the following, we present a detailed proof for our main result. We are going to prove that for any given $\epsilon > 0$, there is a $\tilde{g} \in D^0(\mathbb{R}^2)$, such that $\|\tilde{f} - \tilde{g}\|_0 < \epsilon$ and $\rho(\tilde{g}) \cap \rho(\tilde{f})^c \neq \emptyset$. So, from now on, we fix an $\epsilon > 0$. As $\tilde{f} \in D^0(\mathbb{R}^2)$ we know that $\rho(\tilde{f})$ is a compact convex subset of the plane, which from the theorem hypothesis, has an extremal point $(t, \omega) \notin \mathbb{Q}^2$. This means, among other things, that there is a line r passing through (t, ω) such that $\rho(\tilde{f})$ does not intersect one of the open connected components of r^c—see Figure 1 for a diagram of this fact. So, if the line r is given by

$$r : \{(a, b) \in \mathbb{R}^2 : L(a, b) = u(a - t) + v(b - \omega) = 0\}, \quad \text{for certain } u, v \in \mathbb{R},$$

then we can suppose without loss of generality that $\rho(\tilde{f}) \subset \{L(a, b) \leq 0\}$.

If we recall Theorem 2, we know that there is an f-invariant, ergodic measure μ with rotation vector $\rho(\mu) = (t, \omega)$, which is non-atomic because an atom of μ would constitute a periodic point with rational rotation vector.

Now let $A \subset T^2$ be the following set: $A = \{x \in \text{supp}(\mu) : x \text{ is recurrent and } \rho(x, \tilde{f}) = (t, \omega)\}$, which has full μ-measure, $\mu(A) = 1$, and is f-invariant, $f(A) = A$.

Let $\phi_{u,v} : T^2 \to \mathbb{R}$ be the following continuous function:

$$\phi_{u,v}(x) = (u, v) \cdot (\phi(x) - (t, \omega)),$$

where $\phi(x)$ is given by expression (2). We recall that (u, v) is the direction of the line r, which passes through (t, ω).

As $\rho(\mu) = (t, \omega)$, we get that

$$\int_{T^2} \phi_{u,v}(x)\, d\mu = 0.$$

So, Theorems 6 and 7 imply that the cocycle $a_{u,v}(n, x)$ and the skew-product $V_{u,v}(x, \alpha)$ associated to f and $\phi_{u,v}$ are, respectively, recurrent and conservative.

Let $\delta = \epsilon/8$ and choose any $x_0 \in A$. As x_0 is recurrent, there is an $n_0 > 0$ such that $f^{n_0}(x_0) \in B_\delta(x_0)$. This means that there is a pair $(l_0, m_0) \in \mathbb{Z}^2$, such that for all

$\tilde{x}_0 \in p^{-1}(x_0)$ we have

$$\|\tilde{f}^{n_0}(\tilde{x}_0) - \tilde{x}_0 - (l_0, m_0)\| < \delta.$$

If $(l_0/n_0, m_0/n_0) \notin \rho(\tilde{f})$, then we apply the C^0-closing lemma (see Lemma 1) in order to obtain a mapping $\tilde{g} \in D^0(\mathbb{R}^2)$, such that $g^{n_0}(x_0) = x_0$, $\rho(x_0, \tilde{g}) = (l_0/n_0, m_0/n_0) \notin \rho(\tilde{f})$ and $\|\tilde{f} - \tilde{g}\|_0 < 2\delta < \epsilon$ and the proof is complete. So we can suppose that $(l_0/n_0, m_0/n_0) \in \rho(\tilde{f})$. Therefore, it is left to consider the case $L(l_0/n_0, m_0/n_0) \leq 0$. Of course $(l_0/n_0, m_0/n_0) \neq (t, \omega)$, because $(t, \omega) \notin \mathbb{Q}^2$. The following technical lemma shows that we can in fact suppose that $L(l_0/n_0, m_0/n_0) < 0$.

LEMMA 3. *Given $\delta > 0$, it is not possible that for all $x \in A$ ($\tilde{x} \in p^{-1}(x)$) and $n > 0$ such that $\|\tilde{f}^n(\tilde{x}) - \tilde{x} - (l, m)\| < \delta$, for some pair $(l, m) \in \mathbb{Z}^2$, $(l/n, m/n) \in \rho(\tilde{f})$ and $L(l/n, m/n) = 0$.*

Proof. The proof of this lemma is driven by an idea analogous to that used in the remaining part of the proof of the main theorem. So we hope this proof will help the reader in understanding the rest of the proof of the main theorem. We just remark that if $\rho(\tilde{f})$ is a polygon or a strictly convex set, then through the extremal point (t, ω) of $\rho(\tilde{f})$ passes a line r which intersects $\rho(\tilde{f})$ only at (t, ω). This implies that if $(l/n, m/n) \in \rho(\tilde{f})$, as $(l/n, m/n) \neq (t, \omega)$, then $L(l/n, m/n) < 0$. So the proof is immediate in this case. The general situation goes as follows.

By contradiction, suppose that for all $x \in A$ ($\tilde{x} \in p^{-1}(x)$) and $n > 0$ such that $\|\tilde{f}^n(\tilde{x}) - \tilde{x} - (l, m)\| < \delta$, for some pair $(l, m) \in \mathbb{Z}^2$, we have $(l/n, m/n) \in \rho(\tilde{f})$ and $L(l/n, m/n) = 0$. As (t, ω) is an extremal point of $\rho(\tilde{f})$, we know that it divides the line r into a closed half-line $r_1 \ni (t, \omega)$ and an open half-line r_2, with $r_2 \cap \rho(\tilde{f}) = \emptyset$; see Figure 1. If we choose some $x_0 \in A$, as x_0 is recurrent, there is an $n_0 > 0$ and a pair $(l_0, m_0) \in \mathbb{Z}^2$ such that for all $\tilde{x}_0 \in p^{-1}(x_0)$ we have

$$\|\tilde{f}^{n_0}(\tilde{x}_0) - \tilde{x}_0 - (l_0, m_0)\| < \frac{\delta}{2}. \tag{5}$$

Our hypothesis implies that $(l_0/n_0, m_0/n_0) \in \rho(\tilde{f}) \cap r_1$. As $(t, \omega) \notin \mathbb{Q}^2$, without loss of generality we can suppose that $\omega \notin \mathbb{Q}$. As r_1 is a closed half-line starting at (t, ω) and containing rational points, we have two possibilities:

(i) any $(a, b) \in r_1 \backslash \{(t, \omega)\}$ satisfies $\omega > b$;

(ii) any $(a, b) \in r_1 \backslash \{(t, \omega)\}$ satisfies $\omega < b$.

As the two possibilities above are analogous, we only analyze the first. As $(l_0/n_0, m_0/n_0) \in r_1$, we have that $\omega > m_0/n_0$ and let $C = n_0\omega - m_0 > 0$. As $f^{n_0}(x_0) \in B_{\delta/2}(x_0)$, there exists $0 < \overline{\delta}_1 < \delta/2$ such that for all $z \in B_{\overline{\delta}_1}(x_0)$ we have $f^{n_0}(z) \in B_{\delta/2}(x_0)$. Let $\delta_1 = \min\{\overline{\delta}_1/10, C/10\}$. As $x_0 \in A$ and $\mu(A) = 1$, we know that $\mu(A \cap B_{\delta_1}(x_0)) = \mu(B_{\delta_1}(x_0)) > 0$.

Now we define a function $\phi_2 : T^2 \to \mathbb{R}$ by

$$\phi_2(x) = p_2 \circ \phi(x) - \omega,$$

where $\phi(x)$ is given by expression (2). As $\rho(\mu) = (t, \omega)$ we get

$$\int_{T^2} \phi_2(x)\, d\mu = 0.$$

So, Theorems 6 and 7 imply that the cocycle $a_2(n, x)$ and the skew-product $V_2(x, \alpha)$ associated to f and ϕ_2 are, respectively, recurrent and conservative.

As $\mu(A \cap B_{\delta_1}(x_0)) = \mu(B_{\delta_1}(x_0)) > 0$, by Lemma 2 we know that there exists $z \in A \cap B_{\delta_1}(x_0)$ such that $f^{n_1}(z) \in B_{\delta_1}(x_0)$ and $|a_2(n_1, z)| < \delta_1$, for some $n_1 > n_0$.

Now we make two observations (as always, $\tilde{z}$ is any point in $p^{-1}(z)$):

(1) as $\delta_1 < \overline{\delta}_1$, $f^{n_0}(z) \in B_{\delta/2}(x_0)$;

(2) there exists $(l_1, m_1) \in \mathbb{Z}^2$ such that

$$\|\tilde{f}^{n_1}(\tilde{z}) - \tilde{z} - (l_1, m_1)\| < 2\delta_1 \quad \text{and} \quad |a_2(n_1, z)| = |p_2 \circ \tilde{f}^{n_1}(\tilde{z}) - p_2(\tilde{z}) - n_1\omega| < \delta_1.$$

The second observation implies that

$$|n_1\omega - m_1| < 3\delta_1 < C = n_0\omega - m_0. \tag{6}$$

So ω satisfies the following inequality:

$$\omega < \frac{m_1 - m_0}{n_1 - n_0}. \tag{7}$$

From the observations above and expression (5), defining $w = f^{n_0}(z) \in B_{\delta/2}(x_0)$, we get that $f^{n_1 - n_0}(w) = f^{n_1}(z) \in B_{\delta_1}(x_0)$ and for any $\tilde{w} \in p^{-1}(w)$,

$$\|\tilde{f}^{n_1 - n_0}(\tilde{w}) - \tilde{w} - (l_1 - l_0, m_1 - m_0)\| < \frac{\delta}{2} + \delta_1 < \delta.$$

Finally, as $z \in A$ and $f(A) = A$ we get that $w \in A$. Our initial hypothesis implies that

$$\left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) \in \rho(\tilde{f}) \quad \text{and} \quad L\left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) = 0.$$

However, this is a contradiction, since from (7) we know that if

$$L\left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) = 0,$$

then

$$\left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) \in r_2 \Rightarrow \left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) \notin \rho(\tilde{f}).$$

So the lemma is proved. □

Thus, from the above lemma we can suppose that $L(l_0/n_0, m_0/n_0) < 0 \Rightarrow u(l_0/n_0 - t) + v(m_0/n_0 - \omega) < 0 \Rightarrow C = u(n_0 t - l_0) + v(n_0\omega - m_0) > 0$.

Let $0 < \overline{\delta}_1 < \delta$ be such that for all $z \in B_{\overline{\delta}_1}(x_0)$, $f^{n_0}(z) \in B_\delta(x_0)$ and let

$$\delta_1 = \min\left\{\frac{\overline{\delta}_1}{10}, \frac{C}{4(|u| + |v| + 1)}\right\}.$$

As $x_0 \in A$ we get that $\mu(B_{\delta_1}(x_0)) > 0$, which implies (by Lemma 2) that there exist $z \in B_{\delta_1}(x_0)$ and $n_1 > n_0$ such that $f^{n_1}(z) \in B_{\delta_1}(x_0)$ and $|a_{u,v}(n_1, z)| < \delta_1$. This means that there exist $(l_1, m_1) \in \mathbb{Z}^2$ such that for any $\tilde{z} \in p^{-1}(z)$ we have

$$\|\tilde{f}^{n_1}(\tilde{z}) - \tilde{z} - (l_1, m_1)\| < 2\delta_1 < \delta \tag{8}$$

$$
\begin{aligned}
|a_{u,v}(n_1, z)| &= \left| \sum_{i=0}^{n_1-1} \phi_{u,v} \circ f^i(z) \right| \\
&= |u(p_1 \circ \tilde{f}^{n_1}(\tilde{z}) - p_1(\tilde{z}) - n_1 t) + v(p_2 \circ \tilde{f}^{n_1}(\tilde{z}) - p_2(\tilde{z}) - n_1\omega)| < \delta_1.
\end{aligned} \tag{9}
$$

Expressions (8) and (9) imply that

$$
|u(n_1 t - l_1) + v(n_1\omega - m_1)| < (2|u| + 2|v| + 1)\delta_1 < C = u(n_0 t - l_0) + v(n_0\omega - m_0). \tag{10}
$$

So $u(n_1 t - l_1) + v(n_1\omega - m_1) < u(n_0 t - l_0) + v(n_0\omega - m_0)$, which gives

$$
u\left(\frac{l_1 - l_0}{n_1 - n_0} - t\right) + v\left(\frac{m_1 - m_0}{n_1 - n_0} - \omega\right) > 0.
$$

The last inequality implies that

$$
\left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) \notin \rho(\tilde{f}).
$$

From the choice of $\delta_1 > 0$, defining $w = f^{n_0}(z) \in B_\delta(x_0)$ we get that ($\tilde{w} \in p^{-1}(w)$)

$$
\|\tilde{f}^{n_1 - n_0}(\tilde{w}) - \tilde{w} - (l_1 - l_0, m_1 - m_0)\| < \delta + \delta_1 < 2\delta.
$$

Thus the C^0-closing Lemma 1 implies that we can obtain a mapping $\tilde{g} \in D^0(\mathbb{R}^2)$, such that

$$
g^{n_1 - n_0}(w) = w, \quad \rho(w, \tilde{g}) = \left(\frac{l_1 - l_0}{n_1 - n_0}, \frac{m_1 - m_0}{n_1 - n_0}\right) \notin \rho(\tilde{f})
$$

and $\|\tilde{f} - \tilde{g}\|_0 < 4\delta < \epsilon$, which completes the proof. □

4. *Consequences of the main result and some questions*

A simple corollary of our main theorem and Theorems 3 and 4 is the following.

COROLLARY 1. *Let $\tilde{f} \in D^0(\mathbb{R}^2)$ be a C^r diffeomorphism ($r \geq 0$) such that $\rho(\tilde{f})$ has an extremal point $(t, \omega) \notin \mathbb{Q}^2$ and* $\operatorname{int}(\rho(\tilde{f})) \neq \emptyset$. *Then, given $\epsilon > 0$, there exists an isotopy $\tilde{g}_t \in D^0(\mathbb{R}^2)$, which is also a C^r diffeomorphism for all $t \in [0, 1]$, $g_0 = f$, $g_1 = g$, with $\|\tilde{g}_t - \tilde{f}\|_0 < \epsilon$ for all $t \in [0, 1]$ and $\rho(\tilde{g}) \cap \rho(\tilde{f})^c \neq \emptyset$. Moreover, as t goes from zero to one, infinitely many periodic points are created for the mappings of the torus.*

Proof. In order to see that infinitely many periodic points are created as t goes from zero to one, we just have to observe that as $\operatorname{int}(\rho(\tilde{f})) \neq \emptyset$, Theorem 4 implies that $\operatorname{int}(\rho(\tilde{g}_t)) \neq \emptyset$ for all $t \in [0, 1]$, if $\epsilon > 0$ is sufficiently small. As $\rho(\tilde{g}) \cap \rho(\tilde{f})^c \neq \emptyset$, $\rho(\tilde{g})$ has infinitely many rational interior points that do not belong to $\rho(\tilde{f})$. Theorem 3 implies that for each of these points, there is a periodic orbit for g with that rotation vector, something that does not hold for f. The rest is easy. □

There are many open questions about this subject, but one of the most natural is the following.

Question 1. Can we prove Theorem 1 in the C^1 topology?

Also, we would like to know what happens when all the extremal points of the rotation set have rational coordinates.

Question 2. Is there an $\tilde{f} \in D^0(\mathbb{R}^2)$ with a rational polygon Q as a rotation set and sequences $\tilde{f}_n^i, \tilde{f}_n^o \in D^0(\mathbb{R}^2)$ such that $\tilde{f}_n^i, \tilde{f}_n^o \overset{n\to\infty}{\to} \tilde{f}$ in the C^0 topology and $\rho(\tilde{f}_n^i)$ is strictly contained in Q and $\rho(\tilde{f}_n^o)$ strictly contains Q?

The above question is motivated by some simple properties of circle homeomorphisms with rational rotation number.

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