A simple computable criteria for the existence of horseshoes

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Abstract

In this note we give a simple computable criteria that assures the existence of hyperbolic horseshoes for certain diffeomorphisms of the torus. The main advantage of our method is that it is very easy to check numerically whether the criteria is satisfied or not.

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1 Introduction

In this note, we present a simple criteria, which gives information on the existence of hyperbolic horseshoes for certain diffeomorphisms (the so called twist diffeomorphisms) of the 2-torus. As is well-known, the existence of a horseshoe for a dynamical system implies "chaotic" behavior, so an important subject in dynamical systems theory is to develop methods that assure the presence of horseshoes for a given system. What we present here is a simple application of some well-known important results to a particular class of diffeomorphisms of the 2-torus, the ones which satisfy a twist condition.

In order to be more precise, let us give some definitions.

Definitions

0) Let $\mathrm{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the flat torus and let $(\phi, I) \in \mathrm{T}^2$ denote coordinates in the torus, $(\widehat{\phi}, \widehat{I}) \in S^1 \times \mathbb{R} = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ denote coordinates in the cylinder, where $\widehat{\phi}$ is defined modulo 1. Let $p_1 : S^1 \times \mathbb{R} \to \mathbb{R}$ and $p_2 : S^1 \times \mathbb{R} \to \mathbb{R}$ be given by: $p_1(\widehat{\phi}, \widehat{I}) = \widehat{\phi}$ and $p_2(\widehat{\phi}, \widehat{I}) = \widehat{I}$ (the corresponding projections in the plane are denoted in the same way).

1) Let $Diff_t^{1+\epsilon}(\mathrm{T}^2)$ be the set of $C^{1+\epsilon}$ (for any $\epsilon > 0$) torus diffeomorphisms $T(\phi, I) = (T_\phi(\phi, I), T_I(\phi, I))$ that are homotopic to the Dehn twist $(\phi, I) \to (\phi + I \ mod\ 1, I \ mod\ 1)$ and satisfy $\partial_I T_\phi \geq K > 0$, where K is a positive number (this is the so called twist condition).

2) Let $D_t^{1+\epsilon}(S^1 \times \mathbb{R})$ be the set of diffeomorphisms of the cylinder $\widehat{T}(\widehat{\phi}, \widehat{I})$ which are lifts of elements from $Diff_t^{1+\epsilon}(\mathrm{T}^2)$. Clearly, such a $\widehat{T}$ satisfies: $\widehat{T}(\widehat{\phi}, \widehat{I} + 1) = \widehat{T}(\widehat{\phi}, \widehat{I}) + 1$.

Now we are ready to state our result:

Theorem 1 *: Given $T \in Diff_t^{1+\epsilon}(\mathrm{T}^2)$, there exists a number $M > 0$, which depends only on T and can be easily computed, such that if for some lift $\widehat{T}$ of T, there are points $\widehat{z}, \widehat{w} \in S^1 \times \mathbb{R}$ and natural numbers $n > 1, k > 1$ such that $p_2 \circ \widehat{T}^n(\widehat{z}) - p_2(\widehat{z}) > M$ and $p_2 \circ \widehat{T}^k(\widehat{w}) - p_2(\widehat{w}) < -M$, then T has a hyperbolic periodic point, whose stable and unstable manifolds intersect transversally.*

The proof of this result will show that M can be computed from T in a very simple way and as $\widehat{T}(\widehat{\phi}, \widehat{I} + 1) = \widehat{T}(\widehat{\phi}, \widehat{I}) + 1$, one just has to iterate a horizontal curve $\widehat{I} = const.$ a sufficiently large number of times in order to see if the curve contains points like $\widehat{z}$ and $\widehat{w}$ as in the theorem.

This paper is organized as follows. In the next section we present the statements of some results we use. In section 3 we present the proof of our result.

2 Basic tools

Here we recall some topological results for twist mappings essentially due to Le Calvez (see [6] and [7]). Let $\widehat{T} \in D_t^{1+\epsilon}(S^1 \times \mathbb{R})$ and $\widetilde{T} \in D_t^{1+\epsilon}(\mathbb{R}^2)$ be its lift to the plane. For every pair (s,q), $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we define the following sets ($\pi : \mathbb{R}^2 \to S^1 \times \mathbb{R}$ is given by $\pi(\widetilde{\phi}, \widetilde{I}) = (\widetilde{\phi} \ mod\ 1, \widetilde{I})$):

$$\begin{gathered} K_{lift}(s,q) = \left\{ (\widetilde{\phi}, \widetilde{I}) \in \mathbb{R}^2 \colon\ p_1 \circ \widetilde{T}^q(\widetilde{\phi}, \widetilde{I}) = \widetilde{\phi} + s \right\} \\ \text{and} \\ K(s,q) = \pi \circ K_{lift}(s,q) \end{gathered} \tag{1}$$

Then we have the following:

Lemma 1 *: For every $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$, $K(s,q) \supset C(s,q)$, a connected compact set that separates the cylinder.*

For all $s \in \mathbb{Z}$ and $q \in \mathbb{N}^*$ we can define the following functions on S^1:

$$\begin{aligned} \mu^-(\widehat{\phi}) &= \min\{p_2(z)\colon\ z \in K(s,q) \text{ and } p_1(z) = \widehat{\phi}\} \\ \mu^+(\widehat{\phi}) &= \max\{p_2(z)\colon\ z \in K(s,q) \text{ and } p_1(z) = \widehat{\phi}\} \end{aligned}$$

And we can define similar functions for $\widehat{T}^q(K(s,q))$:

$$\begin{aligned} \nu^-(\widehat{\phi}) &= \min\{p_2(z)\colon\ z \in \widehat{T}^q \circ K(s,q) \text{ and } p_1(z) = \widehat{\phi}\} \\ \nu^+(\widehat{\phi}) &= \max\{p_2(z)\colon\ z \in \widehat{T}^q \circ K(s,q) \text{ and } p_1(z) = \widehat{\phi}\} \end{aligned}$$

Lemma 2 *: Defining $Graph\{\mu^\pm\}$=$\{(\widehat{\phi}, \mu^\pm(\widehat{\phi})) : \widehat{\phi} \in S^1\}$ we have:*

$$Graph\{\mu^-\} \cup Graph\{\mu^+\} \subset C(s,q)$$

So for all $\widehat{\phi} \in S^1$ we have $(\widehat{\phi}, \mu^\pm(\widehat{\phi})) \in C(s,q)$.

The next lemma is a fundamental result in all this theory:

Lemma 3 *: $\widehat{T}^q(\widehat{\phi}, \mu^-(\widehat{\phi})) = (\widehat{\phi}, \nu^+(\widehat{\phi}))$ and $\widehat{T}^q(\widehat{\phi}, \mu^+(\widehat{\phi})) = (\widehat{\phi}, \nu^-(\widehat{\phi}))$.*

For proofs of the previous results see Le Calvez [6] and [7].

Now we remember some ideas from [8].

Given a triplet $(s,p,q) \in \mathbb{Z}^2 \times \mathbb{N}^*$, if there is no point $(\widetilde{\phi}, \widetilde{I}) \in \mathbb{R}^2$ such that $\widetilde{T}^q(\widetilde{\phi}, \widetilde{I}) = (\widetilde{\phi} + s, \widetilde{I} + p)$, it can be proved that the sets $\widehat{T}^q \circ K(s,q)$ and $K(s,q) + (0,p)$ can be separated by the graph of a continuous function from S^1 to $\mathbb{R}$, essentially because from all the previous results, either one of the following inequalities must hold:

$$\nu^-(\widehat{\phi}) - \mu^+(\widehat{\phi}) > p \tag{2}$$

$$\nu^+(\widehat{\phi}) - \mu^-(\widehat{\phi}) < p \tag{3}$$

for all $\widehat{\phi} \in S^1$, where $\nu^+, \nu^-, \mu^+, \mu^-$ are associated to $K(s,q)$.

3 Proofs

The proof of theorem 1 depends on the next lemma, which is a variation of a result of [1]. First we need some definitions:

Given $\widehat{T} \in D_t^{1+\epsilon}(S^1 \times \mathbb{R})$ and a lift $\widetilde{T} \in D_t^{1+\epsilon}(\mathbb{R}^2)$, it can be written in the following way,

$$\widetilde{T} : \begin{cases} \widetilde{\phi}' = \widetilde{T}_\phi(\widetilde{\phi}, \widetilde{I}) \\ \widetilde{I}' = \widetilde{T}_I(\widetilde{\phi}, \widetilde{I}) \end{cases}$$

and for all $(\widetilde{\phi}, \widetilde{I}) \in \mathbb{R}^2$ we have the following estimates:

$$\exists\ a > 0, \text{ such that } \left|\widetilde{T}_I(\widetilde{\phi}, \widetilde{I}) - \widetilde{I}\right| < a \tag{4}$$

$$\exists\ b > 0, \text{ such that } \left|\frac{\partial \widetilde{T}_\phi}{\partial \widetilde{\phi}}\right| < b \tag{5}$$

$$\exists\ K > 0, \text{ such that } \frac{\partial \widetilde{T}_\phi}{\partial \widetilde{I}} \geq K \text{ (twist condition)} \tag{6}$$

Now, let $M = \left[3 + \frac{(2+b)}{K}\right] + a > 0$.

Lemma 4 *: Let $\widetilde{T} \in D_t^{1+\epsilon}(\mathbb{R}^2)$ be such that, there exist points $z, w \in [0,1]^2$ and natural numbers $n > 1, k > 1$, such that $p_2 \circ \widetilde{T}^n(z) - p_2(z) > M$ and $p_2 \circ \widetilde{T}^k(w) - p_2(w) < -M$. Then there are points $z', w' \in [0,1]^2$ and numbers $n', k' \in \mathbb{N}^*$ such that*

$$\begin{cases} \widetilde{T}^{n'}(z') = z' + (s, pos) \\ \widetilde{T}^{k'}(w') = w' + (l, neg) \end{cases} , \textit{ for some } s, l \in \mathbb{Z}, \ pos \geq 1 \textit{ and } neg \leq -1.$$

Proof.

The proof of existence of w' is analogous to the one for z', so we omit it.

By contradiction, suppose that for all $x \in [0,1]^2$ (as $\widetilde{T}$ is the lift of a torus mapping homotopic to the Dehn twist, we can restrict ourselves to points in the unit square) and $n > 0$ such that $\widetilde{T}^n(x) = x + (s, m)$, for some $s \in \mathbb{Z}$, we have $m < 1$.

A very natural thing is to look for a point z' as above, in the line segment $r = \left\{(\widetilde{\phi}, \widetilde{I}) \in [0,1]^2 : \ \widetilde{\phi} = p_1(z) = \widetilde{\phi}_z\right\}$.

First, let us define

$$Max.H.L(\widetilde{T}^n(r)) = \sup_{\widetilde{I}, \widetilde{I}' \in [0,1]} \left|p_1 \circ \widetilde{T}^n(\widetilde{\phi}_z, \widetilde{I}) - p_1 \circ \widetilde{T}^n(\widetilde{\phi}_z, \widetilde{I}')\right|. \tag{7}$$

It is clear that

$$Max.H.L(\widetilde{T}^n(r)) \geq \left|p_1 \circ \widetilde{T}^n(\widetilde{\phi}_z, 0) - p_1 \circ \widetilde{T}^n(\widetilde{\phi}_z, 1)\right| = n \overset{n\to\infty}{\to} \infty. \tag{8}$$

So, for all $n > 1$, $\exists$ at least one $s \in \mathbb{Z}$, such that $\widetilde{\phi}_z + s \in p_1\left(\widetilde{T}^n(r)\right)$.

The hypothesis we want to contradict implies that for all $n > 0$ and $x \in r$, such that

$$p_1 \circ \widetilde{T}^n(x) = \widetilde{\phi}_z \ (mod\ 1), \tag{9}$$

we have:

$$p_2 \circ \widetilde{T}^n(x) < 1 + p_2(x) < 2 \tag{10}$$

As $p_2 \circ \widetilde{T}^n(z) > \left[3 + \frac{(2+b)}{K}\right] + a$, $\exists\ z_1 \in r$ such that:

$$\begin{array}{c} p_2 \circ \widetilde{T}^n(z_1) = 3 + a \\ \text{and} \\ \forall\ x \in\ \overline{z_1 z} \subset r, \\ p_2 \circ \widetilde{T}^n(x) \geq 3 + a \end{array}$$

The reason why such a point z_1 exists is the following: As $n > 1$, $\exists$ at least one $s \in \mathbb{Z}$ such that $\widetilde{\phi}_z + s \in p_1\left(\widetilde{T}^n(r)\right)$. Thus, from (9) and (10), $\widetilde{T}^n(r)$ must cross the line l given by: $l = \left\{(\widetilde{\phi}, 3 + a),\ \text{with}\ \widetilde{\phi} \in \mathbb{R}\right\}$

Also from (9) and (10) we have that:

$$\sup_{x,y \in \overline{z_1 z}} \left| p_1 \circ \widetilde{T}^n(x) - p_1 \circ \widetilde{T}^n(y) \right| < 1$$

Now let $\gamma_n : J \to \mathbb{R}^2$ be the following curve:

$$\gamma_n(t) = \widetilde{T}^n(\widetilde{\phi}_z, t),\ t \in J = \text{interval whose extremes are } p_2(z) \text{ and } p_2(z_1) \tag{11}$$

It is clear that it satisfies the following inequalities:

$$p_2 \circ \gamma_n(p_2(z)) - p_2 \circ \gamma_n(p_2(z_1)) > \tfrac{(2+b)}{K}$$

$$\sup_{t,s \in J} |p_1 \circ \gamma_n(t) - p_1 \circ \gamma_n(s)| < 1$$

Claim 1 : Given a continuous curve $\gamma : J = [\alpha, \beta] \to \mathbb{R}^2$, with

$$\sup_{t,s \in J} |p_1 \circ \gamma(t) - p_1 \circ \gamma(s)| < 1 \tag{12}$$

$$|p_2 \circ \gamma(\beta) - p_2 \circ \gamma(\alpha)| > \frac{(2+b)}{K} \tag{13}$$

Then $\exists\ s \in \mathbb{Z}$, such that $\widetilde{\phi}_z + s \in p_1\left(\widetilde{T} \circ \gamma(J)\right)$.

Proof.

$$\begin{aligned}&\sup_{t,s\in J}\left|p_1\circ\widetilde{T}\circ\gamma(t)-p_1\circ\widetilde{T}\circ\gamma(s)\right|=\\&=\sup_{t,s\in J}\left|\widetilde{T}_\phi\circ\gamma(t)-\widetilde{T}_\phi\circ\gamma(s)\right|\geq\left|\widetilde{T}_\phi\circ\gamma(\beta)-\widetilde{T}_\phi\circ\gamma(\alpha)\right|\geq\\&\geq -b+K.\tfrac{(2+b)}{K}=2\end{aligned}$$

So the claim is proved. ■

$\gamma_n(t)$ (see (11)) satisfies the claim hypothesis, by construction. So $\exists\ s\in\mathbb{Z}$ such that $\widetilde{\phi}_z+s\in p_1\left(\widetilde{T}\circ\gamma_n(J)\right)=p_1\left(\widetilde{T}^{n+1}(\overline{z_1z})\right)$.

As $\inf_{t\in J}p_2\left(\gamma_n(t)\right)=p_2\left(\gamma_n(p_2(z_1))\right)=3+a$, from the choice of $a>0$ we get that $\inf_{t\in J}p_2\left(\widetilde{T}\circ\gamma_n(t)\right)>3$.

So there exists $t'\in J$ and $z'=(\widetilde{\phi}_z,t')\in r$ such that:

$$\begin{aligned}&p_1\circ\widetilde{T}^{n+1}(z')=\widetilde{\phi}_z\ (mod\ 1)\\&p_2\circ\widetilde{T}^{n+1}(z')>3\end{aligned}$$

And this contradicts (9) and (10). So $\exists\ n'>0$ and $z'\in r$, such that $\widetilde{T}^{n'}(z')=z'+(s,pos)$ for some $s\in\mathbb{Z}$, with $pos>1$. ■

Thus from theorem 1 hypotheses, applying lemma 4, we get that there are points $z',w'\in S^1\times[0,1]$ and numbers $n',k'\in\mathbb{N}^*$ such that

$$\begin{array}{l}\widehat{T}^{n'}(z')=z'+(0,pos)\\\widehat{T}^{k'}(w')=w'+(0,neg)\end{array},\ \text{for constants } pos>1 \text{ and } neg<-1.$$

For some $s,l\in\mathbb{Z}$, $z'\in K(s,n')$ and $w'\in K(l,k')$, see definition (1) and lemma 1. If we remember lemmas 2 and 3, we get that $\nu^+(p_1(z'))-\mu^-(p_1(z'))\geq pos>1$. So, we have 2 possibilities:

i) there exists $z^*\in C(s,n')$ such that $\widehat{T}^{n'}(z^*)=z^*+(0,1)$

ii) as $C(s,n')$ is connected, for all $x\in C(s,n')$, $p_2\circ\widehat{T}^{n'}(x)-p_2(x)>1\Rightarrow\nu^-(\widehat{\phi})-\mu^+(\widehat{\phi})>1$, for all $\widehat{\phi}\in S^1$.

So there exists a simple closed curve $\gamma\subset\mathrm{S}^1\times\mathbb{R}$, which is a graph over S^1, that satisfies:

$$\widehat{T}^{n'}(\gamma)\text{ is above }\gamma+(0,1)\tag{14}$$

As $\widehat{T}$ is the lift of a torus mapping homotopic to the Dehn twist, expression (14) implies that $\widehat{T}^{n'}(\gamma+(0,m))$ is above $\gamma+(0,1+m)$, for all $m\in\mathbb{Z}$. So, for all $x\in\mathrm{S}^1\times\mathbb{R}$,

$$\liminf_{n\to\infty}\ \frac{p_2\circ\widehat{T}^n(x)-p_2(x)}{n}\geq\frac{1}{n'}>0.\tag{15}$$

Now we consider $w'\in K(l,k')$. In the same way as above, we get that $\nu^-(p_1(w'))-\mu^+(p_1(w'))\leq neg<-1$. So, again, we have 2 possibilities:

a) there exists $w^*\in C(l,k')$ such that $\widehat{T}^{k'}(w^*)=w^*-(0,1)$. But this clearly contradicts (15).

b) as $C(l,k')$ is connected, for all $x \in C(l,k')$, $p_2 \circ \widehat{T}^{k'}(x) - p_2(x) < -1 \Rightarrow \nu^+(\widehat{\phi}) - \mu^-(\widehat{\phi}) < -1$, for all $\widehat{\phi} \in S^1$.

So there exists a simple closed curve $\alpha \subset \mathrm{S}^1 \times \mathbb{R}$, which is a graph over S^1, that satisfies:

$$\widehat{T}^{k'}(\alpha) \text{ is below } \alpha - (0,1) \tag{16}$$

As above, expression (16) implies that $\widehat{T}^{k'}(\alpha + (0,m))$ is below $\alpha + (0, m-1)$, for all $m \in \mathbb{Z}$. So, for all $x \in \mathrm{S}^1 \times \mathbb{R}$,

$$\limsup_{n\to\infty} \frac{p_2 \circ \widehat{T}^n(x) - p_2(x)}{n} \leq -\frac{1}{k'} < 0,$$

which again contradicts (15). Thus possibility ii) above does not happen and we get that there exists $z^* \in C(s,n')$ such that $\widehat{T}^{n'}(z^*) = z^* + (0,1)$. From the existence of z^*, possibility b) can not happen and so there exists $w^* \in C(l,k')$ such that $\widehat{T}^{k'}(w^*) = w^* - (0,1)$. The orbits of z^* and w^* project to periodic orbits for T, the mapping on the torus. Let O_{z^*} and O_{w^*} denote these periodic orbits and let $Q = O_{z^*} \cup O_{w^*}$. Now blow-up each $x \in Q$ to a circle S_x. Let $\mathrm{T}^2_{\mathrm{Q}}$ be the compact manifold (with boundary) thereby obtained ; $\mathrm{T}^2_{\mathrm{Q}}$ is the compactification of $\mathrm{T}^2 \backslash \mathrm{Q}$, where S_x is a boundary component where x was deleted. Now we extend $T : \mathrm{T}^2\backslash\mathrm{Q} \to \mathrm{T}^2\backslash\mathrm{Q}$ to $T_Q : \mathrm{T}^2_{\mathrm{Q}} \to \mathrm{T}^2_{\mathrm{Q}}$ by defining $T_Q : S_x \to S_x$ via the derivative; we just have to think of S_x as the unit circle in $T_x\mathrm{T}^2$ and define

$$T_Q(v) = \frac{DT_x(v)}{\|DT_x(v)\|}, \text{ for } v \in S_x.$$

T_Q is continuous on $\mathrm{T}^2_{\mathrm{Q}}$ because T is C^1 on T^2. Let $b : \mathrm{T}^2_{\mathrm{Q}} \to \mathrm{T}^2$ be the map that collapses each S_x onto x. Then $T \circ b = b \circ T_Q$. This gives $h(T_Q) \geq h(T)$ (see [5], page 111). Actually $h(T_Q) = h(T)$, because each fibre $b^{-1}(y)$ is a simple point or an S_x and the entropy of T on any of these fibres is 0 (the map on the circle induced from any linear map has entropy 0). This construction is due to Bowen (see [2]). In [1] we proved the following theorem:

Theorem 2 *: The mapping $T_Q : \mathrm{T}^2_{\mathrm{Q}} \to \mathrm{T}^2_{\mathrm{Q}}$ is isotopic to a pseudo-Anosov homeomorphism of $\mathrm{T}^2_{\mathrm{Q}}$.*

In a certain sense, pseudo-Anosov homeomorphisms are the less "complex" mappings in their isotopy classes. One of the many motivations for this imprecise notion is the following theorem (see [3] and [4] for a proof and for more information on the theory):

Theorem 3 *: Let M be a compact, connected oriented surface possibly with boundary, and $f, g : M \to M$ be two isotopic homeomorphisms. If g is pseudo-Anosov, then their topological entropies satisfy: $h(f) \geq h(g) > 0$.*

The above results imply that $h(T) = h(T_Q) > 0$. Finally, in order to conclude the proof, we just have to remember the following result due to Katok, see for instance the appendix of [5].

Lemma 5 *: Let $f : M \to M$ be a $C^{1+\epsilon}$ (for any $\epsilon > 0$) diffeomorphism of a closed surface M. If $h(f) > 0$, then f has a hyperbolic periodic point, whose stable and unstable manifolds intersect transversally.*

And the proof of our criteria is complete.

References

[1] Addas-Zanata S. (2002): On the existence of a new type of periodic and quasi-periodic orbits for twist maps of the torus. *Nonlinearity* **15 (5)**, 1399-1416

[2] Bowen R. (1978): Entropy and the Fundamental Group. *Springer Lec. Notes in Math.* **668,** 21-29

[3] Fathi A., Laudenbach F. and Poenaru V. (1979): Travaux de Thurston sur les surfaces. *Astérisque.* **66-67**

[4] Handel M. and Thurston W. (1985): New proofs of some Results of Nielsen. *Adv. Math.* **56,** 173-191

[5] Katok A. and Hasselblatt B. (1995): *Introduction to Modern Theory of Dynamical Systems.* Cambridge University Press

[6] Le Calvez P. (1986): Existence d'orbites quasi-périodiques dans les attracteurs de Birkhoff. *Comm. Math. Phys.* **106**, 383-394

[7] Le Calvez P. (1991): Propriétés Dynamiques des Difféomorphismes de L'Anneau et du Tore. *Astérisque* **204**

[8] Le Calvez P. (1995): Construction d'orbites périodiques par perturbation d'un difféomorphisme de l'anneau déviant la verticale. *C. R. Acad. Sci. Paris,* **t 321,** p. 463-468