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Mather's regions of instability for annulus diffeomorphisms

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Abstract

Let f be a $C^{1+\varepsilon}$ diffeomorphism of the closed annulus A that preserves orientation and the boundary components, and $\widetilde{f}$ be a lift of f to its universal covering space. Assume that A is a Birkhoff region of instability for f, and the rotation set of $\widetilde{f}$ is a nondegenerate interval. Then there exists an open f-invariant essential annulus A^* whose frontier intersects both boundary components of A, and points z^+ and z^- in A^*, such that the positive (resp., negative) orbit of z^+ converges to a set contained in the upper (resp., lower) boundary component of A^* and the positive (resp., negative) orbit of z^- converges to a set contained in the lower (resp., upper) boundary component of A^*. This extends a celebrated result originally proved by Mather in the context of area-preserving twist diffeomorphisms.

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1 | INTRODUCTION AND STATEMENTS OF THE RESULTS

When studying the dynamics of C^1-area-preserving twist diffeomorphisms of the closed annulus $A = S^1 \times [0,1]$, a celebrated theorem due to Mather [17] states that if f is such a diffeomorphism and A contains no essential f-invariant continuum apart from each boundary component of A, except for continua containing both boundary components, (in other words, A is minimal with respect to the inclusion: its interior contains no proper essential f-invariant open annulus), then there are points z^+, z^- in A such that the α-limit set of z^+ is contained in $S^1 \times \{0\}$, the ω-limit set of z^+ is contained in $S^1 \times \{1\}$, the α-limit set of z^- is contained in $S^1 \times \{1\}$ and the ω-limit set of

z^- is contained in $S^1 \times \{0\}$. Mather's theorem was proved using an intricate variational argument, which gave a lot of insight about what happens in the C^r-generic situation, for all $r \geqslant 1$. Later on, Le Calvez developed a completely topological proof [13], extending Mather's result to twist area-preserving homeomorphisms.

In this paper, our main objective is to prove a version of this result in the $C^{1+\varepsilon}$ world, without the twist and area-preservation hypotheses and under the weaker condition of the annulus being a Birkhoff region of instability, a situation that was widely considered for twist homeomorphisms, see, for instance, [5, 13, 17] and [8]. Namely, we consider $C^{1+\varepsilon}$ diffeomorphisms $f : A \to A$ that preserve orientation and the boundary components, whose rotation sets are nondegenerate intervals and as explained above, we assume that A is a Birkhoff region of instability for f (see below for the precise definition). The nondegeneracy condition on the rotation set is always satisfied by twist maps.

One last remark is that, in the area-preserving world, if A is minimal as explained above, then it is also a Birkhoff region of instability (this was originally proved by Birkhoff for twist maps in [5]), but the two definitions are not equivalent.

To state our main theorems properly, below we present some definitions.

Definitions.

(1) Let $\mathrm{Diff}_0^r(A)$ be the subset of C^r (for any $r \geqslant 0$) diffeomorphisms $f : A \to A$ (when $r = 0$, f is just a homeomorphism) that preserve orientation and the boundary components of $A = S^1 \times [0, 1]$. A lift of f to the universal cover of the annulus $\widetilde{A} = \mathbb{R} \times [0, 1]$, is denoted $\widetilde{f}$, a homeomorphism that satisfies $\widetilde{f}(\widetilde{z} + (1, 0)) = \widetilde{f}(\widetilde{z}) + (1, 0)$ for all $\widetilde{z} \in \widetilde{A}$.
(2) When $r = 1 + \varepsilon$ for some $0 < \varepsilon < 1$ in the above definition, we mean that Df is ε-Holder.
(3) The annulus A is said to be a *Birkhoff region of instability* for some $f \in \mathrm{Diff}_0^0(A)$ if, for all $\varepsilon > 0$, there exist integers $N, M > 0$ such that $f^N(S^1 \times]0, \varepsilon[)$ intersects $S^1 \times]1 - \varepsilon, 1[$ and $f^{-M}(S^1 \times]0, \varepsilon[)$ also intersects $S^1 \times]1 - \varepsilon, 1[$. We say that A is a *Mather region of instability* for f if there are points z^+, z^- in A, such that the α-limit set of z^+ is contained in the lower boundary component of A and its ω-limit set is contained in the upper boundary component of A. Similarly, the α-limit set of z^- is contained in the upper boundary component of A and its ω-limit set is contained in the lower boundary component of A.
(4) Let $p_1 : \widetilde{A} \to \mathbb{R}$ be the projection on the horizontal coordinate and as usual, let $p : \widetilde{A} \to A$ be the covering mapping. Fixed $f \in \mathrm{Diff}_0^0(A)$ and a lift $\widetilde{f}$, the displacement function $\phi : A \to \mathbb{R}$ is defined as

$$\phi(z) = p_1 \circ \widetilde{f}(\widetilde{z}) - p_1(\widetilde{z}),$$

for any $\widetilde{z} \in p^{-1}(z)$.
(5) Given any $f \in \mathrm{Diff}_0^0(A)$ and fixed some lift $\widetilde{f}$, a point $z \in A$ is said to have *rotation number* ρ_0 if the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(z))$ exists and is equal to ρ_0. We say that ρ_0 is *realized by the compact set* $K \subset A$ if K is f-invariant, and all points in K have rotation number ρ_0.
(6) Given any $f \in \mathrm{Diff}_0^0(A)$ and fixed some lift $\widetilde{f}$, the rotation set of $\widetilde{f}$ is defined as

$$\rho(\widetilde{f}) = \Big\{ \omega \in \mathbb{R} : \text{there exists a Borel probability } f\text{-invariant}$$

$$\text{measure } \mu \text{ such that } \omega = \int_A \phi(z) d\mu \Big\}.$$

Clearly, from the convexity of the subset of Borel probability f-invariant measures, $\rho(\widetilde{f})$ is a closed interval, maybe a single point. Moreover, its extremes are realized by ergodic measures, see [18]. In this generality, not much more can be said. There are well-known examples with a nondegenerate interval as a rotation set, for which only the extremes are the rotation numbers of some orbits.

(7) Given $f \in \mathrm{Diff}_0^0(\mathrm{A})$, we say that it satisfies the curve intersection property, if for any homotopically nontrivial simple closed curve $\gamma \subset A$, we have $f(\gamma) \cap \gamma \neq \emptyset$. It is not hard to see that if some $f \in \mathrm{Diff}_0^0(\mathrm{A})$ satisfies the curve intersection property, then f^n also satisfies it, for all integers $n \neq 0$. Also, it is immediate that, in case A is a Birkhoff region of instability for some $f \in \mathrm{Diff}_0^0(\mathrm{A})$, then f satisfies the curve intersection property.

(8) Let $f \in \mathrm{Diff}_0^0(\mathrm{A})$ and γ contained in the interior of A be a homotopically nontrivial simple closed curve. Denote by γ^- the connected component of γ^c that contains $S^1 \times \{0\}$ and analogously, let γ^+ be the connected component of γ^c that contains $S^1 \times \{1\}$. Denote by $\xi^-(\gamma)$ the connected component of the maximal invariant set in the closure of γ^- that contains $S^1 \times \{0\}$ and by $\xi^+(\gamma)$ the connected component of the maximal invariant set in the closure of γ^+ that contains $S^1 \times \{1\}$. Finally, for all integers $n > 1$, denote by $\xi^-_{1/n} = \xi^-(S^1 \times \{1 - 1/n\})$ and $\xi^+_{1/n} = \xi^+(S^1 \times \{1/n\})$.

Note that, in case A is a Birkhoff region of instability,

$$\partial(\xi^-(\gamma))^c \cap S^1 \times \{0\} \neq \emptyset \text{ and } \partial(\xi^+(\gamma))^c \cap S^1 \times \{1\} \neq \emptyset.$$

We are ready to state our main theorem.

Theorem 1. *Let $f \in \mathrm{Diff}_0^{1+\varepsilon}(\mathrm{A})$ for some $\varepsilon > 0$ be such that A is a Birkhoff region of instability and for some fixed lift $\widetilde{f}$, $\rho(\widetilde{f})$ has interior. Then there exists a homotopically nontrivial simple closed curve $\gamma \subset A$ and an f-invariant minimal open annulus $A^* \subset A$ containing γ such that ∂A^* intersects both $S^1 \times \{0\}$ and $S^1 \times \{1\}$ and A^* is a Mather region of instability.*

Remark. Clearly, if A is itself minimal as in the hypothesis of Mather's original theorem, then $A = A^*$ and so, the whole annulus is a Mather region of instability.

Note that, in [6, Proposition 2.19], a result in the same direction was obtained. There, f was an area-preserving homeomorphism, instead of a $C^{1+\varepsilon}$ diffeomorphism, and the thesis obtained was that A^* is a mixed SN region of instability, a condition weaker than being a Mather region of instability.

Let us comment on the hypothesis and thesis of Theorem 1. First, we point out that the result does not hold in case the rotation set of $\widetilde{f}$ is a single point. There are known examples (see, for instance, [4]) of smooth maps $f : A \to A$ having a single rotation number that are both weak-mixing (therefore A is a Birkhoff region of instability), but also rigid (meaning that there is a sequence of positive iterates of f that converges to the identity). So, no subannulus of f can be a Mather region of instability.

A natural question is also to understand, under our hypotheses, if A itself is always a Mather region of instability. But this is false, and we sketch an example of a C^∞ area-preserving diffeomorphism. Take $f : A \to A$, which extends to a smooth area-preserving diffeomorphism $g : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}$ such that the restriction of f to the upper boundary has a single degenerate

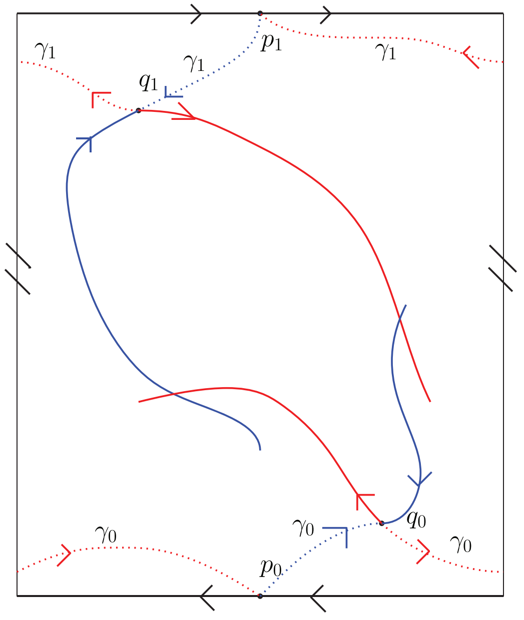

FIGURE 1 Sketch of the smooth diffeomorphism for which A (obtained by gluing the two lateral sides together) is not a Mather region of instability.

topological saddle fixed point p_1, that is, p_1 is a fixed point such that the differential of g at p_1 is the identity, and such that there exists a local topological conjugation between the dynamics of g at p_1 and a linear hyperbolic saddle at the origin. Likewise, we assume that f has a single degenerate topological saddle fixed point p_0 in the lower boundary, so that each boundary of A consists of a single fixed point and a saddle connection. Furthermore, we assume that f has two hyperbolic saddle points, q_0 and q_1, and that there exists saddle connections between a branch of the unstable manifold of q_0 and a branch of the stable manifold of p_0, as well as a connection between a branch of the stable manifold of q_0 and a branch of the unstable manifold of p_0, so that there exists an essential invariant closed curve γ_0 that intersects the lower boundary just at the point p_0. One can do a similar picture, with two saddle connections between p_1 and q_1, and an invariant essential closed curve γ_1 made by these connections and the points p_1 and q_1, which intersects the upper boundary just at p_1. Finally, one can assume that there are transversal heteroclinic intersections between the "free" branches of the unstable manifold of q_0 and the stable manifold of q_1, and between the unstable manifold of q_1 and the stable manifold of q_0, see Figure 1. Finally, we may assume that, for a given lift $\widetilde{f}$ of f, p_0 and p_1 have different rotation numbers.

In this picture, from the λ-lemma one would get that the unstable manifold of q_1 accumulates on p_0 and as the unstable manifold of q_1 is contained in the closure of the future orbit of any neighborhood of p_1, one gets that there exists points arbitrarily close to p_1 whose future orbit lie arbitrarily close to p_0. A similar argument shows that there exists points arbitrarily close to p_0 whose future orbit lie arbitrarily close to p_1, and so A is a Birkhoff region of instability. But we claim A cannot be a Mather region of instability for f. Indeed, if the ω-limit of a point z in the included in $S^1 \times \{1\}$, then either z lies in a stable branch of p_1, in which case the α-limit of z is either $\{p_1\}$ or $\{q_1\}$, or z must lie above the graph determined by $\gamma_1 \cup \{p_1\}$. As the closure of the later region is invariant and disjoint from $S^1 \times \{0\}$, this implies that the α-limit of z is disjoint from $S^1 \times \{0\}$. This shows that there does not exist a point whose α-limit lies in $S^1 \times \{0\}$ and whose ω-limit lies in $S^1 \times \{1\}$.

The next lemma is a crucial step in the proof of Theorem 1. It also explains how the annulus A^* is constructed.

Lemma 1. *Under the hypotheses of Theorem 1, there exists a homotopically nontrivial closed curve γ contained in the interior of A such that $\xi^-(\beta) \cap \gamma = \emptyset$ and $\xi^+(\beta) \cap \gamma = \emptyset$ for any homotopically nontrivial simple closed curve $\beta \subset A$. So, there exist f-invariant continua $K^- \supset S^1 \times \{0\}$ and $K^+ \supset S^1 \times \{1\}$ such that $K^- \cap K^+ = \emptyset$ and for all sufficiently large integers $n > 1$, $\xi^-_{1/n} = K^-$ and $\xi^+_{1/n} = K^+$. Moreover, if A^* is the f-invariant open annulus between K^- and K^+, then ∂A^* intersects both $S^1 \times \{0\}$ and $S^1 \times \{1\}$.*

When A is minimal, in the sense that its interior contains no f-invariant proper essential open sub-annulus, then $\xi^-_{1/n} = S^1 \times \{0\}$ and $\xi^+_{1/n} = S^1 \times \{1\}$, for all integers $n > 1$, but when A is just a Birkhoff region of instability, we cannot avoid considering the sets K^- and K^+ from Lemma 1.

We also provide the following result, which is fundamental in the proof of Theorem 1, but whose interest stands alone for its possible applications:

Theorem 2. *Again, under the hypotheses of Theorem 1, there exists E, an open and dense subset of $\rho(\widetilde{f})$, such that for any rational number $p/q \in E$, there exists a hyperbolic periodic saddle point z contained in the interior of A whose rotation number is p/q and a homotopically nontrivial closed curve $\gamma_{p/q} \ni z$, contained in the union of the stable and unstable manifolds of z.*

Finally, we remark that part of the interest in regions of instability comes from the fact that the dynamics in these regions is usually very rich. For instance, a classical result says that, if $f \in \mathrm{Diff}^0_0(\mathrm{A})$ has the curve intersection property (a condition that is satisfied when A is a Birkhoff region of instability), and if $\rho(\widetilde{f}) = [a, b]$, then for any rational number $a < p/q < b$, $\widetilde{f}^q - (p, 0)$ has a fixed point. In other words, all rationals in the interior of the rotation set are realized by periodic orbits. To prove this statement, note that in [18] it was proved that a and b are realized by ergodic Borel f-invariant probability measures, that is, a and b are equal to the rotation numbers of actual points in A. So, if for some $a < p/q < b$, $\widetilde{f}^q - (p, 0)$ does not have a fixed point, then the version of Brouwer's translation theorem applied to the annulus that appears in a 1928–1929 paper of Kerekjarto [11] implies the existence of a homotopically nontrivial simple closed curve disjoint from its image under f, a contradiction with the curve intersection property.

Moreover, if the twist condition is present, even for any irrational number ρ_0 in the rotation set, one can find an f-invariant compact set K_{ρ_0} that realizes ρ_0, such that the restriction of f to K_{ρ_0} is semiconjugated to the irrational rotation of the circle with the same rotation number. The fact that the full rotation set is realized by compact f-invariant sets was also proved in the absence of the twist condition, for area preserving homeomorphisms [6, 14].

Our final result, a direct consequence of Theorems 1, 2 and [6, Theorem C], says that:

Theorem 3. *Let $f \in \mathrm{Diff}^{1+\varepsilon}_0(\mathrm{A})$ for some $\varepsilon > 0$ be such that A is a Birkhoff region of instability and let $\widetilde{f}$ be a lift of f to its universal covering space. Then there are at most two numbers in $\rho(\widetilde{f})$ that are not realized by compact f-invariant sets.*

The paper is organized as follows. In the next section, we present all the necessary preliminary results, with a brief overview and remainder of the tools needed in this work. Section 3 is dedicated to the proofs of our main results.

2 | PRELIMINARIES

In this section, we describe some theories we use and quote some results.

2.1 | Prime ends compactification of open disks

If D is an open topological disk of an oriented surface S potentially with boundary, such that ∂D is a Jordan curve and f is an orientation preserving homeomorphism of that surface which satisfies $f(D) = D$, then $f : \partial D \to \partial D$ is conjugate to a homeomorphism of the circle, and so a real number $\rho(D)$, the rotation number of $f\mid_{\partial D}$ can be associated to this problem. By the classical properties of rotation numbers, if $\rho(D)$ is rational, then there exists a periodic point in ∂D and if it is not, then there are no such points. This is known since Poincaré. The difficulties arise when we do not assume ∂D to be a Jordan curve.

The prime ends compactification is a way to attach to D a circle called the circle of prime ends of D, obtaining a space $D \sqcup S^1$ with a topology that makes it homeomorphic to the closed unit disk. If, as above, we assume the existence of an orientation preserving homeomorphism f of S such that $f(D) = D$, then $f\mid_D$ extends to $D \sqcup S^1$. The prime ends rotation number of f in D, still denoted $\rho(D)$, is the usual rotation number of the orientation preserving homeomorphism induced on S^1 by the extension of $f\mid_D$. But things may be quite different in this setting. In full generality, it is not true that when $\rho(D)$ is rational, there are periodic points in ∂D and for some examples, $\rho(D)$ is irrational and ∂D is not periodic point free. Anyway, the only result on this subject we need is the following classical lemma (as usual, a point $z \in \partial D$ is said to be accessible if there exists a simple arc $\gamma : [0,1] \to D \cup \partial D$ such that $\gamma([0,1[) \subset D$ and $\gamma(1) = z$) whose proof, for instance, can be found in [9, Theorem 16].

Lemma 2. *Let $f \in \mathrm{Diff}_0^0(\mathrm{A})$ and let $D \subset A$ be an f-invariant open annulus given by the complement of some f-invariant continuum K that contains $S^1 \times \{0\}$ and avoids $S^1 \times \{1\}$. Then the boundary of D has two connected components, one is $S^1 \times \{1\}$ and the other one is some continuum $M \subseteq K$. Moreover, if $z_1, z_2 \in M$ are periodic points, both accessible from D, then they have the same rotation number.*

Remark. As D is not a disk, by prime ends rotation number of D (denoted $\rho(D)$), we mean the following: Contract $S^1 \times \{1\}$ to a point N in order to turn A into a closed disk. Clearly, $f : A \to A$ induces a homeomorphism of this closed disk that fixes N. Now, D becomes an open topological disk and $\rho(D)$ is the prime ends rotation number of this disk.

Some interesting facts related to the previous result, but not necessary for us are the following: $\rho(D)$ is equal to the rotation number (in A) of any accessible periodic point in M. More generally, the main result of [9] says that for any accessible point in $M = \partial D$, either its forward or backward annulus rotation number is equal to $\rho(D)$.

For more information on the theory of prime ends, see, for instance, [12, 17] and [9].

2.2 | Topologically transverse intersections

Let S be a compact orientable surface. We say that a closed topological disk $R \subset S$ is a topological rectangle if its boundary, which is a Jordan curve, is given by the union of four C^1 oriented arcs, $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ such that: the end point of γ_1 is the first point of γ_2, the end point of γ_2 is the first point of γ_3, the end point of γ_3 is the first point of γ_4 and the end point of γ_4 is the first point of γ_1. We also assume that for $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, the intersection between γ_i and γ_j is either empty or C^1-transversal.

Suppose f is a diffeomorphism of S and assume it has a hyperbolic n -periodic saddle point p contained in the interior of S.

Definition of topological transversality: We say that some continuum $K \subset S$ has a topologically transverse intersection with a branch α at p (stable or unstable), if there exists a topological rectangle $R \subset S$, whose boundary is given by arcs $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ as explained above, and there exists an arc $\alpha' \subset \alpha$, whose interior is contained in the interior of R and its extreme points belong, one to the interior of γ_1 and the other one to the interior of γ_3 (the intersections between γ_1 and α' and γ_3 and α' are both C^1-transverse), such that K contains a subcontinuum $K^* \subset R$ that intersects both γ_2 and γ_4, and avoids γ_1 and γ_3.

Such intersections are important because of the following result:

Proposition 1. *In the above setting, if K is a continuum that has a topologically transverse intersection with a stable branch α at p (the unstable case is analogous), then given any $\mu > 0$ and a compact arc $\theta \subset W^u(p)$, there exists $M > 0$ such that for all $m \geqslant M$, $f^{m.2n}(K)$ contains a continuum θ_m that is μ-close to θ for the Hausdorff distance.*

Proof. By considering f^{2n} instead of f, we can assume that p is fixed and all branches at p are f-invariant. From the Hartman–Grobman theorem, there exists V, an open neighborhood of p, $W \subset \mathbb{R}^2$, an open neighborhood of the origin and a homeomorphism $\varphi : V \to W$ that conjugates $f\mid_V$ to the linear model $H(x, y) = (x/2, 2y)$ restricted to W. Suppose, without loss of generality, that, locally, $\varphi(\alpha)$ corresponds to the positive x-axis in W.

Consider the rectangle $R_{c,d} = [-c, c] \times [-d, d]$ for $c, d > 0$ such that $H(R_{c,d}) \cup R_{c,d} \subset W$. As K is topologically transverse to α, there exists a topological rectangle R, such that $K^* \subset R$ is a subcontinuum of K that intersects both γ_2 and γ_4, and avoids γ_1 and γ_3 (see the definition of topological transversality above). It is clearly possible to modify R by choosing γ_2^* and γ_4^* much closer (in a C^1-way) to α' (the connected arc contained in α whose interior is contained in R and whose extreme points are contained, one in the interior of γ_1 and the other, in the interior of γ_3), and choosing γ_1^* and γ_3^* subarcs of γ_1 and γ_3, respectively, in a way that $\gamma_1^*, \gamma_2^*, \gamma_3^*, \gamma_4^*$ form the boundary of a new rectangle R^*, such that for some integer $N \geqslant 0$, $f^N(R^*) \subset V$ and the corresponding rectangle $R' = \varphi(f^N(R^*))$ belongs to $]0, c[\times] - d, d[\subset R_{c,d}$. Furthermore, there exists some subcontinuum K^{**} of K^* that is disjoint from γ_1^* and γ_3^*, intersects both γ_2^* and γ_4^* and is contained in R^*. Related to K^{**}, let us choose real numbers $a, b, \delta > 0$ such that $0 < a < b < c, 0 < \delta < d, R' \subset]a, b[\times] - d, d[$ and $[-c, c] \times [-\delta, \delta] \cap \varphi(f^N(\gamma_2^* \cup \gamma_4^*)) = \emptyset$. Clearly, both $[a, b] \times \{\delta\}$ and $[a, b] \times \{-\delta\}$ intersect R'.

Let $\Gamma \subset [a, b] \times [-\delta, \delta]$ be a connected component of

$$[a, b] \times [-\delta, \delta] \cap \varphi(f^N(K^{**}))$$

which intersects both $[a, b] \times \{-\delta\}$ and $[a, b] \times \{\delta\}$. Clearly, $\Gamma \cap \{a, b\} \times [-\delta, \delta] = \emptyset$.

Claim. For any $\varepsilon > 0$, there exists $M > 0$ such that for all integers $m \geqslant M$, there is a subcontinuum $\Gamma' \subset \Gamma$, depending on ε and m, such that $H^m(\Gamma') \subset [0, \varepsilon[\times[-d, d]$ and $H^m(\Gamma')$ intersects both $[0, \varepsilon[\times\{-d\}$ and $[0, \varepsilon[\times\{d\}$.

Let us, before proving the claim, show that it implies the proposition. Indeed, if it holds, then as $\varepsilon \to 0$, $\varphi^{-1}(H^m(\Gamma')) \subset f^{m+N}(K^{**})$ converges in the Hausdorff topology to $\varphi^{-1}(\{0\} \times [-d, d])$, a local unstable manifold at p.

For θ as in the statement of the proposition, there exists some integer $J > 0$ such that $\theta \subset f^J(\varphi^{-1}(\{0\} \times [-d, d]))$. As $\varepsilon \to 0$, $f^J(\varphi^{-1}(H^m(\Gamma'))) \subset f^{J+m+N}(K^{**})$ converges in the Hausdorff topology to $f^J(\varphi^{-1}(\{0\} \times [-d, d]))$, which contains θ. So, there exists a subcontinuum θ_m of $f^{J+m+N}(K^{**})$ that converges to θ in the Hausdorff topology as $\varepsilon \to 0$.

Therefore, to conclude the proof of Proposition 1, we have to show that the above claim holds. For this, given $\varepsilon > 0$, let $M > 0$ be an integer such that:

- $b/2^M < \varepsilon$;
- $d/2^M < \delta$,

Clearly, $M \to \infty$ as $\varepsilon \to 0$. Moreover, for all $m \geqslant M$, let Γ' be a connected component of $\Gamma \cap [a, b] \times [-d/2^m, d/2^m]$ that intersects both $[a, b] \times \{-d/2^m\}$ and $[a, b] \times \{d/2^m\}$. From the choice of $M > 0$, Γ' is not empty and it clearly satisfies $H^m(\Gamma') \subset [0, \varepsilon[\times[-d, d]$ and $H^m(\Gamma')$ intersects both $[0, \varepsilon[\times\{-d\}$ and $[0, \varepsilon[\times\{d\}$.

This proves the claim and concludes the proof of the proposition. □

Still in the above setting, if θ, the compact subarc of a branch, has a topologically transverse intersection with some other continuum T, then $f^{m.2n}(K)$ also intersects T provided $m > 0$ is large enough. For more about topologically transverse intersections, see, for instance, [2] and [3].

The next result will be used in the proof of Theorem 1.

Lemma 3. *In the above setting, assume K is a continuum that does not have a topologically transverse intersection with the stable branch α at a saddle p. Then, for every $\varepsilon > 0$ and any θ, subarc of α such that K is disjoint from the endpoints of θ, there exists a simple arc θ_ε that is contained in the ε-neighborhood of θ, has the same endpoints as θ, and is disjoint from K.*

Proof. Fix $\varepsilon > 0$ and θ as in the statement of the lemma. Let $f_\theta(t)$, $t \in [0, 1]$ be a parameterization of θ. One can find an ε-neighborhood V of θ, a neighborhood W of $[0, 1] \times \{0\}$ and a C^1 diffeomorphism $\phi : V \to W$ such that $\phi(f_\theta(t)) = (t, 0)$. Also, if $\delta > 0$ is sufficiently small, then $R = [0, 1] \times [-\delta, \delta]$ is a subset of W and $\phi(K \cap V)$ is disjoint from $\{0, 1\} \times [-\delta, \delta]$ and not contained in R. Consider the subset of R, denoted $F := \phi(K \cap V) \cap R$. If it separates $\{0\} \times [-\delta, \delta]$ from $\{1\} \times [-\delta, \delta]$, then, as F is closed, there is a connected component of F that separates the former two sets in R. So, this component intersects both $[0, 1] \times \{-\delta\}$ and $[0, 1] \times \{\delta\}$, something that contradicts the assumption that K does not have a topologically transverse intersection with α.

In this way, there is a connected component B of $R \setminus F$ that contains $\{0\} \times [-\delta, \delta]$ and $\{1\} \times [-\delta, \delta]$. And it is open. So if we pick $\beta : [0, 1] \to B$, a simple arc joining $(0, 0)$ and $(1, 0)$, it suffices to take $\theta_\varepsilon = \phi^{-1}(\beta)$. □

2.3 | Some Pesin theory

In this subsection, assume that $f : A \to A$ is a $C^{1+\epsilon}$ diffeomorphism, for some $\epsilon > 0$. Recall that an f-invariant Borel probability measure μ is hyperbolic if all the Lyapunov exponents of f are nonzero at μ -almost every point (for instance, see the supplement of [10]). Remember that for μ-almost every $z \in A$, there are two Lyapunov exponents $\lambda_+(z) \geqslant \lambda_-(z)$ defined as follows:

$$\lambda_+(z) = \lim_{n\to\infty} \frac{1}{n} \log \|Df^n(z)\| \text{ and } \lambda_-(z) = -\lim_{n\to\infty} \frac{1}{n} \log \|Df^{-n}(z)\|$$

The next paragraphs were taken from [7]. They consist of an informal description of the theory of nonuniformly hyperbolic systems, together with some definitions and lemmas from [7].

Let μ be a nonatomic hyperbolic ergodic f-invariant Borel probability measure. Given $0 < \delta < 1$, there exists a compact set Λ_δ (called Pesin set) with $\mu(\Lambda_\delta) > 1 - \delta$, having the following properties: for every $p \in \Lambda_\delta$, there exists an open neighborhood U_p, a compact neighborhood $V_p \subset U_p$ and a diffeomorphism $F : (-1,1)^2 \to U_p$, with $F(0,0) = p$ and $F([-1/10, 1/10]^2) = V_p$, such that:

- The local unstable manifolds $W^u_{\text{loc}}(q)$, $q \in \Lambda_\delta \cap V_p$, given by the connected component of the set of $z \in U_p$ sucht that $\text{dist}(f^n(z), f^n(q)) \to 0$ as $n \to -\infty$ that contains q, are the images under F of graphs of the form $\{(x, F_2(x)) : x \in (-1,1)\}$, F_2 a function with k-Lipschitz constant, for some $0 < k < 1$. Any two such local unstable manifolds are either disjoint or equal and they depend continuously (in the Hausdorff topology) on the point $q \in \Lambda_\delta \cap V_p$.
- Similarly, local stable manifolds $W^s_{\text{loc}}(q)$, $q \in \Lambda_\delta \cap V_p$, given by the connected component of the set of $z \in U_p$ such that $\text{dist}(f^n(z), f^n(q)) \to 0$ as $n \to \infty$ that contains q, are the images under F of graphs of the form $\{(F_1(y), y) : y \in (-1,1)\}$, F_1 a function with k-Lipschitz constant, for some $0 < k < 1$. Any two such local stable manifolds are either disjoint or equal and they depend continuously (again in the Hausdorff topology) on the point $q \in \Lambda_\delta \cap V_p$.

These are the properties that characterize a Pesin set.

It follows that there exists a continuous product structure in $\Lambda_\delta \cap V_p$: given any $r, r' \in \Lambda_\delta \cap V_p$, the intersection $W^u_{loc}(r) \cap W^s_{loc}(r')$ is transversal and consists of exactly one point, which will be denoted $[r, r']$. This intersection varies continuously with the two points and may not be in Λ_δ. Hence, we can define maps $P^s_p : \Lambda_\delta \cap V_p \to W^s_{loc}(p)$ and $P^u_p : \Lambda_\delta \cap V_p \to W^u_{loc}(p)$ as $P^s_p(q) = [q, p]$ and $P^u_p(q) = [p, q]$.

Let $R^\pm$ denote the set of all points in A that are both forward and backward recurrent. By the Poincaré recurrence theorem, $\mu(R^\pm)$ is equal to 1.

Definition (**Accessible and inaccessible points**). A point $p \in \Lambda_\delta \cap V_p \cap R^\pm$ is inaccessible if it is accumulated on both sides of $W^s_{loc}(p)$ by points in $P^s_p(\Lambda_\delta \cap V_p \cap R^\pm)$ and also accumulated on both sides of $W^u_{loc}(p)$ by points in $P^u_p(\Lambda_\delta \cap V_p \cap R^\pm)$. Otherwise, p is accessible.

After this definition, we can state two lemmas from [7] about accessible and inaccessible points and the relation between these points and nearby hyperbolic periodic points.

Lemma 4. *Let $q \in \Lambda_\delta \cap V_p \cap R^\pm$ be an inaccessible point. Then there exist rectangles enclosing q, having sides along the invariant manifolds of two hyperbolic periodic saddle points in V_p (which are corners of the rectangles) and having arbitrarily small diameter.*

The boundary of such a rectangle is a Jordan curve made up of alternating segments of stable and unstable manifolds, two of each. The segments forming the boundary are its sides and the intersection points of the sides are the corners. As explained above, two of the corners are hyperbolic periodic saddle points and the other two corners are C^1-transverse heteroclinic intersections. A rectangle is said to enclose p if its interior, which is an open topological disk, contains p.

Lemma 5. *The subset of accessible points in $\Lambda_\delta \cap V_p \cap R^\pm$ has μ measure equal to zero.*

Another concept that will be a crucial hypothesis for us is positive topological entropy. In the following, we describe why.

When the topological entropy $h(f|_K)$ is positive, for some compact f-invariant set K, by the variational principle, there exists an f -invariant Borel probability measure μ_0 with supp(μ_0) (the topological support of μ_0) contained in K and positive metric entropy $h_{\mu_0}(f)$. Using the ergodic decomposition of μ_0 we find an extremal point μ of the set of Borel probability f-invariant measures, such that supp(μ) is also contained in K and $h_\mu(f) > 0$. As the extremal points of this set are ergodic measures, μ is ergodic. The ergodicity and the positiveness of the entropy imply that μ has no atoms and applying the Ruelle inequality (which, in our case, says that $\lambda_+(z) = \lambda_+(\mu) \geqslant h_\mu(f) > 0$ for μ-almost every $z \in A$), we get that μ has a positive Lyapunov exponent, see [10]. Working with f^{-1} and using the fact that $h_\mu(f^{-1}) = h_\mu(f) > 0$, we see that f^{-1} must also have a positive Lyapunov exponent with respect to μ, which is the opposite of the negative Lyapunov exponent for f.

Hence, when K is a compact f-invariant set and the topological entropy of $f|_K$ is positive, there always exists an ergodic, nonatomic, invariant measure supported in K, with nonzero Lyapunov exponents, one positive and one negative, the measure having positive entropy: That is, an hyperbolic measure.

The existence of this kind of measure will be important for us because of Lemmas 4 and 5.

2.4 | Some forcing results

This subsection is mostly based on the work of Le Calvez and the second author [15, 16] on forcing theory for surface homeomorphisms, which we restrain to explain in more detail as not to substantially increase the length of this paper. We refer the interested reader to the above works, as well as [6].

Given $f \in \mathrm{Diff}_0^0(\mathrm{A})$ and a lift $\widetilde{f}$, we say that f has a *rotational topological horseshoe* if there exists, for some power $g = f^r$ of f:

- a lift $\widetilde{g} = \widetilde{f}^r - (s, 0)$ to $\widetilde{A}$, where s is an integer,
- a compact g-invariant subset $\Lambda \subset A$,
- a compact subset $\widetilde{\Lambda} \subset \widetilde{A}$ such that $p(\widetilde{\Lambda}) = \Lambda$ and the restriction of p to $\widetilde{\Lambda}$ is a homeomorphism onto its image,
- an integer M_0, a compact metric space Y, a homeomorphism T of Y, and a surjective continuous map $\pi_1 : Y \to \Lambda$ semiconjugating T and g, such that for each $x \in \Lambda$, the cardinality of $\pi_1^{-1}(x)$ is not larger than M_0,
- a continuous surjective map $\pi_2 : Y \to \Sigma_2 := \{0, 1\}^{\mathbb{Z}}$, semiconjugating T to the shift map $\sigma : \Sigma_2 \to \Sigma_2$ defined so that $(\sigma(u))_j = (u)_{j+1}$, $u \in \Sigma_2$,
- also, if $y \in Y$, $x = \pi_1(y)$, $\tilde{x} = p^{-1}(x) \cap \widetilde{\Lambda}$, then $\widetilde{g}(\widetilde{x}) \in \widetilde{\Lambda}$ if $(\pi_2(y))_0 = 0$ and $\widetilde{g}(\widetilde{x}) \in \widetilde{\Lambda} + (1, 0)$ if $(\pi_2(y))_0 = 1$.

In general terms, if f has a rotational topological horseshoe, then modulo some finite extension and taking a power of the dynamics, we obtain a compact invariant set where the displacement of points in the lift can be estimated by a symbolic coding, and for which every possible coding with two symbols is admitted. We remark that it follows directly from the definition, that the rotation set of $\widetilde{g}$ contains the interval $[0,1]$, and so the rotation set of $\widetilde{f}$ contains the interval $[s/r,(s+1)/r]$. We also say that the $\widetilde{f}$-rotation set of the horseshoe contains the interval $[s/r,(s+1)/r]$. But we can obtain a little more, which will be useful:

Lemma 6. *Let $f \in \mathrm{Diff}^0_0(\mathrm{A})$ and let $\widetilde{f}$ be a lift of f. Let $\Lambda, \widetilde{\Lambda}, g, \widetilde{g}, Y, T, \pi_1$ and π_2 be as above. Then there exists a compact g-invariant set $\Lambda_0 \subset \Lambda$, such that the restriction of g to Λ_0 is transitive and such that, for every rational $0 < p/q < 1$, one can find a g^q-invariant compact set $K_{p/q} \subset \Lambda_0$ satisfying:*

(1) *the restriction of g^q to $K_{p/q}$ has strictly positive topological entropy;*
(2) *if $\widetilde{K}_{p/q} = \pi^{-1}(K_{p/q}) \cap \widetilde{\Lambda}$, then $\widetilde{g}^q(\widetilde{K}_{p/q}) = \widetilde{K}_{p/q} + (p,0)$.*

Proof. Let $u_0 \in \Sigma_2$ be a point whose future σ orbit is dense, and for notation sake denote $L_0 = \pi_2^{-1}(u_0)$. We claim that there exists some $y_0 \in L_0$ that is recurrent for T. Indeed, let us consider the set $\mathcal{T}$ of all closed and T-invariant subsets of Y that have nonempty intersection with L_0, which is naturally ordered by inclusion, that is, where for $F_1, F_2 \in \mathcal{T}$ we denote $F_1 \preceq F_2$ if $F_1 \subset F_2$. Note that $\mathcal{T}$ is not empty as Y belongs to it. Consider a chain $(F_i)_{i\in I}$ with each F_i in $\mathcal{T}$ and such that for all i, j in I, either $F_i \preceq F_j$ or $F_j \preceq F_i$. We claim that $F = \bigcap_{i\in I} F_i$ also belongs to $\mathcal{T}$. Indeed, F is compact as it is the intersection of compact sets. We will show that $F \cap L_0$ is nonempty. If not, by compactness of L_0, as the complements of F_i would form an open covering of L_0, there would be finitely many indices $\{i_1, i_2, \dots, i_n\}$ such that $L_0 \subset \bigcup_{j=1}^n Y \setminus F_{i_j}$. But as F is a chain and we chose finitely many indices, there must exists some smallest element $F_k \in \{F_{i_1}, \dots, F_{i_n}\}$. Note that $F_k \cap L_0$ is not empty and F_k is disjoint from $\bigcup_{j=1}^n Y \setminus F_{i_j}$, a contradiction. Therefore, F is nonempty and belongs to $\mathcal{T}$. One verifies trivially that $F \preceq F_i$ for all $i \in I$ and so we can apply Zorn's lemma to obtain that $\mathcal{T}$ has an element $\overline{F}$ that is minimal for inclusion. Let y_0 be a point in $\overline{F} \cap L_0$. Note that, if $\omega(y_0)$ is the ω-limit of y_0 by T, then it is also a compact invariant subset of Y, and as u_0 is recurrent, $\omega(y_0)$ must also intersect L_0. One deduces that $\omega(y_0)$ is also an element of $\mathcal{T}$ which is contained in $\overline{F}$, and thus is equal to $\overline{F}$ by minimality. But this implies $y_0 \in \omega(y_0)$ and the claim is proved.

Let then Y' be the closure of the forward orbit of y_0 by T, which is a compact T-invariant set such that the restriction of T to Y' is both transitive and an extension of the shift σ. Let $\Lambda_0 = \pi_1(Y')$, a compact g-invariant set for which the restriction of g is also transitive. Given p/q as in the statement, consider

$$A_{p/q} = \left\{ u \in \sigma_2;\ \text{for all } j \in \mathbb{Z}, \left(\sum_{i=jq}^{j(q+1)-1} u_i \right) = p \right\}$$

which is invariant by σ^q, and let $Y_{p/q} = \pi_2^{-1}(A_{p/q}) \cap Y'$, and $K_{p/q} = \pi_1(Y_{p/q})$. Note that the restriction of σ^q to $A_{p/q}$ has strictly positive topological entropy (as it is conjugated to the full shift on $\binom{q}{p}$ symbols). This implies that the restriction of T^q to $Y_{p/q}$ has positive topological entropy and, as the cardinality of the fibers of π_1 is uniformly bounded, the same holds for the restriction of g^q to $K_{p/q}$. The second assertion from the lemma follows directly from noticing that, if $y \in Y', x = \pi_1(y)$ and $\widetilde{x} \in p^{-1}(x) \cap \widetilde{\Lambda}$, then $\widetilde{g}^k(\widetilde{x})$ lies in $\widetilde{\Lambda} + (\sum_{i=0}^{k-1} (\pi_2(y))_i, 0)$, concluding the proof. □

The following result is basically contained in [6, subsection 6.1.2]:

Proposition 2. *Let A be a Birkhoff region of instability for some $f \in \mathrm{Diff}_0^0(\mathrm{A})$ with a lift $\widetilde{f}$ whose rotation set is a nondegenerate interval. Then f has a rotational topological horseshoe. Moreover, for any nonempty open interval $J \subset \rho(\widetilde{f})$, there exists a rotational topological horseshoe whose rotation set intersects J.*

Let us just explain how to perform the necessary modifications to that subsection, so as to obtain this result. Using the same language and definitions of [6], the main idea in order to show that f has a rotational topological horseshoe is to apply [16, Theorem M]. For this, consider an open interval $J \subset \rho(\widetilde{f})$ and choose some rational s/r in J such that s/r is not the rotation number for $\widetilde{f}$ of any point in the boundary of A. Let $g = f^r$. Then there exists a *maximal isotopy* I' joining g to the identity, that lifts to a maximal isotopy $\widetilde{I'}$ joining the identity in $\widetilde{A}$ to a lift $\widetilde{g} = \widetilde{f}^r - (s, 0)$ of g. This can be done in such a way that the rotation set of $\widetilde{g}$ is an interval containing the origin, and such that the rotation number of points in the upper boundary of $\widetilde{A}$ is not null. For such maps, one can find a Brouwer–Le Calvez foliation $\mathcal{F}$ for I' that is lifted to a Brouwer–Le Calvez foliation $\widetilde{\mathcal{F}}$ for $\widetilde{I'}$ and one can consider the set of admissible $\widetilde{\mathcal{F}}$-transverse paths, as defined in [15]. To show the existence of a rotational topological horseshoe, it is then sufficient to show that there exists an n-admissible $\widetilde{\mathcal{F}}$-transverse path $\widehat{\gamma'}$ such that $\widehat{\gamma'}$ has a $\widetilde{\mathcal{F}}$-transverse intersection with $\widehat{\gamma'} + (p, 0)$ for some nonnull integer p. This implies that there exists $n > 0$ such that g has a rotational topological horseshoe whose $\widetilde{g}$-rotation set contains $[0, p/n]$ if $p > 0$, or contains $[p/n, 0]$ if p is negative. In any case, this implies that f has a rotational topological horseshoe whose rotation set contains an interval having s/r as an endpoint, and thus intersecting J.

Most of the work contained in [6, subsection 6.1.2] is concerned with estimating the size of the rotation set contained in the rotational topological horseshoe for f, and for such a reason a stronger hypothesis than just asking A to be a Birkhoff region of instability was assumed. But in what concerns us here, which is to show that a rotational topological horseshoe exists without requiring its rotation set to be of any specific length, that extra hypothesis is unnecessary. The curve $\widehat{\gamma'}$ we need is obtained much in the same way as in the quoted subsection: One shows that, assuming without loss of generality that the rotation number of the upper boundary is strictly positive for $\widetilde{g}$, any point $\widetilde{z}$ in $\mathbb{R} \times \{1\}$ has a full $\widetilde{\mathcal{F}}$-transverse trajectory that is equivalent to a $\widetilde{\mathcal{F}}$-transverse simple curve $\widehat{\gamma} : \mathbb{R} \to \widetilde{A}$ satisfying $\widehat{\gamma}(t+1) = \widehat{\gamma}(t) + (1, 0)$. In particular, using that A is a Birkhoff region of instability, there exist positive integers N_0, N_1, a point $\widetilde{Z}_{0,1}$ that is sufficiently close to the lower boundary of $\widetilde{A}$ with $\widetilde{g}^{N_0}(\widetilde{Z}_{0,1})$ sufficiently close to the upper boundary of $\widetilde{A}$, so that its transverse path up to time N_0 contains a subpath equivalent to $\widehat{\gamma}\,|_{[0,2]}$ that starts at a leaf not intersected by $\widehat{\gamma}$, and another point $\widetilde{Z}_{1,0}$ that is sufficiently close to the upper boundary of $\widetilde{A}$ with $\widetilde{g}^{N_1}(\widetilde{Z}_{1,0})$ sufficiently close to the lower boundary of $\widetilde{A}$, and such that its transverse path up to time N_1 contains a subpath equivalent to $\widehat{\gamma}\,|_{[0,2]}$, which ends at a leaf not intersected by $\widehat{\gamma}$. The construction of $\widehat{\gamma'}$ then follows exactly as in [6, subsection 6.1.2].

A direct consequence of the two previous results is the following:

Corollary 1. *Let A be a Birkhoff region of instability for some $f \in \mathrm{Diff}_0^0(\mathrm{A})$ and assume $\rho(\widetilde{f})$ is a nondegenerate interval for some fixed lift $\widetilde{f}$. Then there exists an open and dense subset E of $\rho(\widetilde{f})$ such that for any $p/q \in E$, there exists:*

- *integers s, r, L depending on p/q and a rotational topological horseshoe for $\widetilde{f}$ whose rotation set contains an interval $[s/r, (s+1)/r]$ with $s/r < p/q < p_1/q_1 < (s+1)/r$, where $p_1 = p.r.L + 1, q_1 = q.r.L$;*

- *a compact and transitive f-invariant set $\Lambda' = \Lambda'(p/q)$;*
- *two compact f-invariant sets $G^0 = G^0(p/q)$ and $G^1 = G^1(p/q)$, both contained in Λ', such that the restriction of f to each of them has strictly positive topological entropy;*
- *two compact sets $\widetilde{G^0}, \widetilde{G^1}$ in $\widetilde{A}$ projecting surjectively onto G^0 and G^1, respectively, and such that $\widetilde{f}^{q_1}(\widetilde{G^0}) - (p.r.L, 0) = \widetilde{G^0}$ and $\widetilde{f}^{q_1}(\widetilde{G^1}) - (p_1, 0) = \widetilde{G^1}$.*

Proof. Let E be the union of the interior of the rotation sets of all rotational topological horseshoes of f. Proposition 2 shows that E is dense, and as it is the union of open intervals, it is also open. If p/q is a rational point in E, then by definition one can find a rotational horseshoe whose rotation set contains p/q in its interior, and let then s, r be integers such that $s/r < p/q < (s+1)/r$ and such that $[s/r, (s+1)/r]$ is also contained in the interior of the rotation set of this topological horseshoe. If we then choose L sufficiently large, than taking $p_1 = p.r.L + 1, q_1 = q.r.L$, we get that $p_1/q_1 < (s+1)/r$. This shows the first item.

To get the other items of the proposition, note first that, by setting $p^* = rp - sq, p_1^* = \frac{1}{r} \cdot (rp_1 - sq_1) = p_1 - sqL$, then $p^*/q = r(p/q) - s,\ p_1^*/(Lq) = r(p_1/q_1) - s$ and so $0 < p^*/q < p_1^*/(Lq) < 1$. We apply Lemma 6 with $g = f^r, \widetilde{g} = \widetilde{f}^r - (s, 0)$, giving us a set Λ_0 in A which is transitive for g, as well as two sets $K_{p^*/q}, K_{p_1^*/(Lq)}$, both contained in Λ_0, such that the former is invariant by g^q and the latter is invariant by g^{Lq}, and such that the topological entropy of the restriction of g^{Lq} to either $K_{p^*/q}$ or $K_{p_1^*/(Lq)}$ is strictly positive. Furthermore, there exists sets $\widetilde{K}_{p^*/q}, \widetilde{K}_{p_1^*/(Lq)}$ in $\widetilde{A}$ projecting onto $K_{p^*/q}, K_{p_1^*/(Lq)}$, respectively, and such that $\widetilde{g}^q(\widetilde{K}_{p^*/q}) = \widetilde{K}_{p^*/q} + (p^*, 0)$ and such that $\widetilde{g}^{(Lq)}(\widetilde{K}_{p_1^*/(Lq)}) = \widetilde{K}_{p_1^*/(Lq)} + (p_1^*, 0)$.

Define now $\Lambda' = \bigcup_{i=0}^{r-1} f^i(\Lambda_0)$, and note that, as Λ_0 was an invariant set for $g = f^r$ and the restriction of g to this set was transitive, then Λ' is invariant for f and the restriction of f to it is also transitive. This gives us the second item of the corollary. Define also

$$G^0 = \bigcup_{i=0}^{r-1} f^i\left(\bigcup_{j=0}^{q-1} g^j(K_{p^*/q})\right), \text{ and } G^1 = \bigcup_{i=0}^{r-1} f^i\left(\bigcup_{j=0}^{q_1-1} g^j(K_{p_1^*/(Lq)})\right),$$

which are both subsets of Λ' as they are each contained in the f-orbit of $K_{p^*/q}$ and $K_{p_1^*/(Lq)}$, respectively, and $K_{p^*/q}$ and $K_{p_1^*/(Lq)}$ are contained in Λ' which is invariant. One also observes that G^0 and G^1 are f invariant by construction, and the restriction of f to both this sets has strictly positive topological entropy. This gives us the third item of the proposition.

Finally, set

$$\widetilde{G^0} = \bigcup_{i=0}^{r-1} \widetilde{f}^i\left(\bigcup_{j=0}^{q-1} \widetilde{g}^j(\widetilde{K_{p^*/q}})\right), \text{ and } \widetilde{G^1} = \bigcup_{i=0}^{r-1} \widetilde{f}^i\left(\bigcup_{j=0}^{q_1-1} \widetilde{g}^j(\widetilde{K_{p_1^*/(Lq)}})\right).$$

Note that, as $\widetilde{f}$ and $\widetilde{g}$ commute, we get that

$$\begin{aligned}\widetilde{f}^{rq}(\widetilde{G^0}) &= \widetilde{g}^q(\widetilde{G^0}) + (sq, 0)\\ &= \bigcup_{i=0}^{r-1} \widetilde{f}^i\left(\bigcup_{j=0}^{q-1} \widetilde{g}^j(\widetilde{g}^q(\widetilde{K_{p^*/q}}))\right) + (sq, 0)\end{aligned}$$

$$= \bigcup_{i=0}^{r-1} \widetilde{f}^{i}\left(\bigcup_{j=0}^{q-1} \widetilde{g}^{j}(\widetilde{K_{p^*/q}})\right) + (sq,0) + (p^*,0)$$

$$= \widetilde{G^0} + (rp,0)$$

and so $\widetilde{f}^{q_1}(\widetilde{G^0}) = \widetilde{f}^{L\cdot rq}(\widetilde{G^0}) = \widetilde{G^0} + (L\cdot rp,0)$. A similar computation shows that $\widetilde{f}^{q_1}(\widetilde{G^1}) = \widetilde{G^1} + (sqL + p_1^*,0) = \widetilde{G^1} + (p_1,0)$ ending the claim. □

We remark that from the corollary, every point in the set $G^0(p/q)$ has rotation number p/q for $\widetilde{f}$, and every point in $G^1(p/q)$ has rotation number p_1/q_1 for $\widetilde{f}$.

2.5 | On maximal invariant sets

Let $\gamma \subset S^1\times]0,1[$ be a homotopically nontrivial simple closed curve and, as we defined before, let γ^- be the connected component of γ^c that contains $S^1\times\{0\}$ (similarly for γ^+ and $S^1\times\{1\}$).

If we consider the sets

$$B^s_{0,\gamma} = \bigcap_{n\leqslant 0} f^n(\overline{\gamma^-}),\ B^u_{0,\gamma} = \bigcap_{n\geqslant 0} f^n(\overline{\gamma^-}),$$
$$B^s_{1,\gamma} = \bigcap_{n\leqslant 0} f^n(\overline{\gamma^+}) \text{ and } B^u_{1,\gamma} = \bigcap_{n\geqslant 0} f^n(\overline{\gamma^+}),$$

we get that

$$f(B^s_{0,\gamma}) \subset B^s_{0,\gamma},\ f^{-1}(B^u_{0,\gamma}) \subset B^u_{0,\gamma},$$

$$f(B^s_{1,\gamma}) \subset B^s_{1,\gamma} \text{ and } f^{-1}(B^u_{0,\gamma}) \subset B^u_{0,\gamma}.$$

Denote by $\widehat{B}^s_{0,\gamma}$ the connected component of $B^s_{0,\gamma}$ that contains $S^1\times\{0\}$ and define similarly $\widehat{B}^u_{0,\gamma}$. Analogously, let $\widehat{B}^s_{1,\gamma}$ be the connected component of $B^s_{1,\gamma}$ that contains $S^1\times\{1\}$ and define similarly $\widehat{B}^u_{1,\gamma}$.

The next result appears in Le Calvez [13] and even in Birkhoff's paper [5].

Lemma 7. *Let $f : A \to A$ be an orientation and boundary components preserving homeomorphism, which has the curve intersection property. Then, for any γ as above, $\widehat{B}^s_{0,\gamma}, \widehat{B}^s_{1,\gamma}, \widehat{B}^u_{0,\gamma}$ and $\widehat{B}^u_{1,\gamma}$ intersect γ.*

As $f(\widehat{B}^s_{0,\gamma}) \subset \widehat{B}^s_{0,\gamma}$, $f(\widehat{B}^s_{1,\gamma}) \subset \widehat{B}^s_{1,\gamma}$, $f^{-1}(\widehat{B}^u_{0,\gamma}) \subset \widehat{B}^u_{0,\gamma}$, and $f^{-1}(\widehat{B}^u_{1,\gamma}) \subset \widehat{B}^u_{1,\gamma}$, we get that the maximal invariant sets $\xi^-(\gamma)$ and $\xi^+(\gamma)$ satisfy the following conditions:

$$\cap_{n\geqslant 0} f^{-n}(\widehat{B}^u_{0,\gamma}) = \cap_{n\geqslant 0} f^{n}(\widehat{B}^s_{0,\gamma}) = \xi^-(\gamma)$$
and
$$\cap_{n\geqslant 0} f^{-n}(\widehat{B}^u_{1,\gamma}) = \cap_{n\geqslant 0} f^{n}(\widehat{B}^s_{1,\gamma}) = \xi^+(\gamma)$$

3 | PROOFS

3.1 | Proof of Theorem 2

Let us remember our set of hypotheses: $f \in \mathrm{Diff}_0^{1+\varepsilon}(\mathrm{A})$ for some $\varepsilon > 0$, A is a Birkhoff region of instability and for some fixed lift $\widetilde{f}$, $\rho(\widetilde{f})$ has interior.

Let E be the set from Corollary 1, which we further assume does not contain the rotation number of the boundaries. Fix some $p/q \in E$ and consider, as in Corollary 1, the sets Λ', $G^0(p/q)$, $G^1(p/q)$ as well as the integers r, s, L, p_1 and q_1, all depending on p/q. As $h(f|_{G^0}) > 0$ and $h(f|_{G^1}) > 0$, there exist two hyperbolic ergodic Borel nonatomic f-invariant measures $\mu_{p/q}$ and μ_{p_1/q_1}, such that $\mathrm{supp}(\mu_{p/q}) \subset G^0(p/q)$ and $\mathrm{supp}(\mu_{p_1/q_1}) \subset G^1(p/q)$. Note that, by the choice of $G^0(p/q)$ (respectively, $G^1(p/q)$), every point in it has rotation number p/q (respectively, p_1/q_1). From Lemma 5, pick inaccessible points $z_0 \in \mathrm{supp}(\mu_{p/q})$ and $z_1 \in \mathrm{supp}(\mu_{p_1/q_1})$.

Lemma 4 implies that there are four hyperbolic periodic saddle points, y_0 and y_0', y_1 and y_1', such that z_0 is enclosed by the rectangle determined by compact subarcs of stable and unstable branches at y_0 and at y_0', an analogous statement holding for z_1 and y_1 and y_1'. The corners of each rectangle are either the saddles, or C^1-transverse intersections between stable branches at one saddle and unstable branches at the other. In particular, the λ-lemma implies that for each of these four periodic points y_0, y_0', y_1 and y_1', there are C^1-transverse homoclinic intersections.

As the rectangles can be chosen in an arbitrarily small way, the rotation numbers of y_0 and y_0' are both equal to p/q (but their periods might be larger than q) and the rotation numbers of y_1 and y_1' are both equal to p_1/q_1, which is different from p/q. So, as $\overline{W^u(y_0)} = \overline{W^u(y_0')}$, $\overline{W^u(y_1)} = \overline{W^u(y_1')}$, $\overline{W^s(y_0)} = \overline{W^s(y_0')}$ and $\overline{W^s(y_1)} = \overline{W^s(y_1')}$, and there is a dense orbit in Λ', arbitrarily large positive iterates of the interior of the rectangle enclosing z_0 intersect the interior of the rectangle enclosing z_1 and vice versa. And this implies (see [2, Lemma 24]) that for some integer i, there exists an unstable branch at y_0 that has a topologically transverse intersection with a stable branch at $f^i(y_1)$ and an unstable branch at $f^i(y_1)$ has a topologically transverse intersection with a stable branch at y_0. As the rotation number of y_0 is p/q and the rotation number of $f^i(y_1)$ is not p/q, the theorem follows from the C^0-λ -lemma that holds for topologically transverse intersections, see Proposition 1.

3.2 | Proof of Lemma 1

Under the lemma hypotheses, Theorem 2 implies that for any rational point p/q in the set E, we can find a hyperbolic periodic saddle $z_{p/q}$ with rotation number p/q and unstable and stable branches, $\lambda^u_{p/q}$ and $\lambda^s_{p/q}$, both at $z_{p/q}$, which intersect at a point $w_{p/q}$, such that if we concatenate the arc in $\lambda^u_{p/q}$ from $z_{p/q}$ to $w_{p/q}$ to the arc in $\lambda^s_{p/q}$ from $w_{p/q}$ to $z_{p/q}$, then we get a homotopically nontrivial closed curve $\gamma_{p/q}$ contained in the interior of A (because the rotation numbers on the boundary components do not lie in E).

Proposition 3. *For each $p/q \in E$, $\overline{\lambda^u_{p/q}}$ and $\overline{\lambda^s_{p/q}}$ intersect $S^1 \times \{0\}$ and $S^1 \times \{1\}$.*

Proof. Fixed some $p/q \in E$, let $q.k_{p/q}$ be twice the period of $z_{p/q}$. As A is a Birkhoff region of instability, $\cup_{n\geqslant 0} f^{n.q.k_{p/q}}(\gamma_{p/q})$ accumulates on both boundary components, $S^1 \times \{0\}$ and $S^1 \times \{1\}$.

An analogous statement holds for $\cup_{n\geqslant 0} f^{-n.q.k_{p/q}}(\gamma_{p/q})$. The proposition follows from the $f^{q.k_{p/q}}$-positive invariance of the arc in $\lambda^s_{p/q}$ from $z_{p/q}$ to $w_{p/q}$ and the $f^{q.k_{p/q}}$-negative invariance of the arc in $\lambda^u_{p/q}$ from $z_{p/q}$ to $w_{p/q}$. $\square$

The curve $\gamma_{p/q}$ may not be simple, but in any case, $(\gamma_{p/q})^c$ still has one connected component that contains $S^1 \times \{0\}$, denoted $\gamma^-_{p/q}$, another connected component that contains $S^1 \times \{1\}$, denoted $\gamma^+_{p/q}$ and maybe other contractible components.

Proposition 4. *For any $p/q \in E$ and any integer $n > 1$,*

$$\xi^-_{1/n} \subset \overline{\gamma^-_{p/q}} \text{ and } \xi^+_{1/n} \subset \overline{\gamma^+_{p/q}}.$$

In particular, there exist f-invariant continua $K^- \supset S^1 \times \{0\}$ and $K^+ \supset S^1 \times \{1\}$ such that $\xi^-_{1/n} = K^-$ and $\xi^+_{1/n} = K^+$ for all sufficiently large n.

Proof. First, note that Proposition 3 implies that $\xi^-_{1/n}$ cannot contain $\lambda^u_{p/q}$ or $\lambda^s_{p/q}$ because both branches accumulate on $S^1 \times \{1\}$. Moreover, if for some $n > 1$, $\xi^-_{1/n}$ is not contained in $\overline{\gamma^-_{p/q}}$, then as $\xi^-_{1/n}$ is f-invariant, connected and contains $S^1 \times \{0\}$, $\xi^-_{1/n}$ would contain two sequences of points, one in $\gamma^-_{p/q}$ and one in $(\overline{\gamma^-_{p/q}})^c$, both converging to $z_{p/q}$. At least one of them converges to $z_{p/q}$ through the local quadrant at $z_{p/q}$ adjacent to $\lambda^u_{p/q}$ and $\lambda^s_{p/q}$. So, [1, Proposition 6, item 2] (which says that any f-invariant continuum that contains a hyperbolic saddle periodic point p and accumulates on p through a certain local quadrant Q at p, must contain at least one branch at p adjacent to Q) implies that $\xi^-_{1/n}$ contains either $\overline{\lambda^u_{p/q}}$ or $\overline{\lambda^s_{p/q}}$, a contradiction as explained above. One shows by a similar argument that $\xi^+_{1/n}$ is contained in $\overline{\gamma^+_{p/q}}$.

The above argument implies that, for any homotopically nontrivial simple closed curve γ contained in the interior of A, such that $\gamma^-_{p/q} \subset \gamma^-$, $\xi^-(\gamma)$ must be equal to the connected component of the maximal invariant set contained in the closure of $\gamma^-_{p/q}$ that contains $S^1 \times \{0\}$. A similar statement holds for $\xi^+(\gamma)$. Therefore, if $N > 0$ is such that $S^1 \times \{1 - 1/N\} \subset \gamma^+_{p/q}$ and $S^1 \times \{1/N\} \subset \gamma^-_{p/q}$, then for all $n \geqslant N$, $\xi^-_{1/n} = \xi^-_{1/N} := K^-$ and $\xi^+_{1/n} = \xi^+_{1/N} := K^+$. $\square$

Note that the intersection between K^- and K^+ must be empty because otherwise one could apply as before, [1, Proposition 6, item 2] and get that either K^- or K^+ (maybe both), contains $\overline{\lambda^u_{p/q}}$ or $\overline{\lambda^s_{p/q}}$. As explained above, this is a contradiction.

So, as $(K^-)^c$ and $(K^+)^c$ are both connected, $(K^- \cup K^+)^c := A^*$ is an open f-invariant essential annulus contained in A. As A is a Birkhoff region of instability, ∂A^* intersects both $S^1 \times \{0\}$ and $S^1 \times \{1\}$.

We finish with the following proposition.

Proposition 5. *For all rationals $p/q \in E$, with the exception of at most two points, $\gamma_{p/q}$ is contained in A^*.*

Proof. If, for some p/q in E, $\gamma_{p/q}$ intersects K^-, then as K^- is compact and f-invariant, $z_{p/q} \in K^-$, and it is clearly accessible from $(K^-)^c$. This happens because $K^- \subset \overline{\gamma^-_{p/q}}$ and every point in the upper boundary of $\overline{\gamma_{p/q}^-}$ (which contains $z_{p/q}$) is the endpoint of a C^1 end cut contained in the complement of $\overline{\gamma_{p/q}^-}$. This is a trivial consequence of the fact that $\gamma_{p/q}$ is a piecewise C^1 curve. So, Lemma 2 implies that all accessible periodic points in K^- have p/q as rotation number (the rotation number of the prime ends compactification of $(K^-)^c$ is equal to p/q). A similar argument holds for K^+, and so, with the exception of at most two rational numbers in E, for all other rationals $p/q \in E$, $\gamma_{p/q}$ avoids $K^- \cup K^+$, in other words, $\gamma_{p/q}$ is contained in A^*.

As E is an infinite set, this concludes the proof of the lemma. □

3.3 | Proof of Theorem 1

From Proposition 5, there exists $p/q \in E$ such that $\gamma_{p/q} \subset A^*$. Moreover, $\gamma_{p/q} = z_{p/q} \cup \lambda^u_{\text{comp}} \cup \lambda^s_{\text{comp}}$, where λ^u_{comp} is the closed arc in $\lambda^u_{p/q}$ from $z_{p/q}$ to $w_{p/q}$ and λ^s_{comp} is the closed arc in $\lambda^s_{p/q}$ from $w_{p/q}$ to $z_{p/q}$, where $w_{p/q}$ is a point in the intersection between $\lambda^u_{p/q}$ and $\lambda^s_{p/q}$.

Denote the lower (resp., upper) connected component of the boundary of A^* by $K^{*0} \subset K^-$ (resp., $K^{*1} \subset K^+$). For $\gamma \subset A$ a homotopically nontrivial simple closed curve, which is also contained in A^*, we can define $\xi^{*-}(\gamma)$ as the connected component of the maximal invariant set contained in $\overline{\gamma^-} \cap \overline{A^*}$ that contains K^{*0}. Analogously for $\xi^{*+}(\gamma)$. From the construction of K^- and K^+ in Lemma 1, we get that

$$\xi^{*-}(\gamma) = K^{*0} \text{ and } \xi^{*+}(\gamma) = K^{*1}. \tag{1}$$

Now consider homotopically nontrivial simple closed curves α_1 and α_0, both contained in A^*, such that $\alpha_1 \subset \gamma^+_{p/q}$ and $\alpha_0 \subset \gamma^-_{p/q}$.

Also consider the sets $\widehat{B}^{*s}_{0,\alpha_1}$, $\widehat{B}^{*u}_{0,\alpha_1}$, $\widehat{B}^{*s}_{1,\alpha_0}$ and $\widehat{B}^{*u}_{1,\alpha_0}$ defined in Subsection 2.5, but now with respect to $\overline{A^*}$.

Clearly,

$$\begin{gathered} \cap_{n\geqslant 0} f^n\left(\widehat{B}^{*s}_{0,\alpha_1}\right) = \cap_{n\geqslant 0} f^{-n}\left(\widehat{B}^{*u}_{0,\alpha_1}\right) = K^{*0} \\ \text{and} \\ \cap_{n\geqslant 0} f^n\left(\widehat{B}^{*s}_{1,\alpha_0}\right) = \cap_{n\geqslant 0} f^{-n}\left(\widehat{B}^{*u}_{1,\alpha_0}\right) = K^{*1}. \end{gathered} \tag{2}$$

The above equalities imply that for any $z \in \widehat{B}^{*s}_{0,\alpha_1}$, its ω-limit set is contained in K^{*0} and for any $w \in \widehat{B}^{*u}_{0,\alpha_1}$, its α-limit set is also contained in K^{*0}. And analogously, for any $z \in \widehat{B}^{*s}_{1,\alpha_0}$, its ω-limit set is contained in K^{*1} and for any $w \in \widehat{B}^{*u}_{0,\alpha_1}$, its α-limit set is also contained in K^{*1}. As K^{*0} and K^{*1} are, respectively, the lower and the upper connected components of the boundary of A^*, we will conclude the proof of the theorem by showing that for some integer $n \geqslant 0$, $f^n(\widehat{B}^{*u}_{0,\alpha_1}) \cap f^{-n}(\widehat{B}^{*s}_{1,\alpha_0})$ and $f^n(\widehat{B}^{*u}_{1,\alpha_0}) \cap f^{-n}(\widehat{B}^{*s}_{0,\alpha_1})$ are both nonempty.

Lemma 8. *The sets $\widehat{B}^{*s}_{0,\alpha_1}$ and $\widehat{B}^{*s}_{1,\alpha_0}$ intersect $\lambda^u_{p/q}$ in a topologically transverse way, and analogously, $\widehat{B}^{*u}_{0,\alpha_1}$ and $\widehat{B}^{*u}_{1,\alpha_0}$ intersect $\lambda^s_{p/q}$ also in a topologically transverse way.*

Proof. The proof is analogous in all four cases, for $\widehat{B}^{*s}_{0,\alpha_1}$, $\widehat{B}^{*s}_{1,\alpha_0}$, $\widehat{B}^{*u}_{0,\alpha_1}$ and $\widehat{B}^{*u}_{1,\alpha_0}$. So, without loss of generality, let us only analyze $\widehat{B}^{*s}_{0,\alpha_1}$.

Let Θ be a connected component of the intersection between $\widehat{B}^{*s}_{0,\alpha_1}$ and the closed annulus bounded by α_0 and α_1 that intersects α_1 (see Lemma 7). As Θ intersects α_1, as $\widehat{B}^{*s}_{0,\alpha_1}$ is connected and contains K^{*0}, and as each connected component of $\widehat{B}^{*s}_{0,\alpha_1}$ is entirely contained in the closed annulus bounded by α_1 and the lower boundary of A, one obtains that Θ must also intersect α_0.

From expression (2), $\widehat{B}^{*s}_{0,\alpha_1}$ does not intersect $\lambda^s_{p/q}$. Assume, for a contradiction, that Θ does not intersect $\lambda^u_{p/q}$ in a topologically transverse way. Let $\varepsilon > 0$ be sufficiently small in a way that the ε-neighborhood of λ^u_{comp}, denoted V, is contractible in A^* and disjoint from α_0 and α_1. Then, as the endpoints of λ^u_{comp} do not belong to $\widehat{B}^{*s}_{0,\alpha_1}$, by Lemma 3, one can find a curve μ_ε contained in V, with the same endpoints as λ^u_{comp}, which is disjoint from Θ. But as V is contractible, $\mu_\varepsilon \cup \lambda^s_{\text{comp}}$ is a homotopically nontrivial closed curve separating α_0 and α_1 and disjoint from Θ. And this is a contradiction because Θ is connected and intersects α_0 and α_1. □

Thus, from the above lemma and Proposition 1, $f^n(\widehat{B}^{*u}_{0,\alpha_1})$ and $f^n(\widehat{B}^{*u}_{1,\alpha_0})$ both contain subcontinua that accumulate on compact sub arcs of $W^u(z_{p/q})$ in the Hausdorff topology as $n \to \infty$, and analogously, $f^{-n}(\widehat{B}^{*s}_{0,\alpha_1})$ and $f^{-n}(\widehat{B}^{*s}_{1,\alpha_0})$ both contain subcontinua that accumulate on compact sub arcs of $W^s(z_{p/q})$ in the Hausdorff topology as $n \to \infty$. And this means that, for a sufficiently large $n > 0$,

$$f^n\left(\widehat{B}^{*u}_{0,\alpha_1}\right) \text{ intersects both } f^{-n}\left(\widehat{B}^{*s}_{0,\alpha_1}\right) \text{ and } f^{-n}\left(\widehat{B}^{*s}_{1,\alpha_0}\right),$$
$$\text{and}$$
$$f^n\left(\widehat{B}^{*u}_{1,\alpha_0}\right) \text{ intersects both } f^{-n}\left(\widehat{B}^{*s}_{0,\alpha_1}\right) \text{ and } f^{-n}\left(\widehat{B}^{*s}_{1,\alpha_0}\right).$$

Denoting a point in $f^n(\widehat{B}^{*u}_{0,\alpha_1}) \cap f^{-n}(\widehat{B}^{*s}_{1,\alpha_0})$ by z^+ and a point in $f^n(\widehat{B}^{*u}_{1,\alpha_0}) \cap f^{-n}(\widehat{B}^{*s}_{0,\alpha_1})$ by z^-, the theorem is proved.

3.4 | Proof of Theorem 3

If the rotation set of $\widetilde{f}$ is the singleton $\{a\}$, there is nothing to be done, as it follows easily that each point in the boundary of A has rotation number a (and one can even show that every point in A has rotation number a). So we can assume that the rotation set of $\widetilde{f}$ has nonempty interior and therefore we are in the hypotheses of Theorem 1. Let A^* be given by this result, which is obtained as in Lemma 1. Note that, from Theorem 2 we know that there exists E that is open and dense in $\rho(\widetilde{f})$ such that for any rational p/q in E we find the homotopically nontrivial closed curve $\gamma_{p/q}$ as described there, and from the end of the proof of Lemma 1, we know that all but at most two of these curves are contained in A^*.

As is done for the disk, one can consider the prime ends compactification of A^*, by adding two circles in order to obtain $A' = A^* \sqcup S^1 \sqcup S^1$, so that A' is homeomorphic to A, and the restriction of f to A^* extends continuously to a homeomorphism h of A', with a lift $\widetilde{h}$ to the universal covering of A'. The rotation number of the lower boundary component of A' is the prime ends rotation number of K^{*0} and the rotation number of the upper boundary component of A' is the prime ends rotation number of K^{*1}. Note that A' is a Mather region of instability for h. Note also that, as every rational in E is the $\widetilde{h}$-rotation number of a point in $\overline{A^*}$, then the rotation set of $\widetilde{h}$ must be the same as that of $\widetilde{f}$, as rotation sets are closed. Now, [6, Theorem C] shows that every point in the rotation set of $\widetilde{h}$ is realized by a compact h-invariant set, which implies that, except maybe for the two prime ends rotation numbers of the boundary components of A^*, every point in the rotation set of $\widetilde{f}$ is realized by a compact f-invariant subset in A^*.

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